On hyperbolic perturbations of algebraic links and small Mahler measure

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Abstract

This paper surveys some results surrounding Lehmer's problem in the context of fibered links and Hopf plumbing. Topics addressed here are Mahler measures of fibered links, the relation between iterated Hopf plumbings and Salem-Boyd polynomials, and the question of when monotone growth occurs under iterated plumbing. Explicit calculations for certain "perturbations" of links associated to the ADE singularities are computed.

1 Introduction

The *Mahler measure* of a monic integer polynomial is the absolute value of the product of roots with norm greater than one. Lehmer's problem [Leh] asks whether the Mahler measure of a monic integer polynomial can be made arbitrarily close to but greater than one. So far, there is no known monic integer polynomial with Mahler measure greater than one but less than Lehmer's number $\alpha_L = 1.17628\ldots$, which is the Mahler measure of the polynomial

$$f_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

To solve Lehmer' s problem it is enough to answer the question for Alexander polynomials of fibered links. A polynomial $f(t)$ is *reciprocal* if $f(t) = t^d f(1/t)$, where $d = \deg(f)$. Smyth [Smy] showed that the Mahler measure of irreducible non-reciprocal polynomials not vanishing at zero are bounded below by $\theta_0 = 1.32472\ldots$, a number greater than Lehmer's number. Thus, it remains to search among reciprocal polynomials. Any monic reciprocal polynomial occurs as the Alexander polynomial of a fibered link $K \subset S^3$ up to cyclotomic factors [Kan]. Lehmer's number α_L appears in this context as the Mahler measure of the Alexander polynomial of the $(-2, 3, 7)$-pretzel knot.

The Mahler measure of a fibered link (K, Σ) can be considered to be a weak measure of "hyperbolicity" of the link in the following sense. Let $K \subset S^3$ be a fibered link with monodromy $h : \Sigma \to \Sigma$. Define the Mahler measure $M(K, \Sigma)$ to be the Mahler measure of $\Delta_{(K,\Sigma)}$, where $\Delta_{(K,\Sigma)}$ is the characteristic polynomial of the automorphism on the first singular homology group of Σ

$$h_* : \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}_1(\Sigma; \mathbb{R})$$

induced by h. The Mahler measure $M(K, \Sigma)$ is bounded from below by the leading eigenvalue $\lambda(K, \Sigma)$ of h_*, known as the *homological dilatation* of the monodromy h. If h is isotopic to a pseudo-Anosov map, then $\lambda(K, \Sigma)$ is also a lower bound for the *(geometric) dilatation* of h. In particular, if $\lambda(K, \Sigma) > 1$, and h is irreducible, then h is isotopic to a pseudo-Anosov homeomorphism [Thu] (see also [FLP], [CB]).

As a first guess, it seems natural to expect small Mahler measures to be attained by "small perturbations" of non-hyperbolic links, for example, algebraic links. Here, we will take *small perturbations* to mean Hopf or trefoil plumbing along a suitable path on the fibering surface. For example, the smallest Mahler measures of degrees 2,4,6,8,10 (listed by Lehmer in [Leh]) all arise from Hopf or trefoil plumbings of torus links (see Section 4).

Two problems arise in this approach. The first is that the Alexander polynomial is only a weak indicator of geometric properties of the fibered link; there are examples of hyperbolic links (K, Σ) such that $M(K, \Sigma) = \lambda(K, \Sigma) = 1$. The second is that Mahler measure and homological dilatation are not always monotone increasing or decreasing under iterations of Hopf plumbing. Useful connections between Mahler measure and geometry do hold, however, when we restrict our attention to certain subfamilies of fibered links.

We begin by defining and stating properties of Hopf plumbings in Section 2. In particular, we give a formula for the Alexander polynomials of fibered links obtained by iterated Hopf plumbing. These have the form of Salem-Boyd polynomials introduced in [Sal], developed further in [Boyd], and applied to Hopf plumbings in [Hir2].

In Section 3 we present two families of fibered links with the monotonicity property. The first example is the family of Coxeter links studied in [Hir1]. For Coxeter links, the homological dilatations grow or decrease monotonically with iterations of Hopf plumbing. If the underlying Coxeter graph is a star graph, then the homological dilatation equals the Mahler measure for any associated Coxeter link. Furthermore, Leininger [Lei] showed that for pseudo-Anosov Coxeter links associated to a bi-colored graph, the monodromy is orientable. It follows that the homological and geometric dilatations are equal for these examples [Ryk]. The second example is the family of Salem links. These are fibered links whose homological dilatation is equal to the Mahler measure of the Alexander polynomial. The Coxeter links associated to star graphs are either cyclotomic or Salem links. We give a criterion for a sequence of fibered links obtained by iterated Hopf plumbing to be eventually Salem, and show that for such Salem sequences, the dilatations grow or decrease monotonically.

Section 4 contains examples and speculations.

2 Iterated Hopf plumbings

In this section, we review some basic definitions and properties of fibered links, and their monodromy. Any fibered link can be converted to any other by a finite sequence of Hopf plumbings and deplumbings [Gir]. We recall the definition of Hopf plumbing, and give a formula for the Alexander polynomial of the fibered link for sequences of links obtained by iterated Hopf plumbing.

A link $K \subset S^3$ is *fibered*, with fiber Σ, if for a regular neighborhood $U(K)$ of K in S^3, there is a fibration

$$S^3 \setminus U(K) \to S^1$$

of the complement $U(K)$ in S^3, where Σ is a general fiber, and the boundary of Σ equals K. Let (K, Σ) denote the fibered link. There is a homeomorphism $h : \Sigma \to \Sigma$, so that $S^3 \setminus U(K)$ can be identified with the mapping torus for Σ with respect to h. The map h is called the *(geometric) monodromy* of the fibered link (K, Σ).

Let h_* be the restriction of h to the first homology group $\mathrm{H}_1(\Sigma; \mathbb{R})$. The transformation h_* is the *homological monodromy* of (K, Σ), and its characteristic polynomial is the *Alexander polynomial* $\Delta_{(K,\Sigma)}(t)$ of (K, Σ). This definition of Alexander polynomial is associated to the pair (K, Σ) and not to the link itself; if K has more than one component, the fibering structure is not in general unique, and each fibering structure gives rise to a different Alexander polynomial. The *homological dilatation* of (K, Σ) is the maximum among absolute values of roots of $\Delta_{(K,\Sigma)}(t)$, or eigenvalues of h_*.

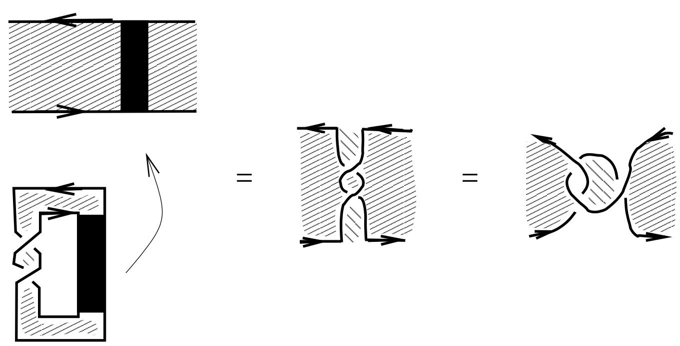

Figure 1: Positive Hopf plumbing.

Let τ be a properly embedded path on Σ. The *positive (negative) Hopf plumbing* on (K, Σ) along τ is obtained by gluing a positive (negative) Hopf band onto Σ along a thickening of τ. Figure 1 shows the result of a positive Hopf plumbing. The *n-th iterated Hopf plumbing on* (K, Σ) *based at* τ is shown in Figure 2. We will write $(K_n^\pm, \Sigma_n^\pm)$ for the result of the n-th iterated Hopf plumbing. By this convention, $(K, \Sigma) = (K_1^\pm, \Sigma_1^\pm)$. If (K, Σ) is a fibered link, so is the result of any Hopf plumbing [Sta]. Thus, $(K_n^\pm, \Sigma_n^\pm)$ is fibered for all n.

As we will show, the Alexander polynomials of links resulting via iterated Hopf plumbings from a fixed (K, Σ) based at a path τ satisfy a simple formula. Before stating the result, we give some definitions.

Given two integer polynomials f and g, we write $f \doteq g$ if there exists cyclotomic polyno-

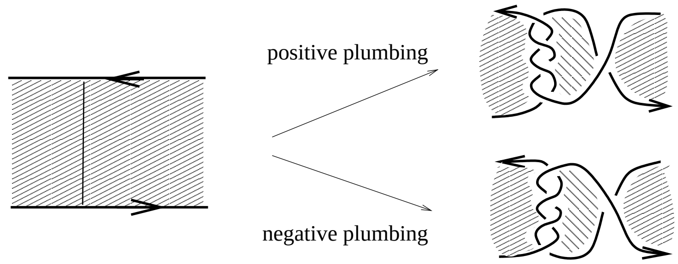

Figure 2: Fourth iterated Hopf plumbing.

mials $c_1, \ldots, c_k$, $d_1, \ldots, d_\ell$, and an integer r such that

$$f(t)c_1(t)\cdots c_k(t) = \pm t^r g(t) d_1(t)\cdots d_\ell(t).$$

If $f(t)$ is a polynomial of degree d, define its *reciprocal*

$$f_*(t) = t^d f(1/t).$$

A polynomial $f(t)$ is said to be a *reciprocal polynomial* if $f = f_*$, and *anti-reciprocal* if $f = -f_*$. If $f(t)$ is anti-reciprocal, then $f \doteq g$, where g is reciprocal. This is because, if $f(t)$ is anti-reciprocal, then $(t-1)$ divides $f(t)$ and $f(t)/(t-1)$ is reciprocal.

The following theorem is proved in [Hir2].

Theorem 1 *Let (K, Σ) be a fibered link, and τ a properly embedded path on Σ. Then there is a polynomial $P = P^{\pm}_{(\Sigma,\tau)}$ depending on Σ, τ and the orientation of the plumbings, such that the Alexander polynomials $\Delta_n(t) = \Delta_{(K_n,\Sigma_n)}$ satisfy*

$$\Delta_n(t) \doteq t^n P(t) + (-1)^{n+r} P_*(t), \tag{1}$$

where r is the number of components of K.

Polynomials of the form given in Equation (1) were studied by Salem [Sal], and Boyd [Boyd] in their investigations of Salem and P-V numbers. We will call Equation (1), the *Salem-Boyd form* of the polynomial Δ_n. Given a polynomial f, let $N(f)$ be the number of roots outside the unit circle, $\lambda(f)$ (called the *radius of f*) the maximum among absolute values of roots of f, and $M(f)$ the Mahler measure of f. The following is proved in [Boyd] (see also, [Hir2]).

Theorem 2 *Let $P(t)$ be a monic integer polynomial and*

$$Q_n(t) = t^n P(t) \pm P_*(t).$$

Then Q_n is reciprocal or anti-reciprocal for all n, and furthermore

(1) $N(Q_n) \le N(P)$ *for all* n*;*

(2) $\lim_{n\to\infty} \lambda(Q_n) = \lambda(P)$*; and*

(3) $\lim_{n\to\infty} M(Q_n) = M(P)$*.*

Analogously define, for a fibered link (K, Σ), $N(K, \Sigma)$ (respectively, $\lambda(K, \Sigma)$, and $M(K, \Sigma)$), to be $N(\Delta_{(K,\Sigma)})$ (respectively, $\lambda(\Delta_{(K,\Sigma)})$, $M(\Delta_{(K,\Sigma)})$). Then Theorem 3 below follows immediately from Theorem 2.

Theorem 3 *Let* (K_n, Σ_n) *be fibered links obtained from* (K, Σ) *by iterated Hopf plumbing. Then* $N(K_n, \Sigma_n)$ *is eventually constant, and* $\lambda(K_n, \Sigma_n)$ *and* $M(K_n, \Sigma_n)$ *are convergent sequences.*

We give two explicit formulae for $P_{(\Sigma,\tau)}$. Before doing this, recall that for any link K and Seifert surface Σ, there is an associated Seifert matrix S with respect to some choice of basis for $\mathrm{H}_1(\Sigma; \mathbb{R})$ (see, for example, [Rolf] for terminology). Then the Alexander polynomial of K with respect to Σ is given by $\Delta_{(K,\Sigma)}(t) = |tS - S^{\mathrm{tr}}|$ up to multiplies of $\pm t$, where $|A|$ denotes the determinant of A and A^{tr} the transpose of A. This definition specializes to our previous definition of Alexander polynomials for fibered links. For an invertible matrix A, let $\mathrm{s}(A)$ be the sign of the determinant of A. For example, if K is a fibered knot with fiber Σ, and S is any invertible Seifert matrix for K, then $\mathrm{s}(S) = \Delta_{(K,\Sigma)}(1)$. Since $\mathrm{s}(S)$ doesn't depend on the choice of basis, we will define $\mathrm{s}(K, \Sigma) = \mathrm{s}(S)$. If (K, Σ) is fibered and S is a Seifert matrix with respect to some choice of basis for $\mathrm{H}_1(\Sigma; \mathbb{R})$, then $S^{-1}S^t$ represents the homological monodromy h_* with respect to this basis.

Let (K, Σ) be a fibered link, and let τ be a properly embedded path in Σ. Let Σ_τ be the surface in S^3 obtained by taking Σ and removing a regular neighborhood of τ. Let K_τ be the boundary of Σ_τ. The first formula is reminiscent of the skein relations, where one keeps track of the associated Seifert surfaces.

$$P_{(\Sigma,\tau)}(t) = \Delta_{(K,\Sigma)}(t) \pm \mathrm{s}(\Sigma)\mathrm{s}(\Sigma_\tau)\Delta_{(K_\tau,\Sigma_\tau)}(t). \tag{2}$$

The second formula is given as a determinant:

$$P_{(\Sigma,\tau)}(t) = |tS - (S^{\mathrm{tr}} \mp vv^{\mathrm{tr}})|, \tag{3}$$

where $v \in \mathrm{H}_1(\Sigma; \mathbb{R})$ is the dual vector to τ considered as an element of $\mathrm{H}_1(\Sigma, \partial\Sigma; \mathbb{R})$.

Remark. Silver and Williams proved the following related result [SW].

Theorem 4 *Let* K *be any link, and let* ℓ *be an unknot disjoint from* K*, whose linking number with* K *is nonzero. Let* $K^{(n)}$ *be obtained by* $1/n$ *surgery on a tubular neighborhood of* ℓ*, and let* $\widetilde{\Delta}_{K^{(n)}}$ *be the multi-variable Alexander polynomial of* $K^{(n)}$*. Then the multi-variable Mahler measures of* $\widetilde{\Delta}_{K^{(n)}}$ *converge to the multi-variable Mahler measure of* $\widetilde{\Delta}_{K\cup\ell}$*.*

If K is a knot, then $K^{(n)}$ is a knot for all n, and we have $\widetilde{\Delta}_{K^{(n)}} = \Delta_{K^{(n)}}$. If (K, Σ) is a fibered knot, and $(K_n^\pm, \Sigma_n^\pm)$ is obtained from (K, Σ) by iterated Hopf plumbing, then $K^{(n)} = K_{2n}^\pm$ is a sequence satisfying the conditions of Theorem 4.

3 Monotone sequences

In general, the sequences described in Theorem 2 are not monotone. This section contains two large families of examples where monotonicity does hold.

3.1 Coxeter links

Let (K, Σ) be the fibered link obtained by positive Hopf plumbing along an ordered system of chords $\ell_1, \ldots, \ell_k$ on an oriented disk in S^3. Let Γ be the dual graph. We say that (K, Σ) is a *Coxeter link* for Γ, if

(1) all plumbings are positive; and

(2) whenever $i < j$, the intersection of ℓ_i with ℓ_j on the disk is negative, with respect to the skew-symmetric intersection form on the disk.

Recall that for any ordered finite graph Γ with no self- or double-edges, there is an associated *simply-laced Coxeter system* (see for example, [Hum]). Let $c(\Gamma)$ be the associated Coxeter element.

The Coxeter element gives important information about the Coxeter link. For example, an irreducible Coxeter system is spherical or affine if and only if $\lambda(c(\Gamma)) = 1$, where $\lambda(c(\Gamma))$ is the leading eigenvalue of $c(\Gamma)$ [Hum], [A'C]. It follows that the Coxeter links whose Mahler measure equals one are those associated to disjoint unions of spherical and affine Coxeter diagrams. In the irreducible case, these are just A_n, D_n, E_6, E_7, and E_8, and their affine extensions. For the irreducible spherical cases, the graphs are trees, and it follows that the Coxeter links are uniquely determined (see [Hir1]), and are the algebraic links associated to the A-D-E plane curve singularities.

For a graph Γ, let $\mu(\Gamma)$ be the leading eigenvalue of the adjacency matrix for Γ, known as the *radius of the graph* Γ. Let $\lambda(\Gamma)$ be the leading eigenvalue of $c(\Gamma)$. Let $\mu = \mu(\Gamma)$, and consider the equation

$$\lambda + \lambda^{-1} = \mu^2 - 2$$

The solutions λ are roots of unity if and only if $\mu \leq 2$, and we set $\lambda(\Gamma) = 1$. Otherwise the solutions are real and positive, and we set $\lambda(\Gamma)$ to be the larger (real) solution.

An *ordered bi-colored graph* is a graph with ordered vertices $\nu_1, \ldots, \nu_k$ such that for some s with $1 \leq s \leq k$, ν_i and ν_j are not connected by an edge whenever $i, j \leq s$ or $i, j > s$. In the following theorem, McMullen shows that $\lambda(c(\Gamma))$ is bounded from below by $\lambda(\Gamma)$ ([Mc] Theorem 1.3).

Theorem 5 *Let Γ be any Coxeter graph. Then*

$$\lambda(\Gamma) \leq \lambda(c(\Gamma)),$$

and equality holds if Γ is bi-colored.

Since $\mu(\Gamma) \mapsto \lambda(\Gamma)$ is order preserving, one can get information about the smallest possible values of $\lambda(c(\Gamma))$ using properties of graph radii.

An *arm* of Γ, is a chain of edges $\xi_1, \ldots, \xi_n$ and vertices $\nu_0, \ldots, \nu_n$, so that

(1) $\deg(\nu_0) = 1$;

(2) $\deg(\nu_i) = 2$ for $i = 1, \ldots, n-1$; and

(3) The end vertices of ξ_i are ν_{i-1} and ν_i for each $i = 1, \ldots, n$.

Choose an edge ξ on Γ connecting vertices γ_1 and γ_2. A graph $\Gamma_{\xi,n}$ is obtained from Γ by extending the edge ξ if $\Gamma_{\xi,n}$ is obtained by replacing ξ on Γ with n edges $\xi_1, \ldots, \xi_n$ and vertices $\nu_1, \ldots, \nu_{n-1}$ so that

(1) ξ_1 connects γ_1 and ν_1;

(2) ξ_i connects ν_i and ν_{i+1} for $i = 2, \ldots, n-1$; and

(3) ξ_{n-1} connects ν_{n-1} with γ_2.

Figure 3 gives an illustration.

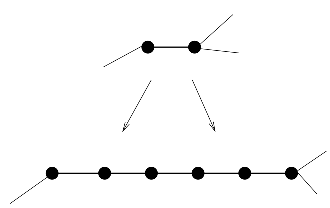

Figure 3: Extending an edge of a graph.

Hoffman proves the following theorem about monotonicty of $\mu(\Gamma)$ and hence of $\lambda(\Gamma)$ with respect to extending edges [Hof].

Theorem 6 *Let ξ be an edge of a graph Γ, and let $\Gamma_{\xi,n}$ be obtained by extending Γ along ξ. There exists N such that*

$$\mu(\Gamma_{\xi,n}) \leq 2$$

if and only if $n < N$. For $n \geq N$, $\mu(\Gamma_{\xi,n})$ is monotone increasing if ξ lies on a free arm of Γ, and $\mu(\Gamma_{\xi,n})$ is monotone decreasing otherwise.

The following property is proved in [Hir1].

Theorem 7 *If (K, Σ) is a Coxeter link associated to Γ, then after a natural identification of underlying vector spaces,*

$$h_* = -c(\Gamma).$$

It follows that in this case $\lambda(K, \Sigma) = \lambda(c(\Gamma))$.

Let (K, Σ) be a Coxeter link associated to a graph Γ. Then extending an edge of Γ corresponds to performing an iterated Hopf plumbing on (K, Σ). Thus, Hoffman's theorem implies the following.

Theorem 8 *Let Γ be a Coxeter graph that is not the union of spherical and affine Coxeter graphs. Let (K, Σ) be an associated Coxeter link, and let (K_n, Σ_n) be obtained by an iterated Hopf plumbing on (K, Σ) associated to extending an edge Γ. Then, for some N, the sequence $\lambda(K_n, \Sigma_n)$, $n > N$, is monotone.*

Lehmer's number α_L occurs as the Mahler measure of the E_{10} Coxeter graph, which is also known as the (2,3,7) star-like graph (cf. [MRS]). The following theorem was proved in greater generality for all Coxeter systems in [Mc], but we give a simpler version here that applies to Coxeter links.

Theorem 9 *If Γ is any connected Coxeter graph, then either Γ is spherical or affine, or*

$$M(E_{10}) = \lambda(E_{10}) \leq \lambda(\Gamma) \leq M(\Gamma).$$

The (-2,3,7)-pretzel knot $K_{2,3,7}$ is a Coxeter link associated to E_{10} (see [Hir1]).

Theorem 10 *If (K, Σ) is a Coxeter link, then either $M(K, \Sigma) = 1$, or*

$$M(K, \Sigma) \geq M(\Delta_{K_{2,3,7}}).$$

If Γ is bi-colored, the monodromy of the Coxeter link is pseudo-Anosov if and only if Γ is connected and the simply-laced Coxeter system associated to Γ is not spherical or affine [Lei]. Furthermore, the invariant stable and unstable foliations are orientable, and hence the homological and geometric dilatations are equal. By Rykken's result [Ryk], we have the following.

Theorem 11 *If (K, Σ) is a Coxeter link associated to a connected bi-colored graph which is not spherical or affine, then the homological and geometric dilatations of (K, Σ) are equal.*

Theorem 12 *Let (K, Σ) be a Coxeter link associated to a non-spherical or affine connected Coxeter graph Γ. Let (K_n, Σ_n) be obtained by iterated Hopf plumbing on (K, Σ) associated to extending an edge of Γ. Then for some $N > 0$, the sequence of geometric dilatations of (K_n, Σ_n) is monotone.*

3.2 Salem sequences

A *Salem number* is a real algebraic integer $\alpha > 1$ such that all other algebraic conjugates lie on or within the unit circle C with at least one on C. The minimal polynomial of a Salem number is always reciprocal. For convenience, we will also include among Salem numbers real quadratic integers $\alpha > 1$ whose other algebraic conjugate equals α^{-1}. With this addition,

α is a Salem number if and only if it is the Mahler measure of a reciprocal monic integer polynomial f and satisfies $N(f) = 1$ (see notation in Section 2). Lehmer's problem is still open for Salem numbers, for example, it is not known if there is a Salem number smaller than Lehmer's number. Furthermore, it is not known whether the minimization problem for Salem numbers is equivalent to the minimization problem for Mahler measures greater than one.

Closely related to Salem numbers are *P-V numbers*, or *Pisot-Vijayaraghavan* numbers. These are algebraic integers $\theta > 1$ all of whose other algebraic conjugates lie strictly within the unit circle. For our purposes we will redefine P-V numbers to be the Mahler measure of a monic integer polynomial f such that $f \neq f_*$, $f \neq -f_*$, and $N(f) = 1$. The set of P-V numbers is closed [Sal] and its smallest element is $\theta_0 = 1.32472\ldots$ [Sie].

If (K, Σ) is a fibered link whose homological dilatation is a Salem number, we say that (K, Σ) is a *Salem (fibered) link*. If (K_n, Σ_n) is a sequence obtained from (K, Σ) by an iteration of Hopf plumbings, and if (K_n, Σ_n) is a Salem link for large enough n, we call (K_n, Σ_n) a *Salem sequence*.

The minimal polynomial of a P-V number will be called a *P-V polynomial*, and the minimal polynomial of a Salem number will be called a *Salem polynomial*. Theorem 2 has a stronger form when restricting to the case when $P(t)$ is a P-V polynomial (see [Sal], [Boyd]).

Theorem 13 *If $P(t)$ is a P-V polynomial, then there exist constants $N_\pm$ such that $M(Q_n^\pm) = 1$ for $n < N_\pm$, and $N(Q_n^\pm) = 1$ for $n \geq N_\pm$. Furthermore, $M(Q_n^\pm)$ converges monotonically to $M(P)$ from below (respectively, above) if and only if $\pm P(0) > 0$ (respectively < 0).*

From Theorem 13 it follows that to each Salem sequence (K_n, Σ_n) there corresponds a P-V number $\theta_{(\Sigma,\tau)} \geq \theta_0$ to which the Salem numbers converge. Furthermore, one has an effective way to find the smallest Salem number occurring in the sequence, as seen in the following corollary.

Corollary 14 *If (K_n, Σ_n) is a Salem sequence associated to a P-V polynomial P, then the values greater than one attained by $M(K_n, \Sigma_n)$ are bounded from below by the minimum of θ_0, and the first nontrivial terms in the sequences $M(K_{2n}, \Sigma_{2n})$ and $M(K_{2n+1}, \Sigma_{2n+1})$.*

Remark. The role of Salem links in studying Mahler measures of fibered links is still mysterious. For Salem links, the homological dilatation and the Mahler measure of $\Delta_{(K,\Sigma)}$ are equal. While both geometric and homological dilatation can be made arbitrarily close to one, a lower bound greater than one for Salem numbers would imply a lower bound greater than one for dilatations for Salem links. This leads to the following problem, which we leave for further research.

Problem 15 *Give a geometric interpretation for the algebraic conjugates of the dilatation of a fibered link.*

4 Small perturbations of A-D-E singularities.

We make use of the Salem-Boyd equations given in Section 3 to find the minimal Mahler measures greater than one occuring in certain families.

The fibered links in this section are obtained by positive or negative Hopf plumbings along an ordered system of chords arranged on a disk in S^3. Let Γ be the dual graph of the chord arrangement. The polynomials $P^{\pm}_{\Sigma,\tau}$ of Theorem 1 are easy to compute from the combinatorics of Γ using Equation 2, especially in the case when Γ is a tree, and the locus of plumbing is one of its nodes. A filled (unfilled) vertex ν corresponds to positive (negative) Hopf plumbing, as shown in Figure 4. We will refer to Γ as the plumbing graph for the associated link.

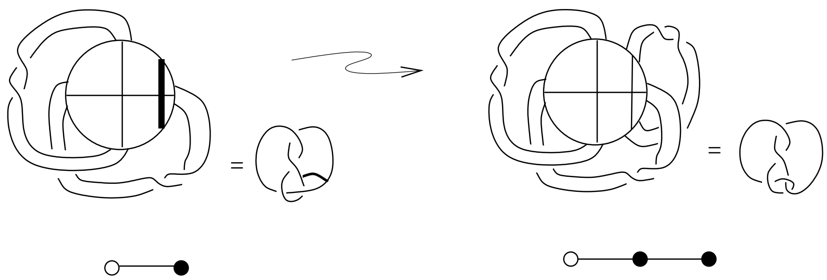

Figure 4: Graphs and plumbing.

If Γ is a tree, then the fibered link associated to any realization is an arborescent link with underlying graph Γ. If Γ is a tree and has no vertices of degree greater than 3, then the link is determined by Γ.

It is not hard to see that for fixed degree, there is a positive gap between 1 and the next smallest Mahler measure. In [Leh], Lehmer lists polynomials with the smallest Mahler measures for non-cyclotomic polynomials in all even degrees up to 10. For degree 2 the minimal Mahler measure is attained by the Figure 8 knot, which can also be thought of a $(2, 3, 1)$-pretzel link. This appears in the sequence described in Section 4.3. For degrees 4,6,8 and 10, the minimal Mahler measures can be obtained by Coxeter links of star graphs (see Section 4.1).

We end by giving an application of Theorem 1, Theorem 2, and Theorem 13, by computing the minimum Salem number occurring for certain positive (Section 4.2) and negative (Section 4.3) perturbations of the algebraic links associated to A_n.

4.1 Coxeter links and pretzel links from star graphs.

The $(-2, m, n)$-pretzel links $K_{-2,m,n}$ and more generally the $(p_1, \ldots, p_k, -1, \ldots, -1)$-pretzel links, where the number of -1's is $k-2$, are Coxeter links associated to $(p_1, \ldots, p_k)$-star graphs [Hir1].

The star graphs are defined as follows. Let A_p be the graph consisting of p nodes $\nu_0, \ldots, \nu_p$ and edges between ν_i and ν_{i+1} for $i = 1, \ldots, p-1$. The vertex ν_0 will be called the *base* of the A_p. A $(p_1, \ldots, p_k)$-*star graph* is a connected tree Γ that is the union of subgraphs isomorphic to $A_{p_1}, \ldots, A_{p_k}$ with their bases identified as in Figure 5.

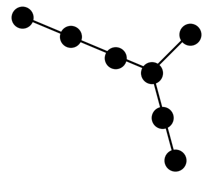

Figure 5: The $(2, 3, 4)$-star graph.

For star graphs with less than or equal to 3 branches, the Coxeter link is an arborescent link completely determined by the graph. If the star graph is one of A_n, D_n, E_6, E_7 or E_8, or their affine extensions, then the links are iterated torus links, and the geometric and homological monodromy equal 1. In all other cases, the fibered links have pseudo-Anosov monodromy with orientable stable and unstable invariant foliations [Lei], and hence the homological and geometric dilatations are also equal [Ryk]. Furthermore, the dilatations are Salem numbers and hence are equal to the Mahler measures of the Alexander polynomials [MRS].

The minimal hyperbolic extensions of D_4, E_6, E_7 and E_8 are respectively the $(2, 2, 2, 3)$, $(3, 3, 4)$, $(2, 4, 5)$, and $(2, 3, 7)$ star links. The Mahler measures for the characteristic polynomials of these links are the minimal ones greater than one in degrees 4,6, 8 and 10 (cf. [Mc], Proposition 7.3 and page 175).

4.2 Positive perturbations of A_n.

For the calculations in this Section, and the next, we will make use of the following Lemma. Let C denote the unit circle $|z| = 1$. Let θ_G, known as the *golden mean*, be the sole root of $t^2 - t - 1$ that is greater than one.

Lemma 16 *Consider the polynomials*

$$f_m^{\pm}(t) = t^m(t^2 - t - 1) \pm 1.$$

Then f_m^- has exactly one root θ_m^- outside C for all $m \geq 1$, and the sequences θ_m^- converge to θ_G monotonically from above. The roots of f_m^+ are roots of unity for $m = 1, 2$, and for $m \geq 3$, they have exactly one root θ_m^+ outside C. The sequences θ_m^+ converge to θ_G monotonically from below.

Proof. To show that f_m^- has at most one root outside C, we will use an argument similar to that of Boyd in [Boyd]. Consider the polynomials

$$F_m^{\pm}(t, s) = t^m(t^2 - t - 1) \pm s$$

where s is a variable ranging in the interval $[0,1]$. Let $\alpha(s)$ be any branch of $F_m^{\pm}(t,s)=0$ considered as curve lying over $[0,1]$. Then $\alpha(s)$ can never lie on C as long as $0 \leq s < 1$, since, on C, $|t^2-t-1|$ is bounded from below by 1. If such an $\alpha = \alpha(s)$ existed, we would have

$$|\alpha^2-\alpha-1| > s = |\alpha^2-\alpha-1|$$

yielding a contradiction. It follows that the number of roots of $f_m^{\pm}(t)$ outside C is bounded from above by $N(t^2-t-1)=1$.

The cases for small m can be checked by hand. Monotonicity follows from the fact that as soon as $f_m^{\pm}(t)$ has a root α outside C, then $f_{m+1}^{\pm}(t)$ is forced to have a root strictly between α and θ_G. □

The Coxeter link K_{A_n} associated to A_n is the torus link $T(2,n+1)$, and the Alexander polynomial is

$$\Delta_{A_n} = \frac{t^{n+1}+(-1)^n}{t+1} = t^n - t^{n-1} + \cdots + (-1)^n. \tag{4}$$

The $(-2,m,n)$-pretzel links $K_{-2,m,n}$ are obtained by positive iterated Hopf plumbing on $K_{A_{m+1}}$ along τ, where $[\tau]^{\mathrm{dual}} = [0,1,0,\ldots,0]$. The link $K_{-2,m,1}$ has one component if m is odd and two components if m is even. Thus, the Alexander polynomial for $K_{-2,m,n}$ is given by

$$\Delta_{K_{-2,m,n}}(t) = t^n P_m(t) + (-1)^{m+n}(P_m)_*(t),$$

where

$$\begin{aligned} P_m(t) &= \Delta_{A_{m+1}}(t) + \Delta_{A_1}(t)\Delta_{A_{m-1}}(t) \\ &= (t^{m+1} - t^m + \cdots + (-1)^{m+1}) + (t-1)(t^{m-1} - t^{m-2} + \ldots + (-1)^{m-1}) \\ &= t^{m+1} - t^{m-1} + t^{m-2} - \ldots + (-1)^m t. \end{aligned}$$

The polynomials $P_m(t)$ satisfy

$$P_m(t) + P_{m+1}(t) = t^{m+2} + t^{m+1} - t^m = t^m(t^2+t-1).$$

Thus

$$\begin{aligned} P_m(t) + (-1)^m P_1(t) &= \sum_{i=1}^{m-1} (-1)^{m-i-1}(P_i(t) + P_{i+1}(t)) \\ &= (t^{m-1} - t^{m-2} + \cdots + (-1)^m t)(t^2+t-1) \end{aligned}$$

and

$$\begin{aligned} P_m(t) &= \frac{(t^{m-1} + (-1)^m)t(t^2+t-1) + (-1)^{m+1}t^2(t+1)}{t+1} \\ &= \frac{t^m(t^2+t-1) + (-1)^{m+1}t}{t+1} \end{aligned}$$

Let $(\overline{P}_m)(t) = P_m(-t)$. Then

$$\overline{P}_m(t) = \frac{(-1)^m \left[t^m(t^2 - t - 1) + t\right]}{t+1},$$

and

$$(\overline{P}_m)_*(t) = \frac{-(P_m)_*(-t)}{t+1}.$$

By Lemma 16, $P_m(-t)$ is cyclotomic for $m = 1, 2$, and is a P-V polynomial for θ_m, for $m \geq 3$ where θ_m converges monotonically to θ_G from below.

Replacing t by $-t$ in the formula for $\Delta_{K_{-2,m,n}}$, we have

$$\begin{aligned} \Delta_{K_{-2,m,n}}(-t) &\doteq t^n \overline{P}_m(t) + (\overline{P}_m)_*(t) \\ &= t^n P_m(-t) - (P_m)_*(-t). \end{aligned}$$

By Theorem 13, all the Salem sequences arising from $(2, m, n)$-stars are monotone increasing. The minimal elements in this family are listed below.

pretzel type	Salem number
$(-2, 3, 7)$	≈ 1.17628
$(-2, 4, 5)$	≈ 1.36

Thus, the $(-2, 3, 7)$-pretzel is minimal in this family.

For the particular case when $m = 3$, we have

$$P_3(t) = t^4 - t^2 + t = t(t^3 - t + 1) = tg(-t),$$

where g is the minimal polynomial for the smallest P-V number θ_0. Lehmer's polynomial $f_L(t)$ can thus be written as

$$f_L(t) = t^8(g(t)) - g_*(t) = \Delta_{K_{-2,3,7}}(-t).$$

4.3 Negative perturbations of A_n.

We now consider the positive $(2, m, n)$-pretzel links. These are not Coxeter links, since they have a negative twist in their plumbing graph as in Figure 6. Just as in the previous example, these links are arborescent links, and the Alexander polynomials are independent of the choice of directions on the plumbing graphs.

We begin with the $(2, m, 1)$-pretzel links. These have plumbing graph as in Figure 7.

Let K_m be the $(2, m, 1)$-pretzel link. When $m = 1, 3, 5, 7$ these links are, respectively, denoted by 4_2, 6_2, 8_2, and 10_2 in Rolfsen's knot table ([Rolf] p. 391–429). The knot 4_2 is more commonly known as the figure eight knot. By Theorem 1, the Alexander polynomials of K_n are given by

$$\Delta_{K_m}(t) = \frac{t^{m+1}P(t) + (-1)^{m+1}P_*(t)}{t+1},$$

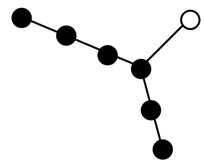

Figure 6: Plumbing graph for the $(2,3,4)$-pretzel.

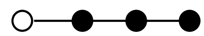

Figure 7: Plumbing graph for the $(2,3,1)$-pretzel.

where

$$\begin{aligned} P(t) &= \Delta_{K_1} + \Delta_{K_0} \\ &= (t^2 - 3t + 1) + (t - 1) \\ &= t^2 - 2t = t(t-2). \end{aligned}$$

It follows that

$$\Delta_{K_m}(t) = t^{m+1} - 3t^m + 3t^{m-1} - \cdots (-1)^m(3t-1).$$

Since $P(t)$ has one root outside C, the K_m are eventually Salem links. Looking at the even and odd subsequences, we see that the only cyclotomic link that occurs is K_2. Thus, the minimal elements in this family are the figure eight knot K_1, and K_4. The sequences are decreasing for n odd and increasing for n even. Thus, the smallest Salem number arising in this sequence is $1.8832\ldots = \alpha(K_4)$.

Let $K_{m,n}$ be the $(2,m,n)$-pretzel link. Then this is an iterated Hopf sequence using the index n, and starting with the $(2,m,1)$-pretzel. We find $P_m(t)$ as follows.

$$\begin{aligned} P_m(t) &= \Delta_{K_m}(t) + \Delta_{A_1}(t)\Delta_{A_{m-1}}(t) \\ &= \Delta_{K_m}(t) + (t-1)(t^{m-1} - t^{m-2} + \cdots + (-1)^{m-1}) \\ &= t^{m+1} - 2t^m + t^{m-1} - t^{m-2} + \cdots + (-1)^m t \end{aligned}$$

Adding consecutive functions, yields the formula

$$P_m(t) + P_{m+1}(t) = t^m(t^2 - t - 1).$$

Thus,

$$\begin{aligned} P_m(t) + (-1)^{m-1}P_1(t) &= \sum_{i=1}^{m-1}(-1)^{m-i-1}(P_i(t) + P_{i+1}(t)) \\ &= \sum_{i=1}^{m-1}(-1)^{m-i-1}t^i(t^2 - t - 1). \end{aligned}$$

Isolating $P_m(t)$, we get

$$\begin{aligned} P_m(t) &= \frac{(t^{m-1} + (-1)^m)t(t^2 - t - 1) + (-1)^{m-1}t(t-2)(t+1)}{t+1} \\ &= \frac{t^m(t^2 - t - 1) + (-1)^m t}{t+1}. \end{aligned}$$

By Lemma 16, $P_m(t)$ has exactly one root θ_m outside C for $m = 1$ and $m \geq 3$, and θ_m tends to the root θ_G of $P_G(t) = t^2 - t - 1$ from above (for odd m) and below (for even m).

The number r of components of K_m is 1 if m is odd and 2 if m is even. We thus have,

$$\Delta_{K_{m,n}}(t) \doteq P_m(t) + (-1)^{m+n}(P_m)_*(t).$$

and the leading coefficient of $(-1)^{m+n}P_m(t)$ is $(-1)^n$. It follows from an argument similar to that in the proof of Lemma 16 that $M(K_{m,2n+1})$ is monotone decreasing and $M(K_{m,2n})$ is monotone increasing.

Since the $(2,4,4)$- and all $(2,2,n)$-pretzel links are cyclotomic, the minimal elements of $(2,m,n)$-pretzel knots with respect to trefoil plumbing are those listed below.

pretzel type	Salem number
$(2,1,1)$	≈ 2.61803
$(2,1,4)$	≈ 1.8832
$(2,4,6)$	≈ 1.36

Of these only the $(2,4,6)$-pretzel gives Salem number smaller than θ_G. Thus, $M(K_{4,6}) \approx 1.36$ is the minimal Mahler measure greater than one among the $(2,m,n)$-pretzel links.

References

[A'C] N. A'Campo. Sur les valeurs propres de la transformation de Coxeter. *Invent. Math.* **33** (1976), 61–67.

[Boyd] D.W. Boyd. Small Salem numbers. *Duke Math. J.* **44** (1977), 315–328.

[CB] A. Casson and S. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston.* Cambridge University Press, 1988.

[FLP] A. Fathi, F. Laudenbach, and V. Poenaru. *Travaux de Thurston sur les surfaces*, volume 66-67. Société Mathématique de France, Paris, 1979.

[Gir] E. Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 405–414, Beijing, 2002. Higher Ed. Press.

[Hir1] E. Hironaka. Chord diagrams and Coxeter links. *J. London Math. Soc.* **69** (2004), 243–257.

[Hir2] E. Hironaka. Salem-Boyd sequences and Hopf plumbing. *to appear in Osaka J. Math.* (2006).

[Hof] A. Hoffman. On limit points of spectral radii of non-negative symmetric integral matrices. In *Proc. Conf., Western Michigan Univ., Kalamazoo, Mich., 1972*, volume 303, pages 165–172. Springer-Verlag, Berlin, 1972.

[Hum] J. Humphreys. *Reflection groups and Coxeter groups.* Cambridge University Press, Cambridge, 1990.

[Kan] T. Kanenobu. Module d'Alexander des nœuds fibrés et polynôme de Hosokawa des lacements fibrés. *Math. Sem. Notes Kobe Univ.* **9** (1981), 75–84.

[Leh] D. H. Lehmer. Factorization of certain cyclotomic functions. *Ann. of Math.* **34** (1933), 461–469.

[Lei] C. Leininger. On groups generated by two positive multi-twists: Teichmuller curves and Lehmer's number. *Geometry & Topology* **88** (2004), 1301–1359.

[MRS] J. F. McKee, P. Rowlinson, and C.J. Smyth. Salem numbers and Pisot numbers from stars. In *Number theory in progress, Vol. 1 (Zakopane-Kościelisko, 1997)*, pages 309–319. de Gruyter, Berlin, 1999.

[Mc] C. McMullen. Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 151–183.

[Rolf] D. Rolfsen. *Knots and Links.* Publish or Perish, Inc, Berkeley, 1976.

[Ryk] E. Rykken. Expanding factors for pseudo-Anosov homeomorphisms. *Michigan Math. J.* **46** (1999), 281–296.

[Sal] R. Salem. A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan. *Duke Math. J.* **11** (1944), 103–108.

[Sie] C. L. Siegel. Algebraic integers whose conjugates lie on the unit circle. *Duke Math. J.* **11** (1944), 597–602.

[SW] D.S. Silver and S.G. Williams. Mahler measure of Alexander polynomials. *J. London Math. Soc.* **69** (2004), 767–782.

[Smy] C. J. Smyth. On the product of the conjugates outside the unit circle of an algebraic integer. *Bull. London Math. Soc.* **3** (1971), 169–175.

[Sta] J. Stallings. Constructions of fibered knots and links. *Proc. Symp. Pure Math.* **27** (1975), 315–319.

[Thu] W. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)* **19** (1988), 417–431.