Minimum dilatation problem for pseudo-Anosov mapping classes

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47th Spring Topology and Dynamics Conference
Central Connecticut State University
March 23, 2013

Minimum dilatation problem

Let S be a compact surface with $\chi(S) < 0$, and let $\phi : S \to S$ a homeomorphism.

Loosely speaking, ϕ is pseudo-Anosov if ϕ is "well-mixing" .

The dilatation $\lambda(\phi)$ is the "average distortion" of the map.

These notions only depend on the isotopy class of the map.

Problem 1: For fixed S what is the least dilatation of a pseudo-Anosov map?

Problem 2: How does least dilatation depend on the complexity of the surface S (e.g. genus, topological Euler characteristic)?

Problem 3: What do the minimizing pseudo-Anosov maps look like?

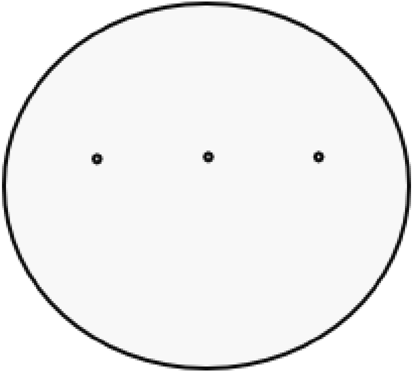
Outline

In this talk, we consider two examples:

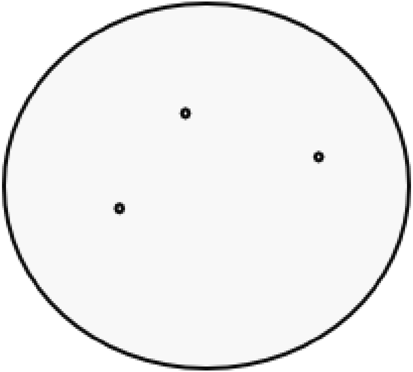
- the simplest pseudo-Anosov braid monodromy and its "deformations"
- an example of Penner

Using these examples, we formulate some conjectural answers to Problems 2 and 3.

Example 1: Simplest pseudo-Anosov braid monodromy



simplest pseudo-Anosov braid:



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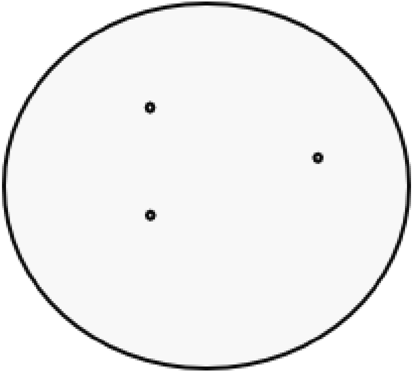

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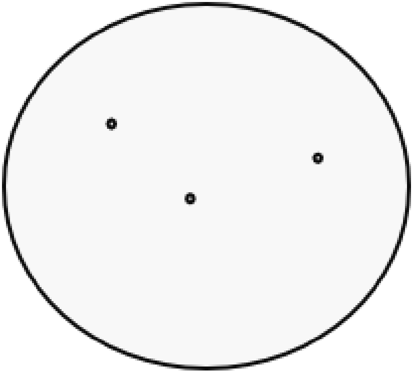

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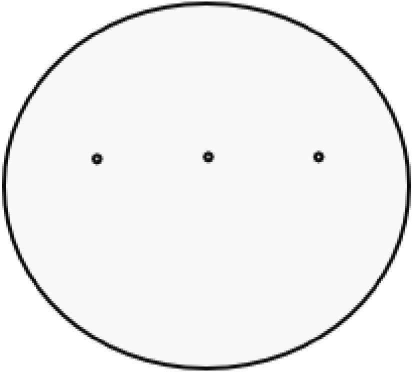

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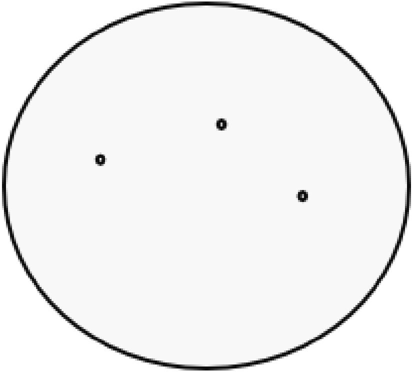

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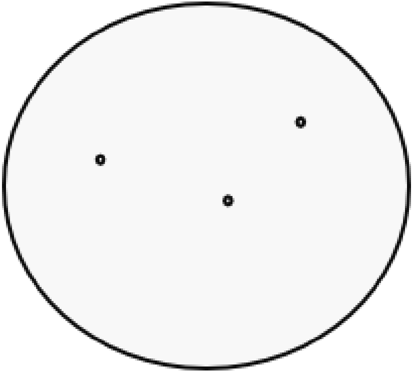

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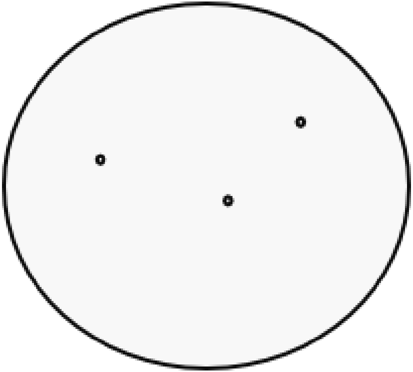

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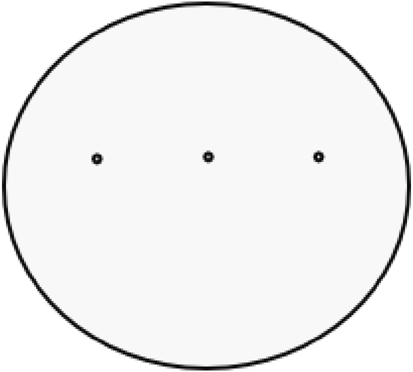

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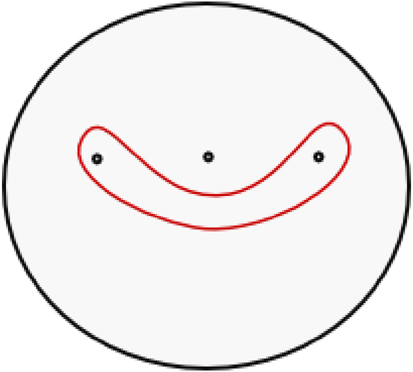
Action of the mapping class

Action of the mapping class on a simple closed curve.

Action on a simple closed curve:



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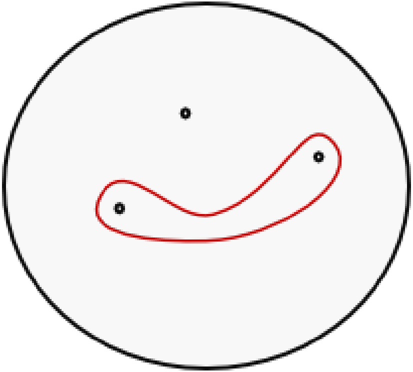

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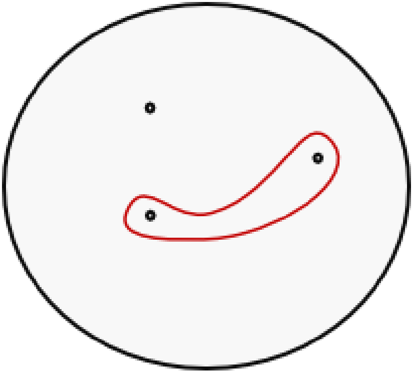

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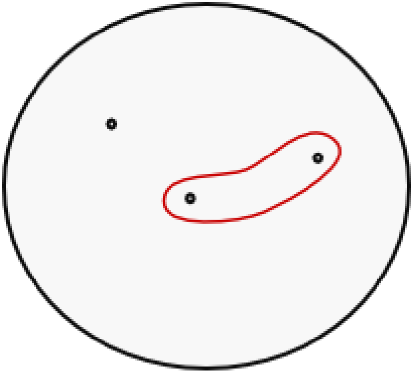

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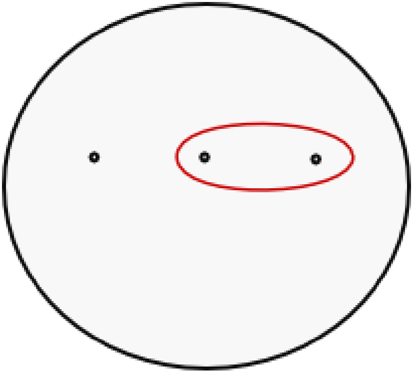

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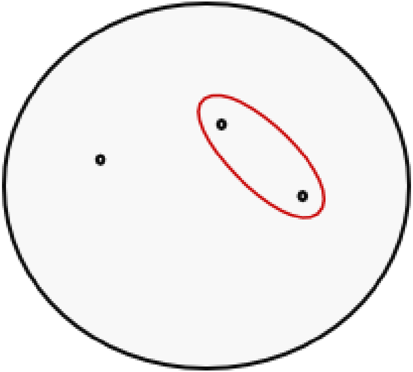

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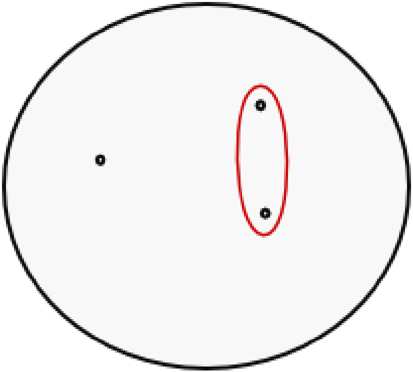

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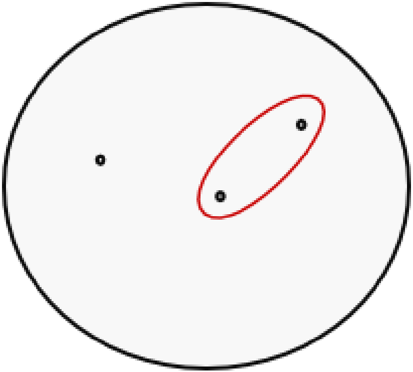
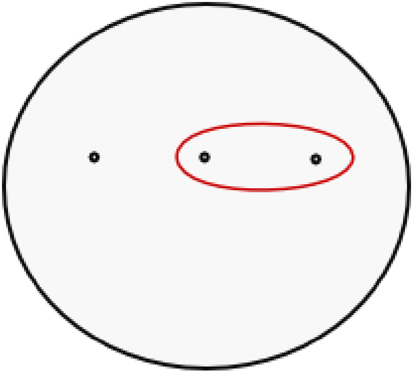


Action on a simple closed curve (one application of map):



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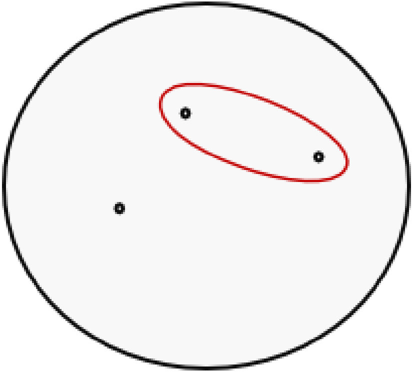

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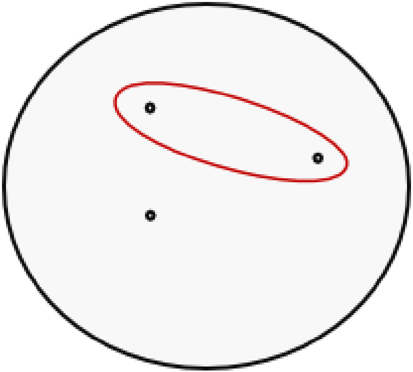

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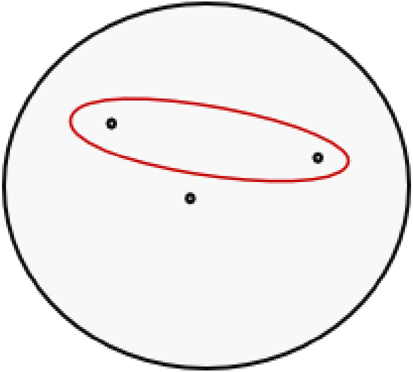

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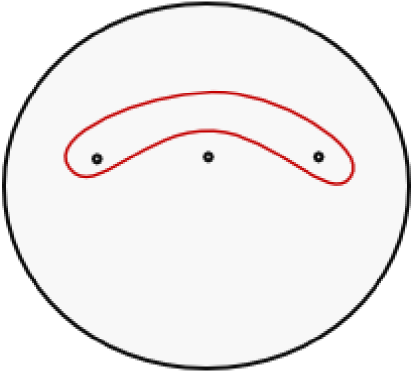

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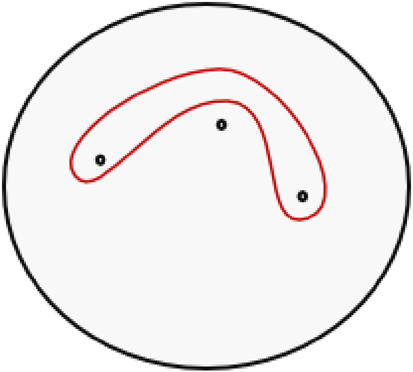

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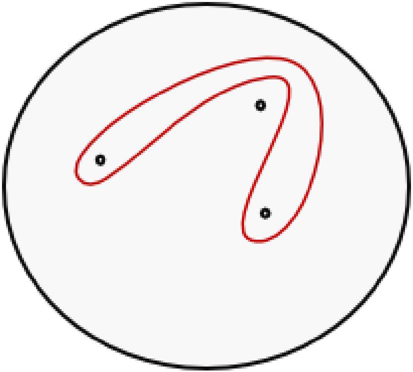

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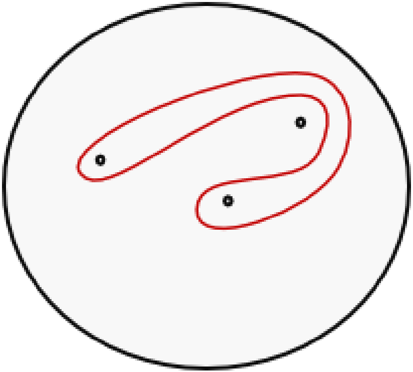
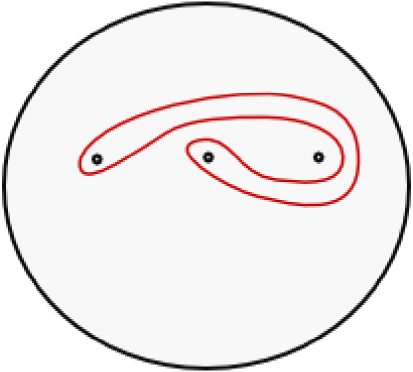
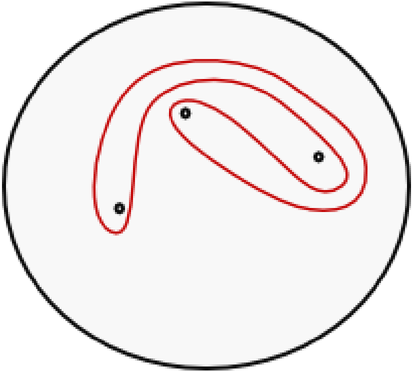


Action on a simple closed curve (2 applications of map):



Action on a simple closed curve:



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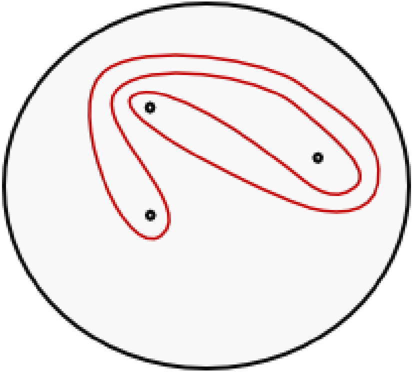

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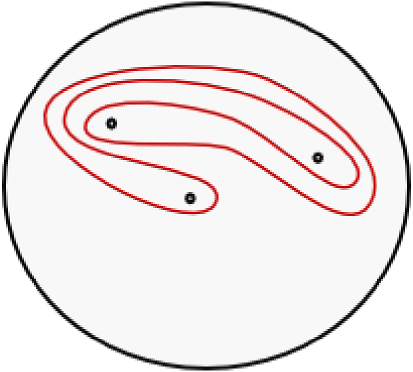

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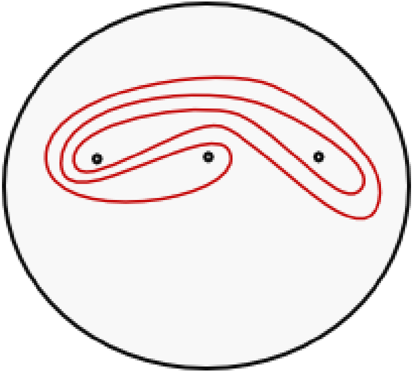

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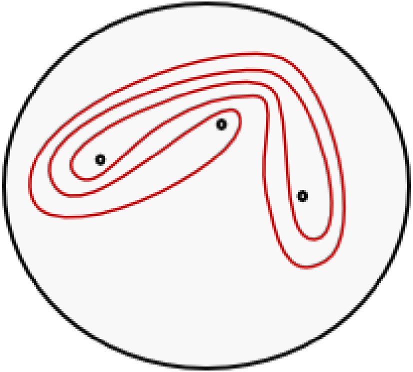

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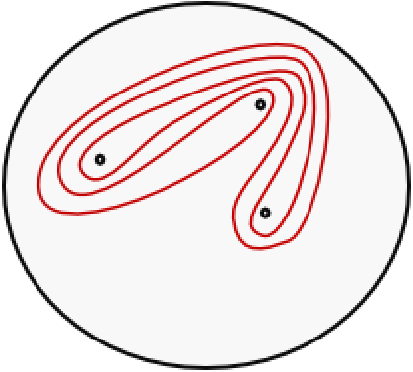

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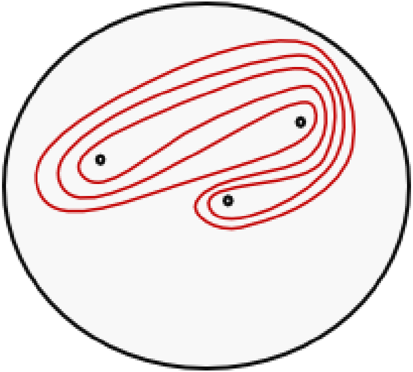
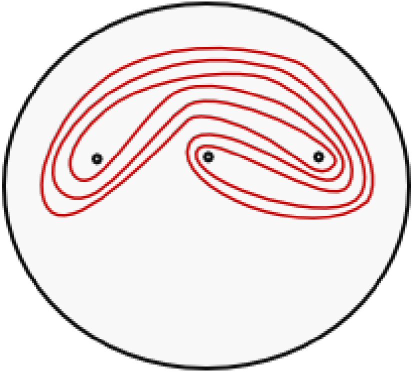


Action on a simple closed curve (3 applications of map):



back to start

Definitions

A homeomorphism $\phi : S \to S$ is *pseudo-Anosov* if there is a pair of ϕ-invariant transverse measured singular foliations $(\mathcal{F}^{\pm}, \nu^{\pm})$ on S and a $\lambda > 1$ so that the action of ϕ on S acts on the measures by $\phi\nu^{\pm} = \lambda^{\pm 1}\nu^{\pm}$.

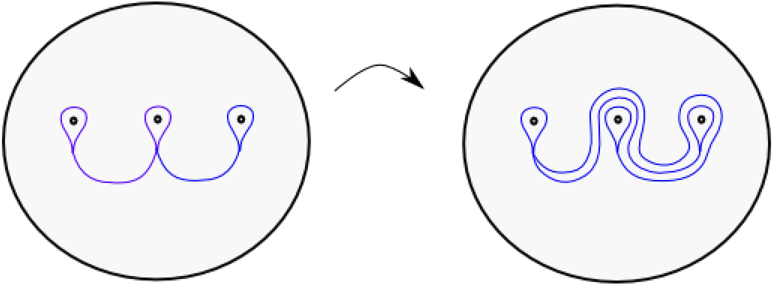
Equivalently:

ϕ is pseudo-Anosov if for any Riemannian metric ω on S and any essential simple closed curve $\gamma \subset S$, the growth rate of $\ell_\omega(\phi^n(\gamma))$ is $\lambda > 1$, where λ does not depend on γ or ω.

$\lambda(\phi) = \lambda$ is called the dilatation of ϕ.

An associated train track map

A train track τ is an embedded graph on S, with "smoothings" of the edges along vertices.
It fills S if the complement components are either disks or boundary parallel annuli.
An essential simple closed curve is carried on a train track if it can be moved isotopically so that it lies smoothly on τ.
If ϕ is pseudo-Anosov, then there is a train track τ on S such that for any essential simple closed curve γ on S, $\phi^n(\gamma)$ is carried on τ for large enough n.



Computing the dilatation of the simplest hyperbolic braid monodromy

A train track τ defines a vector space W of "virtual curves" carried by the train track.
Any pseudo-Anosov map ϕ that is compatible with τ induces a linear map on $T : W \to W$.

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

The dilatation λ equals the spectral radius of T:

$$|x^2 - 3x + 1| = \frac{3 + \sqrt{5}}{2} = (\text{golden mean})^2$$

Minimization problem I

If (S, ϕ) is pseudo-Anosov, then

- $\lambda(\phi)$ is an algebraic integer, in fact, a Perron number,
- the degree of $\lambda(\phi)$ is bounded in terms of the topology of S,

It follows that for a fixed surface S the set of $\lambda(\phi)$ forms a discrete set of algebraic integers.

Problem 1: For fixed S what is the smallest dilatation?

For example, for $S = S_{0,4}$, the simplest pseudo-Anosov braid monodromy has smallest dilatation.

The minimum is also known for $S_{0,n}$ for $n = 5, 6, 7, 8$, and $S_{1,1}, S_{2,0}$ [Ko-Los-Song'02, Ham-Song'05, Cho-Ham'08, H_-Kin'06, Aaber'06, Lanneau-Thiffeault'11]

Minimization Problems II, III

Let $\mathcal{P}$ be the set of all pseudo-Anosov mapping classes (S is allowed to vary).

The *normalized dilatation* of (S, ϕ) is defined to be

$$L(S, \phi) = \lambda(S)^{|\chi(\phi)|}.$$

Problem 2: What is the smallest accumulation point of L?

Problem 3: What do mapping classes with bounded L look like?

Deformations of pseudo-Anosov mapping classes

By Thurston's theory of fibered faces, $\mathcal{P}$ partitions into families associated to *fibered faces* parametrized by rational points $F_{\mathbb{Q}}$ on a convex Euclidean polyhedron F. The mapping classes belonging to a single $F_{\mathbb{Q}}$ correspond to transversal (recurrent) surfaces to a single pseudo-Anosov flow on a hyperbolic 3-manifold, and thus have related dynamics.

This gives a decomposition of $\mathcal{P}$:

$$\mathcal{P} = \bigcup F_{\mathbb{Q}}$$

as opposed to the more usual

$$\mathcal{P} = \bigcup_{S} \mathcal{P}_S.$$

In the former, one has a notion of deformation of a mapping class on a stratum, while in the latter each stratum is discrete.

Deformations of pseudo-Anosov mapping classes

Theorem (Fried '82, Matsumoto '87, McMullen '00)

The normalized dilatation function

$$L(S, \phi) = \lambda(S)^{|\chi(S)|}$$

defined on $F_{\mathbb{Q}}$ extends to a continuous convex function on F.

Corollary

For each F, L has a unique minimum on F.

Remark: (Hongbin Sun), the minimum is not necessarily attained by an element of $F_{\mathbb{Q}}$.

Describing pseudo-Anosov maps with bounded L

Let $\mathcal{P}^0 \subset \mathcal{P}$ be the set of pA maps with no interior singularities.

Theorem (Farb-Leininger-Margalit'08)

For any $P > 1$, there is a finite collection of fibered faces $F_1, \ldots, F_k$ such that for any $(S, \phi) \in \mathcal{P}^0$ such that $L(S, \phi) < P$, we have $(S, \phi) \in F_i$ for some i.

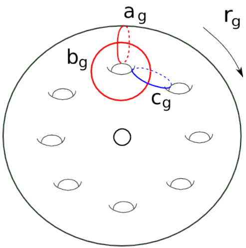
Consequences and remarks:

- To describe the small dilatation maps it suffices to describe what the monodromy on single fibered faces look like.
- For the moment, there is no good bound on the number k. It would be nice to be able to relate geometric information about the a fibered 3-manifold to the size of minimum normalized dilatation.
- Opposite approach: look at natural families of small dilatation maps, and the fibered faces they determine.

Penner's Example

(Penner '91) First explicit example of a small dilatation family:

$$\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}$$



Penner: $\lambda(S_g)^g$ is bounded.

More generally, for each $\frac{k}{m}$, with $m \geq 2$ and $k \geq 1$, we can consider

$$\phi_{k,m} = r_m^k \delta_c \delta_b^{-1} \delta_a.$$

This gives a family of pA maps parameterized by rational points on an open interval.

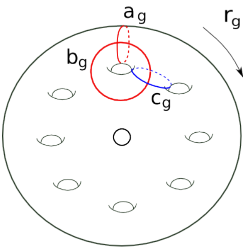
Convergence of Penner's sequence

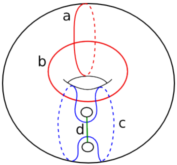
Theorem (H_'12)

Each Penner-type family is a one-dimensional linear section of a fibered face.

Application: Penner's sequence is a convergent sequence on a fibered face F

$$\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g} \qquad \longrightarrow \qquad \phi = \delta_c \delta_b^{-1} \delta_a.$$





and $L(S_g, \phi_g) \to L(S, \phi) = |u^2 - 7u + 1|^2 = 46.9787\ldots$

Deformations of the simplest pseudo-Anosov braid monodromy

(Thurston'80s, McMullen'00, H_'09)

- The fibered face associated to the simplest pA braid is 1-dimensional.

- One can parameterize the fibered face by an open interval $(-1,1)$ so that $\frac{k}{m} \in (-1,1)$ corresponds to a pseudo-Anosov map (S,ϕ), where
$$\chi(S) = -m$$
and
$$\lambda(\phi) = |x^{2m} - x^{m+k} - x^m - x^{m-k} + 1|.$$

- The minimum normalized dilatation occurs at 0, and
$$L(S_0,\phi_0) = \left(\frac{3+\sqrt{5}}{2}\right)^2 \approx 6.8541\ldots$$

Train track automata (Ko-Los-Song '04)

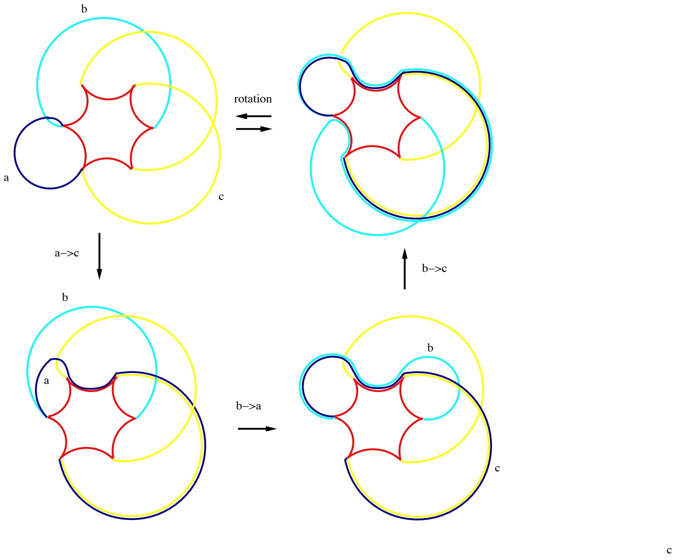
Train track maps can be decomposed into a composition of folding maps, where two edges meeting at a cusp get identified.

This changes the train track to a new one that is still compatible with the same pseudo-Anosov map.

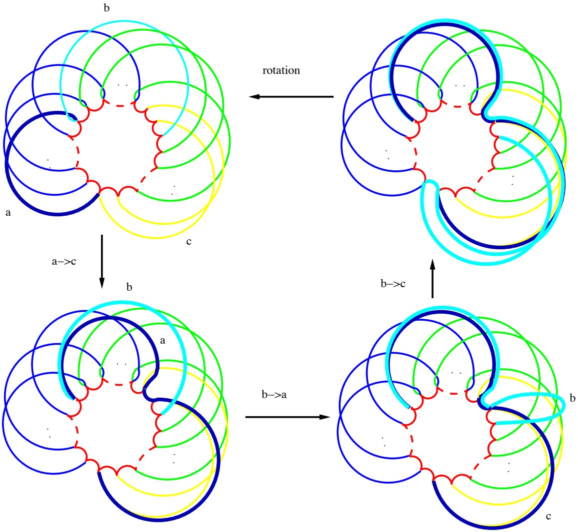
One can define an automaton, where the train tracks are the vertices, and there is a directed edge from a train track to the result of one folding.

(also studied by Ham-Song, Cho-Ham, Lanneau-Thiffeault)

Train track map for the deformation of the simplest pA braid at $k/m = 1/2$



Train track map for $1/m$ (where $3 \nmid m$)



Murasugi-sum of mapping classes

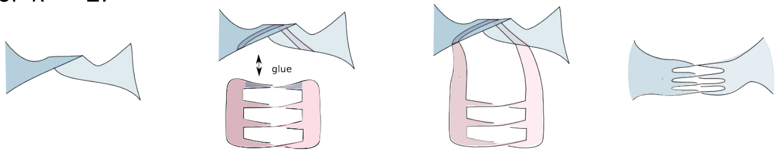
Observation: the maps $(S_{\frac{1}{m}}, \phi_{\frac{1}{m}})$ can be obtained from $(S_{\frac{1}{2}}, \phi_{\frac{1}{2}})$ by a sequence of Murasugi-sum with mapping classes that are periodic relative to their boundary.

(H₋) For deformations of the simplest pA braid monodromy, they can be described using mixed-sign Coxeter graphs, and as twisted maps.

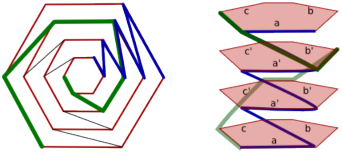
Twisted maps

Let (S, ϕ) be a pA map, where S is a surface with boundary, Let $P_k \subset S$ be a $2k$-gon, such that every other boundary edge lies in ∂S, and the rest of P_k lies in the interior of S. Then we can define a family of twisted maps (S_n, ϕ_n) by

For $k = 2$:



For $k = 3$:



Conjectural answers to Minimizations Problems II and III

Conjecture

The smallest accumulation point for L on $\mathcal{P}$ equals

$$\left(\frac{3+\sqrt{5}}{2}\right)^2.$$

Conjecture

For $P > 1$ there is a constant C (depending on P), such that for every (S, ϕ) with $L(S, \phi) < P$, we have

- *a subsurface $Y \subset S$ with $|\chi(Y)| < C$, and a mapping class $\widehat{\phi}$ supported on Y,*
- *a subsurface $\Sigma \subset S$, and a mapping class R supported on Σ that is periodic relative to the boundary of Σ,*

such that $\phi = R \circ \widehat{\phi}$.

Thank you.