Fibered Faces and Dynamics of Mapping Classes II
Branched Coverings, Degenerations, and Related Topics 2012
Hiroshima University

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   1. Introduction
   2. Visualizing pseudo-Anosov mapping classes
   3. Train tracks
   4. Minimum dilatation problem

II. Fibered faces and applications
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   2. Fibered face theory
   3. Alexander and Teichmüller polynomials
   4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations
   1. Introduction
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Introduction

Setup:
- $S$ compact oriented surface, $g$ genus, $n$ number of boundary components, $\chi(S) = 2 - 2g - n < 0$.
- $\text{Mod}(S)$ mapping class group, $\text{Mod}(S) = \text{Homeo}^+ / \sim$
- $\mathcal{P}_S$ pseudo-Anosov elements of $\text{Mod}(S)$

Goal in this lecture:
Set $\mathcal{P} = \bigcup_S \mathcal{P}_S$
- Describe an embedding $\mathcal{P} \hookrightarrow \bigcup F_\alpha$, where $F_\alpha$ are some convex polyhedra in Euclidean space (fibered faces), and the image of $\mathcal{P}$ are the union of rational points in the interiors of $F_\alpha$.
- Describe invariants of $\mathcal{P}$ like homological and geometric dilatations from this point of view.
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Horizontal and vertical theory of mapping classes

Think of

\[ \mathcal{P}_S \leftrightarrow horizontal \ theory \ of \ pA \ maps \]

(Mapping class groups, Teichmüller space, moduli space)

\[ F_\alpha \leftrightarrow vertical \ theory \ of \ pA \ maps \]

(3-manifold geometry and topology)

This point of view stems from work of W. Thurston, D. Fried, C. McMullen
Vertical Theory for $\mathcal{P}$
The **mapping torus** $M_\phi$ of a mapping class $(S, \phi)$ is the 3-manifold:

$$M_\phi = S \times [0, 1]/ \sim$$

where $(x, 1) \sim (\phi(x), 0)$.

- The homeomorphism type of $M_\phi$ is determined by the isotopy class of $\phi$.
- $M_\phi$ is hyperbolic $\iff \phi$ is pseudo-Anosov.

Let $\Phi(M) = \{(S, \phi) \mid M_\phi = M\}$, the **monodromies** of $M$. 
{Φ(M)} defines partitions of

\[ \text{Mod} = \bigcup_{\text{S}} \text{Mod}(\text{S}) \quad \text{and} \quad \mathcal{P} = \bigcup_{\text{S}} \mathcal{P}_\text{S}. \]

Add some structure on Φ(M).
Fix a 3-manifold $M$.

Given a connected subsurface $\Sigma \subset M$, define $\chi_-(\Sigma) = \max\{0, -\chi(\Sigma)\}$.

For $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$, $\Sigma_i$ connected, define $\chi_-(\Sigma) = \sum_{i=1}^k \chi_-(\Sigma_i)$.

For $\alpha \in H^1(M; \mathbb{Z})$, $\|\alpha\| = \min\{\chi_-(\Sigma) \mid [\Sigma] \in H_2(M; \mathbb{Z}) \text{ is dual to } \alpha\}$.

$\|\| \|$ extends to the *Thurston (semi-) norm* on $H^1(M; \mathbb{R})$.

If $M$ is hyperbolic, the Thurston norm extends to a norm on $H^1(M; \mathbb{R})$. (Assume hereafter that $M$ is hyperbolic.)
Fibered faces

The *Thurston norm ball*

\[ \{ \alpha \in H^1(M; \mathbb{R}) : \|\alpha\| \leq 1 \} \]

is a convex polyhedron with integer vertices. For each top dimensional face \( F \), let \( C_F \) be the positive cone over \( F \). Let \( H^1(M; \mathbb{Z})^{\text{prim}} \) be the set of *primitive* elements \( \Leftrightarrow \) has a connected dual surface.

Then either:

- \( \Phi(M) \cap C_F = H^1(M; \mathbb{Z})^{\text{prim}} \cap C_F \); or
- \( \Phi(M) \cap C_F = \emptyset \).

In the former case we say \( F \) is a *fibered face* of \( M \).
Vertical partition 2

The sets
\[ \Phi(M, F) = \Phi(M) \cap C_F \subset \Phi(M) \]

define a subpartition of \( \mathcal{P} \).

- Identify each \( \Phi(M, F) \) with rational points on \( F \).
- Identify closure \( \overline{\mathcal{P}} \) with the disjoint union of closures of fibered faces. These are homeomorphic to closed disks of dimension
  \[ d = \dim H^1(M; \mathbb{R}) - 1. \]
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Invariants of mapping classes

We are interested in the following invariants of \((S, \phi) \in \mathcal{P}:\)

- **Dilatation** \(\lambda(\phi)\)
- **Homological dilatation**
  \[
  \lambda_{\text{hom}}(\phi) = \text{Spec.Rad.}(\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})).
  \]
- **Normalized dilatation** of \((S, \phi)\)
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  L(S, \phi) = \lambda(\phi)\chi(S).
  \]

(Useful for studying families of mapping classes with asymptotically small dilatations.)
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Specializing Laurent polynomials

Let $G = \mathbb{Z}^d$, and let

$$f = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$$

be a Laurent polynomial. Here we take $a_g \in \mathbb{Z}$ to be nonzero for only a finite number of $g$.

Let $\psi : G \to \mathbb{Z}$. The specialization of $f$ at $\psi$ is:

$$f^\psi = \sum_{g \in G} a_g t^{\psi(g)}.$$

Given a single variable polynomial $f(t) \in \mathbb{R}[t]$, the house is:

$$|f| = \max\{|\mu| : f(\mu) = 0\}.$$
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Alexander and Teichmüller polynomials

\[ G = H_1(M_\phi; \mathbb{Z}) / \text{Torsion} \cong \mathbb{Z}^d \]

\[ \Delta \in \mathbb{Z}[G] \text{ the Alexander polynomial of } M_\phi. \] (E.g., use Fox calculus.)

For \((S, \phi_a) \in \Phi(M_\phi, F_\phi)\), let \(\psi_a : G \to \mathbb{Z}\) be the map associated to the corresponding fibration \(M_{\phi_a} \to S^1\). Then we have:

\[ \lambda_{\text{hom}}(\phi_a) = |\Delta^{\psi_a}|. \]

Analogously,...

(C. McMullen ’00) Given a pseudo-Anosov mapping class \((S, \phi)\) and associated fibered face \((M, F) = (M_\phi, F_\phi)\), there is a Teichmüller polynomial \(\Theta \in \mathbb{Z}[G]\) such that for all \((S_\alpha, \phi_\alpha) \in \Phi(M, F)\)

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Consequences:

- On connected components of $\text{int}(\overline{P})$, it is possible to find defining equations for homological and geometric dilatations so that the coefficient strings of defining polynomials for homological and geometric dilatations are the same.

- The algebraic integers that realize the homological and geometric dilatations belong to particular kinds of algebraic families.

- This property can help to identify when two mapping classes lie on the same fibered face.
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Normalized dilatation and fibered faces

(D. Fried ’82, C. McMullen ’00) The normalized dilatation

\[ L : \mathcal{P} \rightarrow \mathbb{R} \]

\[(S, \phi) \mapsto \lambda(\phi) |\chi(S)| \]

extends to a continuous convex function on \( \text{int}(\overline{\mathcal{P}}) \) that goes to infinity toward the boundary of \( \overline{\mathcal{P}} \).
Effect of puncturing at singularities:

For \((S, \phi) \in \mathcal{P}\), let \(S^0 = S \setminus \text{Sing}(\phi)\) and \(\phi^0 = \phi|_{S^0}\).

Equivalence relation on \(\mathcal{P}\):
Write \((S_1, \phi_1) \sim (S_2, \phi_2)\) if \((S_1^0, \phi_1^0) = (S_2^0, \phi_2^0)\).

Lemma
If \((S_1, \phi_1) \sim (S_2, \phi_2)\), then \(\lambda(\phi_1) = \lambda(\phi_2)\).

Remark: One is tempted to mod \(\mathcal{P}\) out by this equivalence relation. One problem: the normalized dilatation \(L(S, \phi)\) does not behave well with respect to this equivalence relation. (Subject of further study.)
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Universal Finiteness theorem

(Farb-Leininger-Margalit ’09, Agol ’10) For any $L > 1$, there is a finite collection $(M_i, F_i)$, $i = 1, \ldots, k$ so that

$$L(S, \phi) \leq L \quad \Rightarrow \quad (S^0, \phi^0) \in \Phi(M_i, F_i) \quad \text{for some } i.$$ 

(Penner) $\Rightarrow$ The minimum dilatation mapping classes on genus $g$ surfaces have mapping tori coming from a finite collection of 3-manifolds.
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Some immediate consequences and questions:

(Brinkman, Penner) As $g \to \infty$,

$$\delta(S_g) = \min\{\lambda(\phi) ; \phi \in \mathcal{P}_S\} \to 1$$

(In fact, $\log(\delta(S_g)) \asymp \frac{1}{g}$.)

**UFT** ⇒ the minimum value $\ell$ for $L$ is greater than one.

**Question 1:** Is the minimum $\ell$ of $L$ attained by some $(S, \phi)$?

(H) $\limsup \ell(S_g) \leq \left(\frac{3+\sqrt{5}}{2}\right)^2 = L(S_0\phi_0)$, where $(S_0, \phi_0)$ is the simplest pseudo-Anosov braid. (see also Kin-Takasawa, Aaber-Dunfield)

**Golden Mean Conjecture:** $\lim_{g \to \infty} \ell(S_g) = L(S_0, \phi_0)$.

**Question 2 (McMullen):** Are the local minima of $L$ in $\overline{\mathcal{P}}$ attained at rational points (i.e., points in $\mathcal{P}$)?

If Question 2 is true, then UFT would imply Question 1.
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Families with asymptotically small dilatations

A family $\mathcal{F} \subset \mathcal{P}$ is said to have *asymptotically small dilatation elements* if $\mathcal{F}$ contains a subfamily $\mathcal{F}_1 = \{(S, \phi)\}$ where

- $\chi(S)$ is unbounded, and
- normalized dilatation $L(S, \phi) = \lambda(\phi)|\chi(S)|$ is bounded.

i.e.,

$$\log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.$$ 

**Problem:** Which natural subsets of $\mathcal{P}_S$ have asymptotically small dilatation elements?

**Examples:**

- Torelli subgroups $\cap \mathcal{P}$? No (Farb-Leininger-Margalit '08)
- Hyperelliptic elements of $\mathcal{P}$? Orientable elements of $\mathcal{P}$? Yes (H-Kin '06)
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  \log(\lambda(\phi)) \asymp \frac{1}{|\chi(S)|}.
  \]

**Problem:** Which natural subsets of \( \mathcal{P}_S \) have asymptotically small dilatation elements?

**Examples:**

- Torelli subgroups \( \cap \mathcal{P} \)? No (Farb-Leininger-Margalit ’08)
- Hyperelliptic elements of \( \mathcal{P} \)? Orientable elements of \( \mathcal{P} \)? Yes (H-Kin ’06)
I. Pseudo-Anosov mapping classes
   1. Introduction
   2. Visualizing pseudo-Anosov mapping classes
   3. Train tracks
   4. Minimum dilatation problem

II. Fibered faces and applications
   1. Introduction
   2. Fibered face theory
   3. Alexander and Teichmüller polynomials
   4. First application: Orientable mapping classes

III. Families of mapping classes with small dilatations
   1. Introduction
   2. Deformations of mapping classes on fibered faces
   3. Penner sequences and applications
   4. Quasiperiodic mapping classes
Orientable examples

\((S, \phi) \in \mathcal{P}\) is an **orientable mapping class** if the stable foliation \(\mathcal{F}^s\) is orientable

(Rykken) For any \((S, \phi) \in \mathcal{P}\)

\[
\lambda_{\text{hom}}(\phi) \leq \lambda(\phi),
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with equality if and only if \(\phi\) is orientable.

Let \(\delta^+(S_g)\) be the smallest dilatation amongst orientable elements of \(\mathcal{P}_{S_g}\).
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LT polynomials

**LT-polynomials:**

\[ LT_{a,n}(x) = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1. \]

Let \( \lambda_{a,n} = |LT_{n,a}| \) be the house of \( LT_{n,a} \).

(Lanneau-Thiffeault ’09) For \( g = 2, 3, 4, 6, 8, \)

\[ \lambda_{1,g} \leq \delta^+(S_g). \]

with equality for \( g = 2, 3, 4 \).

**LT-Question:** Is it true that for all even \( g \),

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Other known minima for $\delta^+(S_g)$

Using Lanneau and Thiffeault’s lower bounds (for $g \leq 8$) and examples:

- (Lanneau-Thiffeault) $\delta^+(S_5) = \lambda_{1,6}$
  ($= \text{Lehmer's number} \approx 1.17628$).

- (H '10) $\delta^+(S_8) = \lambda_{1,8}$

- (Aaber-Dunfield '10, Kin-Takasawa '11) $\delta^+(S_7) = \lambda_{2,9}$. 
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Let \((S, \phi)\) be the simplest pseudo-Anosov braid, let \((M, F)\) be its mapping torus and fibered face, so that \((S, \phi) \in \Phi(M, F) \subset F\).

The meridians \(\mu_1, \mu_2\) of \(K_1\) and \(K_2\) determine coordinates \((t, u)\) for \(H^1(M; \mathbb{Z})\).
Simplest pseudo-Anosov braid revisited

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The meridians \(\mu_1, \mu_2\) of \(K_1\) and \(K_2\) determine coordinates \((t, u)\) for \(H^1(M; \mathbb{Z})\).
The Alexander polynomial and Teichmüller polynomials are given in terms of these coordinates by:

\[ \Delta(t, u) = u^2 - u(1 - t - t^{-1}) + 1 \]

and

\[ \Theta(t, u) = u^2 - u(1 + t + t^{-1}) + 1. \]

and their specializations

\[ \Delta^{(a,b)}(x) = \Delta(x^a, x^b) \]

\[ \Theta^{(a,b)}(x) = \Theta(x^a, x^b) = LT_{a,b}(x). \]

It follows that \( \phi_{a,b} \) is orientable for \( a \) odd and \( b \) even.
Consequences

Evidence for LT-Question:

Theorem (H$_-$)

For $g$ even and $6 \nmid g$, there is a sequence of orientable mapping classes $\phi_g$ defined on a closed genus $g$ surfaces so that with $\lambda(\phi_g) = \lambda_{1,g}$.

Evidence for Golden Mean Conjecture:

Theorem (H$_-$)

There is an infinite sequence of mapping classes $(S_g, \phi_g)$ where $S_g$ is a closed genus $g$ surface, such that

$$\lim_{g \to \infty} L(S, \phi_g) = \lim_{g \to \infty} \lambda(\phi_g)^{2g} = \left(\frac{3 + \sqrt{5}}{2}\right)^2$$
Consequences

Evidence for LT-Question:

Theorem (H\(_-\))

*For g even and 6 \(\nmid\)\(g\), there is a sequence of orientable mapping classes \(\phi_g\) defined on a closed genus g surfaces so that with \(\lambda(\phi_g) = \lambda_{1,g}\).*

Evidence for Golden Mean Conjecture:

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*There is an infinite sequence of mapping classes \((S_g, \phi_g)\) where \(S_g\) is a closed genus g surface, such that*

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\]
Fibered face for simplest pseudo-Anosov braid

level sets of Thurston norm

Fibered cone

Thurston norm ball

orientable

smaller dilatation non-orientable