MULTI-POLYNOMIAL INVARIANTS FOR PLANE ALGEBRAIC CURVES

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1. Abelian Coverings

The geometry of embeddings of curves $\mathcal{C}$ in the complex projective plane $\mathbb{P}^2$ can be studied by looking at invariants of abelian coverings of $\mathbb{P}^2$ branched along $\mathcal{C}$. The coverings themselves also provide a rich source of examples. Some of the invariants, for example, the Chern numbers c_1^2 and c_2 and the Kodaira dimension κ, are easy to find in terms of local data, while others, such as the Betti numbers, are usually more difficult to compute and take into account positions of singularities in the branch locus. The first Betti number b_1, together with c_1^2, c_2 (hence also the topological Euler characteristic χ_{top}) and κ, are the invariants used to classify minimal surfaces under the Kodaira-Enriques classification. These four invariants also determine many other important surface invariants, such as the rest of the Betti numbers, the Hodge numbers, the irregularity and the geometric and arithmetic genera. In this paper we will mainly concentrate on computations and properties of b_1.

In the algebraic geometric language, a branched covering is a finite surjective morphism between normal varieties $\rho : X \to Y$. These have the property that over the complement of a finite union of irreducible hypersurfaces $\mathcal{C} \subset Y$, the restriction map ρ^u is an unbranched covering (with respect to the usual complex topology.)

A well known result (see [G-R]) states that for any unbranched covering $\rho^u : X^u \to Y \setminus \mathcal{C}$ there is a unique normal completion X of X^u and an extension map ρ of ρ^u which is a finite surjective morphism considered as a map on varieties. Furthermore, the covering automorphisms extend uniquely, and one can identify the group of covering automorphisms of the unbranched covering with the group of automorphisms of X preserving fibers, and also with the Galois group of the corresponding field extension of $K(Y)$.

From the above and basic topology, a surjective homomorphism to a finite abelian group

$$\pi_1(Y - \mathcal{B}) \to G$$

is determining data for an abelian branched covering.

Instead of b_1 of the covering X itself, we will study b_1 of a desingularization $\widetilde{X}$. The value of b_1 will not depend on the desingularization, as it is a birational invariant of smooth varieties.

There is a large body of work on the computation of b_1 of abelian coverings of algebraic surfaces, some of which we will not be able to touch on here. For a general survey see [Sa].

2. Polynomial Periodicity

Let $\mathcal{B}$ be a finite union of irreducible hypersurfaces on Y. Consider, for each integer $n > 1$, the canonical group homomorphisms

$$\pi_1(Y \setminus \mathcal{B}) \to \mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$$

induced by the Hurewicz map. As explained above, we get a sequence of branched coverings $\rho_n : X_n \to Y$ branched along $\mathcal{B}$. Let $\widetilde{X}_n$ be desingularizations of X_n. Then one can think of the function $f(n) = b_1(\widetilde{X}_n)$ as an invariant of the embedding $\mathcal{B} \subset Y$.

As we will see later in this section, several well-known examples give evidence that the function $f(n)$ exhibits both polynomial and periodic behavior.

Definition. A function $f : \mathbb{N} \to \mathbb{N}$ is *polynomial periodic* if there are polynomials $p_1, \ldots, p_N \in \mathbb{Z}[x]$ so that

$$f(n) = p_i(n) \qquad \text{if } i \equiv n \mod N.$$

Polynomial periodicity of first Betti numbers of coverings has been shown in the general topological setting of CW complexes branched along a codimension two subcomplex. Sarnak shows in [Sr], in which he coins the term polynomial periodic, that for unbranched coverings of finite CW complexes (i.e., when the branch locus is empty), $f(n)$ is polynomial periodic. In the case when the base space is an algebraic surface and the branch locus is a finite union of irreducible curves, we show polynomial periodicity in [Ho2] and [Ho3]. In the most general setting of finite CW complexes, Sakuma shows polynomial periodicity in [Su].

The above results are limited because they do not give an easy algorithm for computing the first Betti numbers, and there is still little known about properties of the polynomials, their degrees and the periodicity.

We present here three representative examples of families of coverings which have been studied extensively. The various methods and results surrounding these examples suggest different ways of approaching the general problem. In the following sections we will further discuss some of the techniques which have been used to study the Betti numbers, both in the general topological and algebraic geometric context and what they say about the polynomials.

Example 1. Riemann Surfaces. The simplest example is when Y is a Riemann surface of genus g and $\mathcal{B} = \{p_1, \ldots, p_k\}$. In this case, the result follows from the Riemann-Hurwitz formula, but the analysis is very straight forward and we will give it here.

Assume $k \geq 2$. Since $\mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$ has order n^{2g+k-1}, the covering $X_n \to Y$ has degree n^{2g+k-1} and has branching index n above each branch point. Since the topological Euler characteristic of an unbranched covering is just that of the base multiplied by the degree of the covering, and the number of points added in above each branch point equals the degree of the covering divided by the order of branching, we have

$$\chi_{\mathrm{top}}(X_n) = n^{2g+k-1}\chi_{\mathrm{top}}(Y - \mathcal{B}) + n^{2g+k-2}k.$$

From the formula $b_1(X_n) = 2 - \chi_{\mathrm{top}}(X_n)$, it follows that $f(n)$ is a polynomial in n.

Example 2. Cyclic Multiple Planes. In the 1920's Zariski studied smooth projective completions of the surfaces given by $z^n - f(x, y)$, where f is a polynomial in the coordinates x, y. This is the same as the $\widetilde{X}_n$ described above, when $Y = \mathbb{P}^2$ and $\mathcal{B} = C \cup L$. Here C is the completion of $\{f(x, y) = 0\}$ in $\mathbb{P}^2$ and L is the line at infinity with respect to the affine coordinates x, y. In the case when C has only ordinary nodes and cusps and is in general position with respect to L, Zariski[Za2] showed that

$$b_1(\widetilde{X}_n) = \begin{cases} c & \text{if } 6|n \\ 0 & \text{if } 6 \not| n \end{cases}$$

The constant c depends on the *superabundance* of curves of a given degree satisfying certain adjoint conditions at each of the singularities.

Zariski's computations led him to discover that there are two disjoint components in the family of sextic plane curves with six ordinary cusps and no other singularities [Za1]. This shows that the first Betti number is an invariant which is sensitive to positions of singularities.

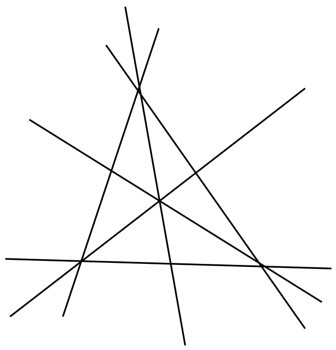
For literature on computations of the first Betti number of abelian coverings in relation to adjoint linear systems which generalize Zariski's results, see [D], [E], [Lib1],[Lib2] and [V].

Example 3. Line Configurations. The final example also shows how computations of invariants of chains of coverings can help give geometric information about the branch locus. In the mid 1980's, Hirzebruch began a study of abelian branched coverings of $\mathbb{P}^2$ which are locally n-fold cyclically branched along components of a configuration $\mathcal{L}$ of lines [Hz]. That is, these are the surfaces $\widetilde{X}_n$ where $Y = \mathbb{P}^2$ and $\mathcal{B} = \mathcal{L}$.

Applying the Miyaoka-Yau inequality to his formulas for the Chern numbers of the coverings Hirzebruch found relations on the numbers of s-fold intersections in a line configuration, which had earlier not been found by more elementary methods.

Hirzebruch also used his computations of the Chern numbers to show that 3 of the coverings he obtains are uniformized by the complex ball. To do this he shows that the Chern numbers for these examples satisfy $c_1^2 = 3c_2$, the extreme case of the Miyaoka-Yau inequality.

Several papers have been written on computing the first Betti number of these coverings including [Ho2],[I], [Gl] and [Zu]. In particular, Gläser showed that for the line configuration of 6 lines through 4 points in general position



the first Betti number of $\widetilde{X}_n$ is $5(n-1)(n-2)$, supporting the idea that Betti numbers should be polynomial. Zuo, however, computes the Betti numbers for the dual Hesse-configuration, which is the dual configuration tothe tangent lines to flexes of a smooth planar cubic. Note that by Sylvester's Theorem this configuration is never defined by real equations. His answer is $b_1 = 8(n-1)(n-2)$ if n is relatively prime to a finite set of integers including 3 and $b_1 = 8(n-1)(n-2) - 2$ otherwise. This shows that the Betti numbers need not strictly be polynomials in n.

3. Generalization to the Topological Setting.

Let Σ be a finite CW complex and Γ a codimension 2 subcomplex. In [F], Fox defines canonical completions of unbranched coverings of $\Sigma \setminus \Gamma$ over all of Σ. His definition is motivated by and generalizes the situation for normal varieties.

Sarnak [Sr] shows polynomial periodicity for unbranched coverings of any finite CW complex Σ:

Theorem 1. *(Sarnak) Let $\pi_1(\Sigma) \to H_1(\Sigma; \mathbb{Z}/n\mathbb{Z})$ be the canonical surjective homomorphisms. Let Σ_n be the associated unbranched coverings. Then $b_1(\Sigma_n)$ is polynomial periodic.*

Roughly, the idea of the proof is to look at the action of $G_n = \mathrm{H}_1(\Sigma; \mathbb{Z}/n\mathbb{Z})$ on the 1-chains in the covering space. Given a presentation of $\pi_1(\Sigma)$ with k generators, one can represent 1-chains (we will use complex coefficients for convenience) on the covering space by elements of $\mathbb{C}[G_n]^k$. One can show the relation matrix M_G with entries in $\mathbb{C}[G]$ giving relations on $\mathbb{C}[G]^k$ induce to relation matrices M_n for the 1-chains in $\mathbb{C}[G_n]^k$. It then follows from the theory of representations of abelian groups that the rank of $\mathrm{H}_1(\Sigma_n)$ can be obtained by counting points on the intersection of G_n^* with certain determinental varieties in G^*. The rest follows from the following key lemma due to Laurent [Lau].

Lemma. *(Laurent) Let T be the set of torsion points on $(\mathbb{C}^*)^k$. For V be an algebraic subset of $(\mathbb{C}^*)^k$ there exist a finite set of planes or translates of planes, $P_1, \dots, P_k$ so that $V \cap T = \cup(P_i \cap T)$.*

We now state Sakuma's theorem, which we mentioned earlier.

Theorem 2. *(Sakuma) Let Σ be any finite CW complex, let Γ be any finite subcomplex, and let Σ_n be the branched coverings of Σ defined by the maps $\pi_1(\Sigma \setminus \Gamma) \to H_1(\Sigma \setminus \Gamma; \mathbb{Z}/n\mathbb{Z})$. Then $b_1(\Sigma_n)$ is polynomial periodic.*

In his proof, Sakuma shows how to incorporate the relations added to the 1-chains when putting the branch locus back in.

4. Intersection Matrices

In [Ho1] and [Ho3], we show the following result:

Theorem 3. *Let Y be a smooth complex projective surface and $\mathcal{C}$ a finite union of irreducible curves on Y. Let $\rho_n : X_n \to Y$, be the corresponding chain of coverings. Then $b_1(\widetilde{X_n})$ is polynomial periodic.*

Theorem 3 is a consequence of Theorem 2, but the techniques we used were different. While Sakuma's proof was more combinatorial, we study the intersection matrix for curves above the branch locus. While it restricts us to the algebraic surface situation, it has the extra benefit that one gets information about the Neron-Severi group of $\widetilde{X}_n$ as we will explain in this section.

Definition. Define null($\mathcal{C}$) to be the nullity of the intersection matrix for the irreducible components of $\mathcal{C}$. One can think of this also as $k - \rho$, where k is the number of irreducible components of $\mathcal{C}$ and ρ is the dimension of the image of $\mathcal{C}$ in $\mathrm{H}_1(X; \mathbb{R})$ (or in the real Neron-Severi group $\mathrm{NS}(X) \otimes \mathbb{R}$.)

Lemma. *Let X be a smooth complex projective surface and $\mathcal{C}$ a finite union of irreducible curves on X. Then $b_1(X) = b_1(X \setminus \mathcal{C}) - null(\mathcal{C})$.*

See [E] or [Ho1] for proof.

Given a smooth complex projective surface Y and $\mathcal{C} \subset Y$, let $G = \mathrm{H}_1(Y \setminus \mathcal{C}; \mathbb{Z})$ and $G_n = \mathrm{H}_1(Y \setminus \mathcal{C}; \mathbb{Z}/n\mathbb{Z})$. Let $\rho_n : X_n \to Y$ be the coverings associated to

$\pi_1(Y \setminus \mathcal{C}) \to G_n$. Let $\sigma : \widetilde{X}_n \to X_n$ be a desingularization of X_n. Let $\mathcal{C}_n = \rho_n^{-1}(\mathcal{C})$ and let $\widetilde{\mathcal{C}}_{n,\sigma}$ be the inverse image of the branch locus $\mathcal{C}$ under the composition map $\widetilde{\rho}_n = \rho_n \circ \sigma$. By Sarnak's theorem it suffices to show that $\mathrm{null}(\widetilde{\mathcal{C}}_{n,\sigma})$ is polynomial periodic.

Under Mumford's definition of intersections on normal varieties [M], $\mathrm{null}(\widetilde{\mathcal{C}}_{n,\sigma}) = \mathrm{null}(\mathcal{C}_n)$ for any desingularization σ. Thus, we need only look at curves on X_n, and don't have to worry about the choice of the desingularization.

For every irreducible component $C \subset \mathcal{C}$, there is an associated subgroups $H_C \subset G$ so that the image H_{C_n} of H_C in G_n is the stabilizer subgroup for irreducible components of $\rho_n^{-1}(C)$. Given these subgroups, $\mathcal{C}_n$ can be identified with a union of quotient groups $\cup_C G_n/H_{C,n}$ and the intersection matrix for $\mathcal{C}_n$ is a form on the vector space $\mathcal{V} = \oplus_C \mathbb{C}[G_n/H_{C,n}]$.

In [Ho3] Proposition 3.1, we give this form explicitly for the case when $\mathcal{C}$ contains smooth curves with normal crossings. (It is enough to do this case, since replacing the base space by blowups along points on the branch locus and replacing the branch locus with its preimage will not change the complement of the branch locus. Thus, the new coverings will be birational to the old ones and the first Betti numbers will be the same.)

Let $\mathcal{T}$ be the power set for the set of irreducible components of $\mathcal{C}$. For each $\tau \in \mathcal{T}$, let $S_{\tau,n} = \sum_{C \in \tau} H_{C,n}$ and let $R_{\tau,n} = G_n/S_{\tau,n}$. These are reductions modulo n of $S_\tau = \sum_{C \in \tau} H_C$ and $R_\tau = G/S_\tau$. The intersection form acts diagonally on a decomposition of $\mathcal{V}$, $\oplus_{\tau \in \mathcal{T}} \mathcal{U}_{\tau,n}$, where $\mathcal{U}_{\tau,n}$ is isomorphic to an ideal in $\mathbb{C}[R_{\tau,n}]^{|\tau|}$. The form is induced by a form on $\mathbb{C}[R_\tau]^{|\tau|}$. This reduces the problem to one similar to that of Sarnak, and the Theorem follows.

In order to use the techniques described above toward better descriptions of $b_1(\widetilde{X}_n)$, one direction is to try to describe the subvarieties of G^* which correspond to the relation matrix on $\mathrm{H}_1(\widetilde{X}_n; \mathbb{C})$ and the intersection matrix of $\mathcal{C}$.

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