Minimum dilatation problem for pseudo-Anosov mapping classes

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Minimum dilatation problem

Let $S$ be a compact surface with $\chi(S) < 0$, and let $\phi : S \to S$ a homeomorphism.

Loosely speaking, $\phi$ is pseudo-Anosov if $\phi$ is “well-mixing”.

The dilatation $\lambda(\phi)$ is the “average distortion” of the map.

These notions only depend on the isotopy class of the map.

**Problem 1:** For fixed $S$ what is the least dilatation of a pseudo-Anosov map?

**Problem 2:** How does least dilatation depend on the complexity of the surface $S$ (e.g. genus, topological Euler characteristic)?

**Problem 3:** What do the minimizing pseudo-Anosov maps look like?
In this talk, we consider two examples:

- the simplest pseudo-Anosov braid monodromy and its "deformations"
- an example of Penner

Using these examples, we formulate some conjectural answers to Problems 2 and 3.
Example 1: Simplest pseudo-Anosov braid monodromy
simplest pseudo-Anosov braid:
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Action of the mapping class on a simple closed curve.
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Action on a simple closed curve (one application of map):
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Action on a simple closed curve (2 applications of map):
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Action on a simple closed curve:
Action on a simple closed curve (3 applications of map):
A homeomorphism \( \phi : S \to S \) is pseudo-Anosov if there is a pair of \( \phi \)-invariant transverse measured singular foliations \((\mathcal{F}^\pm, \nu^\pm)\) on \( S \) and a \( \lambda > 1 \) so that the action of \( \phi \) on \( S \) acts on the measures by \( \phi \nu^\pm = \lambda^\pm \nu^\pm \).

Equivalently:

\( \phi \) is pseudo-Anosov if for any Riemannian metric \( \omega \) on \( S \) and any essential simple closed curve \( \gamma \subset S \), the growth rate of \( \ell_\omega(\phi^n(\gamma)) \) is \( \lambda > 1 \), where \( \lambda \) does not depend on \( \gamma \) or \( \omega \).

\( \lambda(\phi) = \lambda \) is called the dilatation of \( \phi \).
An associated train track map

A train track $\tau$ is an embedded graph on $S$, with "smoothenings" of the edges along vertices. It fills $S$ if the complement components are either disks or boundary parallel annuli.

An essential simple closed curve is carried on a train track if it can be moved isotopically so that it lies smoothly on $\tau$.

If $\phi$ is pseudo-Anosov, then there is a train track $\tau$ on $S$ such that for any essential simple closed curve $\gamma$ on $S$, $\phi^n(\gamma)$ is carried on $\tau$ for large enough $n$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{train_track_diagram.png}
\end{figure}
Computing the dilatation of the simplest hyperbolic braid monodromy

A train track \( \tau \) defines a vector space \( W \) of “virtual curves” carried by the train track. Any pseudo-Anosov map \( \phi \) that is compatible with \( \tau \) induces a linear map on \( T : W \rightarrow W \).

\[
T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\]

The dilatation \( \lambda \) equals the spectral radius of \( T \):

\[
|x^2 - 3x + 1| = \frac{3 + \sqrt{5}}{2} = (\text{golden mean})^2
\]
Minimization problem I

If $(S, \phi)$ is pseudo-Anosov, then

- $\lambda(\phi)$ is an algebraic integer, in fact, a Perron number,
- the degree of $\lambda(\phi)$ is bounded in terms of the topology of $S$,

It follows that for a fixed surface $S$ the set of $\lambda(\phi)$ forms a discrete set of algebraic integers.

**Problem 1:** For fixed $S$ what is the smallest dilatation?

For example, for $S = S_{0,4}$, the simplest pseudo-Anosov braid monodromy has smallest dilatation.

The minimum is also known for $S_{0,n}$ for $n = 5, 6, 7, 8$, and $S_{1,1}, S_{2,0}$ [Ko-Los-Song’02, Ham-Song’05, Cho-Ham’08, H-Kin’06, Aaber’06, Lanneau-Thiffeault’11]
Minimization Problems II, III

Let \( \mathcal{P} \) be the set of all pseudo-Anosov mapping classes (\( S \) is allowed to vary).

The \textit{normalized dilatation} of \((S, \phi)\) is defined to be

\[
L(S, \phi) = \lambda(S)|\chi(\phi)|.
\]

**Problem 2:** What is the smallest accumulation point of \( L \)?

**Problem 3:** What do mapping classes with bounded \( L \) look like?
Deformations of pseudo-Anosov mapping classes

By Thurston’s theory of fibered faces, $\mathcal{P}$ partitions into families associated to fibered faces parametrized by rational points $F_{\mathbb{Q}}$ on a convex Euclidean polyhedron $F$. The mapping classes belonging to a single $F_{\mathbb{Q}}$ correspond to transversal (recurrent) surfaces to a single pseudo-Anosov flow on a hyperbolic 3-manifold, and thus have related dynamics.

This gives a decomposition of $\mathcal{P}$:

$$\mathcal{P} = \bigcup F_{\mathbb{Q}}$$

as opposed to the more usual

$$\mathcal{P} = \bigcup_{S} \mathcal{P}_S.$$  

In the former, one has a notion of deformation of a mapping class on a stratum, while in the latter each stratum is discrete.
Deformations of pseudo-Anosov mapping classes

Theorem (Fried '82, Matsumoto '87, McMullen '00)

The normalized dilatation function

\[ L(S, \phi) = \lambda(S)^{|\chi(S)|} \]

defined on \( F_\mathbb{Q} \) extends to a continuous convex function on \( F \).

Corollary

For each \( F \), \( L \) has a unique minimum on \( F \).

Remark: (Hongbin Sun), the minimum is not necessarily attained by an element of \( F_\mathbb{Q} \).
Describing pseudo-Anosov maps with bounded $L$

Let $\mathcal{P}^0 \subset \mathcal{P}$ be the set of pA maps with no interior singularities.

**Theorem (Farb-Leininger-Margalit’08)**

*For any $P > 1$, there is a finite collection of fibered faces $F_1, \ldots, F_k$ such that for any $(S, \phi) \in \mathcal{P}^0$ such that $L(S, \phi) < P$, we have $(S, \phi) \in F_i$ for some $i$.***

Consequences and remarks:

- To describe the small dilatation maps it suffices to describe what the monodromy on single fibered faces look like.

- For the moment, there is no good bound on the number $k$. It would be nice to be able to relate geometric information about the a fibered 3-manifold to the size of minimum normalized dilatation.

- Opposite approach: look at natural families of small dilatation maps, and the fibered faces they determine.
Penner’s Example

(Penner ’91) First explicit example of a small dilatation family:

\[ \phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g} \]

Penner: \( \lambda(S_g)^g \) is bounded.

More generally, for each \( \frac{k}{m} \), with \( m \geq 2 \) and \( k \geq 1 \), we can consider

\[ \phi_{k,m} = r_m^k \delta_c \delta_b^{-1} \delta_a. \]

This gives a family of pA maps parameterized by rational points on an open interval.
Convergence of Penner’s sequence

Theorem (H.’12)

Each Penner-type family is a one-dimensional linear section of a fibered face.

Application: Penner’s sequence is a convergent sequence on a fibered face $F$

$$
\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g} \quad \longrightarrow \quad \phi = \delta_c \delta_b^{-1} \delta_a.
$$

and $L(S_g, \phi_g) \to L(S, \phi) = |u^2 - 7u + 1|^2 = 46.9787 \ldots$
Deformations of the simplest pseudo-Anosov braid monodromy

(Thurston’80s, McMullen’00, H_’09)

- The fibered face associated to the simplest pA braid is 1-dimensional.

- One can parameterize the fibered face by an open interval \((-1, 1)\) so that \(\frac{k}{m} \in (-1, 1)\) corresponds to a pseudo-Anosov map \((S, \phi)\), where

\[
\chi(S) = -m
\]

and

\[
\lambda(\phi) = |x^{2m} - x^{m+k} - x^m - x^{m-k} + 1|.
\]

- The minimum normalized dilatation occurs at 0, and

\[
L(S_0, \phi_0) = \left(\frac{3 + \sqrt{5}}{2}\right)^2 \approx 6.8541\ldots
\]
Train track automata (Ko-Los-Song ’04)

Train track maps can be decomposed into a composition of folding maps, where two edges meeting at a cusp get identified.

This changes the train track to a new one that is still compatible with the same pseudo-Anosov map.

One can define an automaton, where the train tracks are the vertices, and there is a directed edge from a train track to the result of one folding.

(also studied by Ham-Song, Cho-Ham, Lanneau-Thiffeault)
Train track map for the deformation of the simplest pA braid at $k/m = 1/2$
Train track map for $1/m$ (where $3 \parallel m$)
Murasugi-sum of mapping classes

Observation: the maps \((S_{\frac{1}{m}}, \phi_{\frac{1}{m}})\) can be obtained from \((S_{\frac{1}{2}}, \phi_{\frac{1}{2}})\) by a sequence of Murasugi-sum with mapping classes that are periodic relative to their boundary.

\(\text{(H)}\) For deformations of the simplest pA braid monodromy, they can be described using mixed-sign Coxeter graphs, and as twisted maps.
Twisted maps

Let \((S, \phi)\) be a pA map, where \(S\) is a surface with boundary. Let \(P_k \subset S\) be a \(2k\)-gon, such that every other boundary edge lies in \(\partial S\), and the rest of \(P_k\) lies in the interior of \(S\). Then we can define a family of twisted maps \((S_n, \phi_n)\) by

For \(k = 2\):

For \(k = 3\):
Conjectural answers to Minimizations Problems II and III

Conjecture

The smallest accumulation point for $L$ on $\mathcal{P}$ equals

\[
\left(\frac{3 + \sqrt{5}}{2}\right)^2.
\]

Conjecture

For $P > 1$ there is a constant $C$ (depending on $P$), such that for every $(S, \phi)$ with $L(S, \phi) < P$, we have

- a subsurface $Y \subset S$ with $|\chi(Y)| < C$, and a mapping class $\hat{\phi}$ supported on $Y$,

- a subsurface $\Sigma \subset S$, and a mapping class $R$ supported on $\Sigma$ that is periodic relative to the boundary of $\Sigma$, such that $\phi = R \circ \hat{\phi}$. 
Thank you.