Minimum dilatation problem and quasi-periodicity conjecture

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Overview of Talk

Minimum dilatation problem and quasi-periodicity conjecture

I. Pseudo-Anosov mapping classes and their dilatations

II. Minimum dilatation problem

III. Two models for small dilatation mapping classes.
I. Pseudo-Anosov mapping classes and their dilatations
Background and Notation

Objects of study:

$S = S_{g,n}$ a compact oriented surface with $\chi_{\text{top}}(S) < 0$

$\phi : S \rightarrow S$, mapping class, an isotopy class of orientation preserving homeomorphisms, $(S, \phi)$

$\text{Mod}(S)$ the mapping class group

Goal:

Describe the dynamics of minimum and “small” dilatation pseudo-Anosov mapping classes.

Some references:

Work of Thurston [F-L-P], A Primer on Mapping Class Groups [F-M]
Classification of mapping classes

\((\text{Nielsen, Thurston})\) \(\phi \in \text{Mod}(S)\) is either

- **periodic** \((\exists k, \phi^k = \text{id})\)

Example: a rotation

- **reducible** \((\exists k, \exists \gamma \subset S \text{ ess. s.c.c., such so that } \phi^k(\gamma) = \gamma.)\)

Example: Dehn twist

(\text{Remark: These generate } \text{Mod}(S))

- **pseudo-Anosov** ...
pseudo-Anosov mapping classes

\((S, \phi)\) is \textbf{pseudo-Anosov} if there is a local flat structure outside a finite set of \textit{singularities} preserved by \(\phi\), defined by a transverse measured singular foliations \((\mathcal{F}^\pm, \nu^\pm)\), such that \(\phi\) permutes the singularities, and \(\phi_*(\mathcal{F}^\pm, \nu^\pm) = (\mathcal{F}^\pm, \lambda^\pm \nu^\pm)\).

Near smooth points:

\[
\begin{array}{c}
\hspace{1cm}
\[
\begin{array}{c}
\hspace{1cm}
1
\end{array}
\end{array}
\hspace{1cm}
\begin{array}{c}
\hspace{1cm}
\frac{1}{\lambda}
\end{array}
\]
\]

Near singularities and boundary components:
Properties of pseudo-Anosov maps

- Homological versus geometric dilatation

- Horizontal and vertical theories of mapping classes
Homological versus geometric dilatation

\((S, \phi)\) is pseudo-Anosov iff \(\forall\) ess. s.c.c. \(C\) on \(S\), and \(\forall\) Riemannian metric on \(S\),

\[|\phi^n(C)|^{1/n} \to \lambda(= \lambda_{\text{geo}}) = \text{spectral radius}(T) > 1,\]

where \(T\) is a Perron-Frobenius matrix.

Compare: the homological dilatation of a mapping class \((S, \phi)\)

\[\lambda_{\text{hom}} := \text{spectral radius}(\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})).\]

These can be computed using the Alexander polynomial.

Properties:

- \(\lambda_{\text{hom}}(\phi) \leq \lambda_{\text{geo}}(\phi)\)
- \(\lambda_{\text{geo}}\) is a Perron unit.
- The house of any algebraic integer can be realized as \(\lambda_{\text{hom}}(\phi)\) for some \((S, \phi)\) (Kanenobu)
Fix $S$:

$$\mathcal{M}(S) = \mathcal{T}(S)/\text{Mod}(S).$$

There is a one-to-one correspondence between pseudo-Anosov elements $\mathcal{P}(S)$ and closed Teichmüller geodesics in $\mathcal{M}(S)$, and

$$\{\lambda(\phi) : \phi \in \mathcal{P}(S)\} \xrightarrow{\log} \{\text{Teich. lengths of closed geodesics in } \mathcal{M}(S)\}$$

Problem: How to translate between properties of elements of $\mathcal{P}$ and Teichmüller geodesics.
Vertical Theory: Thurston’s theory of fibered faces

Let \( \mathcal{M} \text{od} = \bigcup_S \text{Mod}(S) \), \( \mathcal{P} = \bigcup_S \mathcal{P}(S) \).

Given a fibered 3-manifold \( M \), let

\[
\Phi(M) = \{(S, \phi) \mid M = M_\phi := \text{Mapping Torus}(S, \phi)\}.
\]

Theorem: \( M \) is hyperbolic \( \iff \Phi(M) \subset \mathcal{P} \iff \Phi(M) \cap \mathcal{P} \neq \emptyset \).

(Assume hyperbolic from now on.)

There are natural inclusions:

\[
\Phi(M) \hookrightarrow H^1(M; \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z})
\]
\[
\hookrightarrow H^1(M; \mathbb{R}) = \mathbb{R}^{b_1(M)}
\]

i.e., \( \Phi(M) \) can be identified with a subset of the integer lattice points in \( \mathbb{R}^{b_1} \).
Thurston norm ball

Thurston norm: \( \exists \) a norm \( ||\cdot|| \) on \( H^1(M; \mathbb{R}) \) with the properties:

- If \( (S, \phi) \in \Phi(M) \), then \( ||(S, \phi)|| = |\chi(S)| \).
- Thurston norm ball \( \{ \alpha \in H^1(M; \mathbb{R}) : ||\alpha|| \leq 1 \} \) is a convex polyhedron with integer vertices.
- For each top-dimensional face \( F \) of \( H^1(M; \mathbb{R}) \), let

\[
\Phi(M, F) = (F \cdot \mathbb{R}^+) \cap \Phi(M).
\]

Then \( \Phi(M, F) \) is either empty or

\[
\Phi(M, F) = (F \cdot \mathbb{R}^+) \cap H^1(M; \mathbb{Z})^{\text{prim}}.
\]

In the latter case, \( F \) is called a fibered face.
Fibered faces and dilatations

The monodromies $\Phi(M, F)$ of a hyperbolic 3-manifold $M$ associated to a fibered face $F$ define a subpartition of $\mathcal{P}$ with topological structure.

- $\Phi(M, F) \longleftrightarrow$ rational points on $F$.
- The fibered faces $F$ are polyhedra of dimension $d = \dim H^1(M; \mathbb{R}) - 1$.

(Fried, McMullen) $L(S, \phi) = \lambda(\phi)|\chi(S)|$ extends to a continuous, convex function on $F$, going to infinity toward the boundary of $F$. 
Example: Simplest pseudo-Anosov braid monodromy

Identify $S = S_{0,4}$ with a disk with three interior boundary components.

Let $(S, \phi_0)$ be the braid monodromy corresponding to the braid $\sigma_1 \sigma_2^{-1}$ in terms of the standard braid generators.

Action of 3 iterations of $\phi$ on an essential simple closed curve.
Example: Fibered face for simplest hyperbolic braid

Mapping torus:
\[ M = S \times [0, 1]/(x, 1) \sim (\phi_0(x), 0) \]

\[ H_1(M; \mathbb{R}) \]
II. Minimum dilatation problem
Minimum dilatation problem.

**Question:** (Penner, Farb) What is the minimum dilatation

\[ \delta(S) = \min \{ \lambda(\phi) \mid \phi \in \mathcal{P}(S) \} \]

(Ko-Los-Song: \( S_{0,5} \), Ham-Song: \( S_{0,6} \), Cho-Ham: \( S_{2,0} \),
Lanneau-Thiffeault: orientable mapping classes)

**Question:** (Penner, McMullen, Farb) What is the behavior of the normalized dilatation

\[ \delta(S_{g,0})^g \]

or more generally

\[ \delta(S_{g,n})^{\chi(S_{g,n})} \]

**Problem:** Describe the “shape” of “small” pseudo-Anosov mapping classes.
(dynamics, number theoretic properties of dilatations)
Asymptotic behavior

The normalized minimum dilatations $\delta(S_{g,n})|\chi(S_{g,n})|$ are bounded along blue/green rays, and unbounded along red rays.
Small dilatation maps

We will say \((S, \phi)\) has \(P\)-small dilatation, if \(\lambda(\phi)|\chi(S)| \leq P\).

(Farb-Leininger-Margalit ’08) To understand all \(P\)-small dilatation mapping classes, it suffices to study the monodromy of a finite number of hyperbolic 3-manifolds.

Smallest known accumulation point for \(\delta(S_g,n)\chi(S_g,n)\) is

\[
\ell_0 = \left(\frac{3 + \sqrt{5}}{2}\right)^2 = (1 + \text{golden mean})^2
\]

achieved by simplest pseudo-Anosov braid. (H, A-D, K-T ’09-’10)

Question: Is \(\ell_0\) the smallest accumulation point?
III. Two models for small dilatation mapping classes
Theorem (Penner, Valdivia, H)

Let \((S_m, \phi_m)\) be the mapping class with \(\phi_m = R_m \circ \delta_C \circ \hat{\phi}\). Then if for some \(m\) \((S_m, \phi_m)\) is pseudo-Anosov, then they all are, and the normalized dilatation \(\lambda(\phi_m)^m\) is bounded.
Example: Penner’s sequence (sketch of proof)

The mapping classes \((S_m, \phi_m)\) form a convergent sequence on a fibered face.

Let \(\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}\), \(\phi = \delta_c \delta_b^{-1} \delta_a\).

\[
(S_{g,2}, \phi_g) \rightarrow (S_{1,2}, \phi)
\]
Associated sequence on fibered face

Polynomial invariants:

Alexander polynomial $\Delta = \Delta_M$

$\lambda_{\text{hom}}(\phi_\alpha) = |\Delta^\alpha|$ 

Teichmüller polynomial

$\Theta = \Theta_{M,F}$ (McMullen)

$\lambda_{\text{geo}}(\phi_\alpha) = |\Theta^\alpha|$. 

For Penner’s sequence:

$\lambda(\phi_g) = |x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1|$

$\lambda(\phi_g)^g \rightarrow \frac{7 + 3\sqrt{5}}{2}$.

(Fried, McMullen)
Type II: Murasugi sum and twisted mapping classes

For $i = 1, 2$, let $(S_i, \phi_i)$ be two mapping classes. Let $P_{2k}$ be a $2k$-gon. Let $\alpha_i : P_{2k} \hookrightarrow S_i$ embeddings so that the $j$th edge of $P_{2k}$ maps into the boundary of $S_i$ whenever $i = j \pmod{2}$.

The Murasugi sum $(S, \phi)$ is given as follows:

$$S = S_1 \cup_{\alpha_1(x) = \alpha_2(x)} S_2,$$

$$\phi = \tilde{\phi}_2 \circ \tilde{\phi}_1,$$

where $\tilde{\phi}_i$ are the extensions to $S$ by the identity on $S \setminus S_i$. 

\[\begin{array}{ccc}
\phi_1 & \rightarrow & \phi_2 \\
S_1 & \cup & S_2 \\
\end{array}\]
Twisted mapping classes

Let \((\Sigma_m, \sigma_m)\) be a sequence of mapping classes, such that

\[(\sigma_m)^m = \delta_{b_1} \cdots \delta_{b_s}\]

where \(b_1, \ldots, b_s\) are boundary components of \(\Sigma_m\).

Let \((S_m, \phi_m)\) be the mapping classes obtained by Murasugi sum of \((\Sigma_m, \sigma_m)\) and some fixed \((S, \hat{\phi})\). We call \((S_m, \phi_m)\) a twisted sequence of mapping classes.
Theorem (H)

There exists \((S, \hat{\phi})\) and \((\Sigma_m, \sigma_m)\) so that the twisted sequence \((S_m, \phi_m)\) satisfies

- \((S_m, \phi_m) \in \Phi(M, F)\), for a single 3-manifold \(M\) and fibered face \(F\);
- for \(m > 1\), \((S_m, \phi_m)\) defines (orientable) pseudo-Anosov maps on closed genus \(g = 6m - 1\) surfaces;
- The normalized dilatations converge:

\[
\lim_{m \to \infty} \frac{\lambda(\phi_m) |\chi(S_m)|}{2} = \frac{(3 + \sqrt{5})}{2}.
\]
Alternate manifestations of this sequence of examples

1. Coxeter mapping classes:

   ![Coxeter mapping classes diagram]

2. A sequence on a fibered face of the complement of the $6_{22}$ link converging to the simplest pseudo-Anosov braid.

   ![Fibered face diagram]
Quasi-periodicity conjecture/question (Farb-Leininger-Margalit, McMullen) Is it possible to decompose all $P$-small elements of $\mathcal{P}$ as a composition of a periodic map and a map that is identity outside a subsurface $\Sigma$ with $|\chi(\Sigma)| < K_P$?

**Question:** Can any $P$-small mapping class be built from a combination of the constructions of Type I and II, where $(S, \hat{\phi})$ has bounded Euler characteristic depending on $P$?
Thank you!