Boundary Manifolds of Line Arrangements

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Abstract

While the boundary 3-manifold of a line arrangement in the complex plane depends only on the incidence correspondence of the line arrangement, the homotopy type of the complement depends on the relative positions of incidences. In this paper we describe the homotopy type of line arrangement complements in terms of an associated plumbed graph. For pseudo-real line arrangements this method provides an explicit description of the homotopy type and fundamental group of the complement in terms of its ordered incidence graph. The method also extends to a larger class of "unknotted" line arrangements.

1 Introduction

Let $\mathcal{L} \subset \mathbb{C}^2$ be a finite union of complex lines in the complex plane. The *incidence graph* associated to a line arrangement $\mathcal{L}$ is the graph with vertices corresponding to the lines and points of intersection of $\mathcal{L}$, and edges connecting a line-vertex and a point-vertex whenever the point is on the line. We will think of the *combinatorics* of a line arrangement $\mathcal{L}$ as the information which is captured in the incidence graph of the arrangement.

Denote by $M(\mathcal{L})$ the boundary of a regular neighborhood of $\mathcal{L}$ in $\mathbb{C}^2$. It is not hard to see that the boundary manifold $M(\mathcal{L})$ is completely determined by the combinatorics of $\mathcal{L}$, and can be pieced together as a graph manifold over the incidence graph of the configuration.

The complement $E(\mathcal{L})$ of $\mathcal{L}$ in $\mathbb{C}^2$, on the other hand, depends on the relative positions of the incidences of the lines [Ryb]. The influence of the positions can be seen, for example, in terms of the superabundances of algebraic curves of certain degrees passing through the incidences with prescribed multiplicities [Zar] [Lib], or in terms of special pencils [Zuo] [Ara].

In this paper, we give a new description of the homotopy type of the complement $E(\mathcal{L})$ in terms of the boundary manifold $M(\mathcal{L})$. This approach helps to measure the dependence of $E(\mathcal{L})$ on non-combinatorial information in terms of what is needed to construct $E(\mathcal{L})$ from $M(\mathcal{L})$.

When $\mathcal{L}$ is defined over the reals, it is called a *(complexified) real arrangement.* There are line arrangements which are not even combinatorially equivalent to a real one. An

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example is the set of lines through pairs of flexes on a smooth cubic plane curve. Sylvester's theorem implies that the points of intersection of these lines cannot lie in the real plane, since the line through any pair of points in this set goes through a third point in the set. It is not yet known whether there are two real line arrangements whose complements have distinct homotopy types, but which have the same incidence graphs.

If a line arrangement $\mathcal{L}$ is real with respect to a given coordinate system, the coordinate system induces a partial ordering on the directed edges of the incidence graph (see Section 3). Our main result is a description of the homotopy class of $E(\mathcal{L})$ in terms of $M(\mathcal{L})$ and the ordered incidence graph $\Gamma^o_{\mathcal{L}}$. For any manifold X and subset $Y \subset X$, let

$$X/Y$$

be the topological quotient of X with respect to Y. Up to homotopy, this is equivalent to the space obtained by gluing the cone over Y to X along $Y \subset X$.

Theorem 1.1 *Let $\mathcal{L}$ be a real arrangement. Then there is a continuous map*

$$\alpha : \Gamma_{\mathcal{L}} \hookrightarrow M(\mathcal{L}),$$

where $\Gamma_{\mathcal{L}}$ is considered as a 1-complex, such that

$$E(\mathcal{L}) \simeq M(\mathcal{L})/\alpha(\Gamma_{\mathcal{L}}).$$

Furthermore, the map α can be described explicitly in terms of $\Gamma^o_{\mathcal{L}}$.

Other explicit descriptions of the fundamental group and homotopy type of $E(\mathcal{L})$, when $\mathcal{L}$ is real, have been made using different approaches, but using essentially the same information as the ordered incidence graph (see, for example, [Falk], [B-Z], and [Cor].)

Since $M(\mathcal{L})$ is a graph manifold, its fundamental group is a graph of groups. Theorem 1.1 thus leads to an explicit presentation of the fundamental group of $E(\mathcal{L})$.

Corollary 1.2 *When $\mathcal{L}$ is a real arrangement, the kernel of the epimorphism*

$$\pi_1(M(\mathcal{L})) \to \pi_1(E(\mathcal{L}))$$

is generated by the image of a map

$$\pi_1(\Gamma_{\mathcal{L}}) \to \pi_1(M(\mathcal{L}))$$

which can be explicitly described in terms of the ordered graph $\Gamma^o_{\mathcal{L}}$.

In Section 2, we review the definitions of plumbing graph, graph manifold, and graph of groups, which are relevant to line arrangements. Section 3 contains a proof of Theorem 1.1. In Section 4, we give some generalizations of Theorem 1.1 for arbitrary line arrangements.

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2 Plumbed graphs and boundary manifolds

Let $\mathcal{L}$ be a line arrangement in the complex plane $\mathbb{C}^2$, let $\mathcal{A}$ be the set of lines in $\mathcal{L}$, and let $\mathcal{P}$ be the points of intersection on $\mathcal{L}$. When $\mathcal{L}$ is the union of n parallel lines, the boundary manifold is a disjoint union of n solid tori, and the incidence graph is a collection of n vertices with no edges. In the rest of this paper, we will assume that $\mathcal{L}$ is connected.

Incidence graph. The *(point/line) incidence graph* $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ is a bipartite graph with *point-vertices*

$$v_P, \qquad P \in \mathcal{P}$$

and *line-vertices*

$$v_L, \qquad L \in \mathcal{A}.$$

The edges of $\Gamma_{\mathcal{L}}$ are of the form

$$y(P, L) \text{ or } y(L, P), \qquad P \in \mathcal{P}, L \in \mathcal{A}, \text{ and } P \in L.$$

The graph $\Gamma_{\mathcal{L}}$ is a directed graph. The *initial point* of $y = y(P, L)$ is defined to be $i(y) = v_p$ and the *terminal point* is defined to be $t(y) = v_L$. Similarly, if $y = y(L, P)$, then $i(y) = v_L$ and $t(y) = v_P$. We say that $y(L, P)$ and $y(P, L)$ are *conjugates* of each other and write $y(L, P) = \overline{y(P, L)}$. When we consider $\Gamma_{\mathcal{L}}$ as a one-complex, we identify conjugate edges.

Plumbed graphs and graph manifolds. We use the notation and definitions in [Neu] to describe the graph manifold associated to an incidence graph $\Gamma_{\mathcal{L}}$. Let Γ be a graph with vertices $\mathcal{V}$ and edges $\mathcal{Y}$, such that for each directed edge $y \in \mathcal{Y}$, there is a unique conjugate edge $\overline{y} \in \mathcal{Y}$, satisfying $i(y) = t(\overline{y})$ and $t(y) = i(\overline{y})$. A *plumbing* on Γ attaches to each vertex $v \in \mathcal{V}(\Gamma)$, an integer triple $(g(v), e(v), n(v))$, where

$$g(v) \geq 0, \quad \text{and} \quad n(v) \geq \deg(v),$$

and attaches to each edge $y \in \mathcal{Y}(\Gamma)$ an element $h(y) \in \mathrm{SL}(2, \mathbb{Z})$ so that $h(\overline{y}) = h(y)^{-1}$. A graph with a plumbing is called a *plumbed graph.*

The *graph manifold* $M(\Gamma)$ associated to a plumbed graph Γ is a collection

$$(\{M_v\}, \{M_y\})$$

where

(i) For each $v \in \mathcal{V}(\Gamma)$, M_v is the S^1-bundle over a genus $g(v)$ surface with $n(v)$ boundary components, whose euler number with respect to fixed trivializations over the boundary components is $e(v)$.

(ii) A subset of the boundary components of M_v is identified with the set of edges $y \in \mathcal{Y}(\Gamma)$, with labelings

$$M_y$$

for $y \in \mathcal{Y}(\Gamma)$.

(iii) Each of the M_y has a trivialization

$$M_y = S^1 \times S^1$$

given by the local trivialization of $M_{i(y)}$ at the corresponding boundary component. That is, the fibration on this trivialized M_y is projection onto the second component.

The underlying space $\mathcal{M}$ associated to a plumbed graph is the manifold obtained by gluing together the vertex manifolds M_v along their boundary components according to the maps

$$\begin{array}{ccc} M_y & \xrightarrow{h_y} & M_{\overline{y}} \\ \| & & \| \\ S^1 \times S^1 & \xrightarrow{h(y)} & S^1 \times S^1 \end{array} .$$

The manifold $\mathcal{M}$ is determined up to diffeomorphism on the above information.

Boundary manifold. Plumbed graphs can be used to describe the boundary manifold of the complement of any algebraic curve on a normal complex projective surface (*surface-curve pair*) as a graph manifold (see [Neu], or [Hir].) We will describe

$$M_B(\mathcal{L}) = M(\mathcal{L}) \cap B$$

where B is a closed ball in $\mathbb{C}^2$ containing all of $\mathcal{P}$ in its interior. This will be enough for our purposes, since we are ultimately interested in finding $E(\mathcal{L})$ up to homotopy.

Proposition 2.1 *Let $\mathcal{L} \subset \mathbb{C}^2$ be a connected line arrangement. Then $M_B(\mathcal{L})$ is diffeomorphic to the graph manifold associated to the incidence graph $\Gamma_{\mathcal{L}}$ together with the plumbing data:*

(i)

$$g(v) = 0 \textit{ for all } v \in \mathcal{V}(\Gamma_{\mathcal{L}}) \quad ;$$

(ii)

$$e(v) = \begin{cases} 1 - \#\{P \in L \cap \mathcal{P} \ : \ \deg(v_P) > 2\} & \textit{if } v = v_L \\ -1 & \textit{if } v = v_P \end{cases} \quad ;$$

(iii)

$$n(v) = \begin{cases} \deg(v_L) + 1 & \textit{if } v = v_L \\ \deg(v_P) & \textit{if } v = v_P \end{cases} \quad ; \text{ and}$$

(iv)

$$h(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Proof. As explained in [Neu], one way to find a plumbed graph for the boundary manifold of a surface-curve pair is by blowing up points on the curve until the total transform is smooth with normal crossings. By replacing the original surface-curve pair with the desingularized surface and the total transform of the curve, one maintains the isomorphism type of the boundary manifold. The new plumbed graph will have

(i) a vertex for each irreducible component of the total transform of the curve;

(ii) Euler numbers defined by the self intersections of the curves;

(iii) genera equal to the genera of the curves; and

(iv) attaching map given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In the case of line arrangements, it is only necessary to blow up the points of triple and higher intersection once for the total transform to be smooth with normal crossings. Thus, the total transform will contain proper transforms of the original lines together with exceptional curves, one for each point of multiplicity greater than two. Since the intersections are normal crossings, the patching maps h will interchange meridians and longitudes as in the statement of the proposition.

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Remark. In the statement above, we could also make $e(v_L) = 0$, for all $L \in \mathcal{A}$, since $n(v_L) > \deg(v_L)$ implies there is a free boundary component in M_{v_L}. What is actually being described in Proposition 2.1 comes from the boundary manifold of the closure of $\mathcal{L}$ in the complex projective plane $\mathbb{CP}^2$.

Graphs of groups. An advantage of describing the boundary manifold as a graph manifold is that one can now describe the fundamental group as a graph of groups.

A *graph of groups* $G(\Gamma)$ over a directed graph Γ is a collection of

(i) groups G_v, for each vertex $v \in \mathcal{V}$;

(ii) groups $G_y = G_{\overline{y}}$, for each edge $y \in \mathcal{Y}$; and

(iii) group endomorphisms

$$\psi_y : G_y \to G_{t(y)}.$$

A graph of groups determines a group $\mathcal{G}$ by a combination of amalgamations, and HNN extensions according to the maps ψ_y. We will call $\mathcal{G}$ the *realization* of $G(\Gamma)$.

A *maximal tree* on $\Gamma_{\mathcal{L}}$ is a subgraph of $\Gamma_{\mathcal{L}}$ passing through all the vertices of $\Gamma_{\mathcal{L}}$, such that there is a unique path up to homotopy between any two vertices. In terms of a maximal tree, the elements of $\mathcal{G}$ can be represented as words of the form

$$r_0 y_1 r_1 y_2 \cdots y_k r_k,$$

where each $r_j \in G_{t(y_j)}$ and $r_0 \in G_{i(y_1)}$. The relations are generated by relations in the vertex groups, and the relations

$$yr\overline{y} = r'$$

whenever there exists an $s \in G_y$ so that $r = \psi_y(s)$ and $r' = \psi_{\overline{y}}(s)$ (see [Ser] or [Hir] for more details.)

Proposition 2.2 *Let $M(\Gamma)$ be a graph manifold over a graph Γ. The collection of fundamental groups of the vertex and edge manifolds of $M(\Gamma)$ forms a graph of groups $G(\Gamma)$ over Γ. The realization $\mathcal{G}$ of $G(\Gamma)$ is isomorphic to the fundamental group of the realization $\mathcal{M}$ of $M(\Gamma)$.*

A more detailed description of the graph of groups associated to a surface-curve pair can be found in [Hir]. As a consequence of this construction one can easily test if the fundamental group of $M(\Gamma)$ is torsion free or residually finite (see also [Ser] and [Hemp].)

3 Real line arrangements

Let $\mathcal{L} \subset \mathbb{C}^2$ be a connected line arrangement which is defined by real equations with respect to coordinates x, y in $\mathbb{C}^2$. Let $\rho : \mathbb{C}^2 \to \mathbb{C}$ be projection onto the x-axis. Assume by changing coordinates if necessary that no line in $\mathcal{L}$ is parallel to the y-axis, and hence the restriction of ρ to any line in $\mathcal{L}$ is onto. Let $\mathcal{A}$ be the set of lines, $\mathcal{P} = \mathrm{Sing}(\mathcal{L})$ the set of multiple points, and $\Gamma_{\mathcal{L}}$ the incidence graph of $\mathcal{L}$.

Boundary manifold. We begin by describing the boundary manifold of real line arrangements concretely. Let $d(z_1, z_2)$ be the usual Euclidean metric on $\mathbb{C}^2$. Given a closed set X, let $B(X, \delta)$ be the set of points whose minimum distance to X is less than or equal to δ. Choose $\delta_P > 0$ and $\delta_L > 0$, for each $P \in \mathcal{P}$ and $L \in \mathcal{A}$, so that the intersections of the boundaries of $B(P, \delta_P)$ and $B(L, \delta_L)$, for pairs $P \in \mathcal{P}$ and $L \in \mathcal{A}$, are transversal (or empty). Let ϵ be the minimum of the δ_L, for $L \in \mathcal{A}$. Then the above property still holds with the δ_L replaced by ϵ.

Let

$$B(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} B(P, \delta_P)$$

and

$$B(\mathcal{A}) = \bigcup_{L \in \mathcal{A}} B(L, \epsilon).$$

Let S_P be the boundary of $B(P, \delta_P)$, and let S_L be the boundary of $B(L, \delta_L)$. Thus S_P is homeomorphic to a 3-sphere and S_L is homeomorphic to a solid torus. Let $M_P = S_P \setminus B(\mathcal{A})$ and $M_L = S_L \setminus B(\mathcal{P})$. The boundary manifold $M_B(\mathcal{L})$ is then the union

$$\bigcup_{P \in \mathcal{P}} \overline{M_P} \cup \bigcup_{L \in \mathcal{A}} \overline{M_L}.$$

This is also the decomposition of $M_B(\mathcal{L})$ as a graph manifold over $\Gamma_{\mathcal{L}}$.

Ordered incidence graph. We order the directed edges emanating from a single vertex. Given $P \in \mathcal{P}$, the edges $y \in \mathcal{Y}(\Gamma_{\mathcal{L}})$ with $i(y) = v_P$ are ordered by the slope of the line L, where $t(y) = v_L$. Given $L \in \mathcal{A}$, the edges $y \in \mathcal{Y}(\Gamma_{\mathcal{L}})$ with $i(y) = v_L$ are ordered by the x-coordinate of the point P, where to $t(y) = v_P$. The incidence graph $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ together with these orderings is called the *ordered graph* associated to $\mathcal{L}$. For example, Figure 1 gives the incidence graph of the Ceva arrangement, with orderings near v_{L_1} and v_{P_5}.

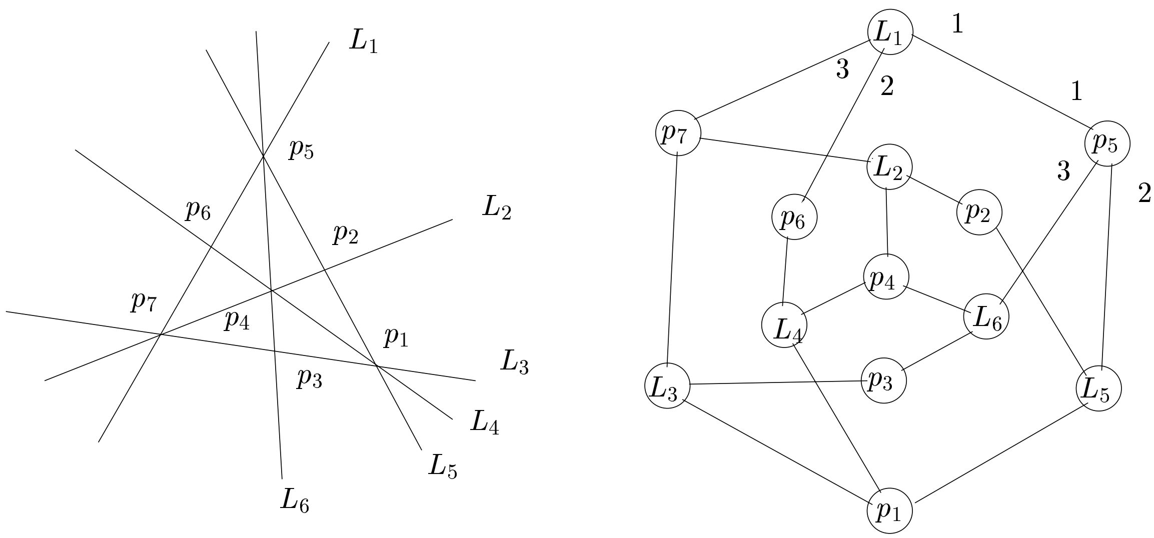

Figure 1. Ordered graph associated to the Ceva arrangement.

We now begin the proof of Theorem 1.1.

Step 1: Existence. Let $M(\Gamma_{\mathcal{L}})$ be the graph manifold associated to $\Gamma_{\mathcal{L}}$. We need to show that there is a continuous map

$$\alpha : \Gamma_{\mathcal{L}} \to M(\Gamma_{\mathcal{L}})$$

where $\Gamma_{\mathcal{L}}$ is considered as a 1-complex, giving a homotopy equivalence, so that

$$E(\mathcal{L}) \simeq M(\Gamma_{\mathcal{L}})/\alpha(\Gamma_{\mathcal{L}}).$$

We will define the skeleton Σ of a line arrangement, which is homotopy equivalent to $\Gamma_{\mathcal{L}}$ as a 1-complex, and has an embedding

$$\sigma : \Sigma \to M(\mathcal{L})$$

with the desired properties.

Skeleton. The *skeleton* of a real line arrangement $\mathcal{L}$ is the 1-complex

$$\Sigma_{\mathcal{L}} = B \cap \mathbb{R}^2 \cap \mathcal{L}.$$

Thus, the skeleton of the Ceva arrangement is the left most picture in Figure 1.

Lemma 3.1 *There is an embedding*

$$\sigma : \Sigma_{\mathcal{L}} \to M_B(\mathcal{L})$$

so that

(i) the image of σ *contracts in* $E(\mathcal{L})$*; and*

(ii) $E(\mathcal{L}) \simeq M_B(\mathcal{L})/\sigma(\Sigma_{\mathcal{L}})$.

Proof. Roughly speaking, we define the map σ by

$$\sigma(x, y) = (x, y + i\epsilon(x, y)),$$

for $(x, y) \in \Sigma_{\mathcal{L}}$, where $\epsilon(x, y)$ is the smallest positive real number such that $\sigma(x, y) \in M_B(\mathcal{L})$. The image of σ then contracts in $E(\mathcal{L})$, since it is contained in

$$\Pi = \mathbb{R} \ \times \ (\mathbb{C} \setminus \mathbb{R}) \cap B$$

which is contractable, and has empty intersection with $\mathcal{L}$.

Here is a more precise description. If $(x, y) \in \Sigma_{\mathcal{L}} \setminus B(\mathcal{P})$, then define

$$\sigma(x, y) = (x, y + i\epsilon).$$

Otherwise, at least one of the points

$$\left(x \ , \ y + i\sqrt{\delta_P^2 - d((x,y), P)^2}\right) \qquad \text{(which is in } S_P\text{)},$$

and

$$(x \ , \ y + i\epsilon) \qquad \text{(which is in } S_L \text{, for some } L\text{)},$$

is contained in $M_B(\mathcal{L})$. If both are contained in M_L, then they must agree and lie on the common boundary fo M_P and M_L. In either case, let $\sigma(x, y)$ be this point. Since the map σ defined above agrees on the boundaries of the M_P and M_L, σ is continuous.

As before, let $\rho : \mathbb{C}^2 \to \mathbb{C}$ be projection onto the first coordinate. Let $\rho_{\mathcal{L}}$ be the restriction of ρ to $E(\mathcal{L})$. Then $\rho_{\mathcal{L}}$ is a topological fibration over the complement of $\mathcal{Q} = \rho(\mathcal{P})$, with fiber a plane minus n points, where n is the order of the set $\mathcal{A}$.

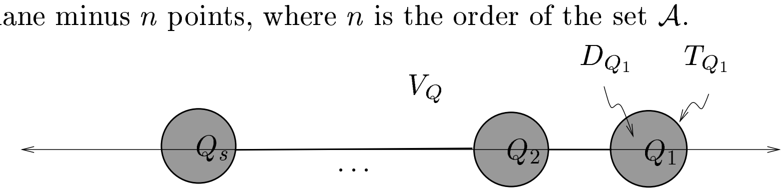

Figure 2. Contraction of the x-coordinate plane to V_Q.

Choose $\delta > 0$ so that for each $Q \in \mathcal{Q}$ the disks

$$D_Q = \{x \in \mathbb{C} \ : \ |x - Q| \le \delta\}$$

are disjoint. Let T_Q be the boundary of D_Q, and let $V_{\mathcal{Q}}$ be the union of the real line and the D_Q as in Figure 2. The deformation retraction of $\mathbb{C}$ onto $V_{\mathcal{Q}}$ extends to a deformation retraction of $E(\mathcal{L})$ onto $\rho_{\mathcal{L}}^{-1}(V_{\mathcal{Q}})$.

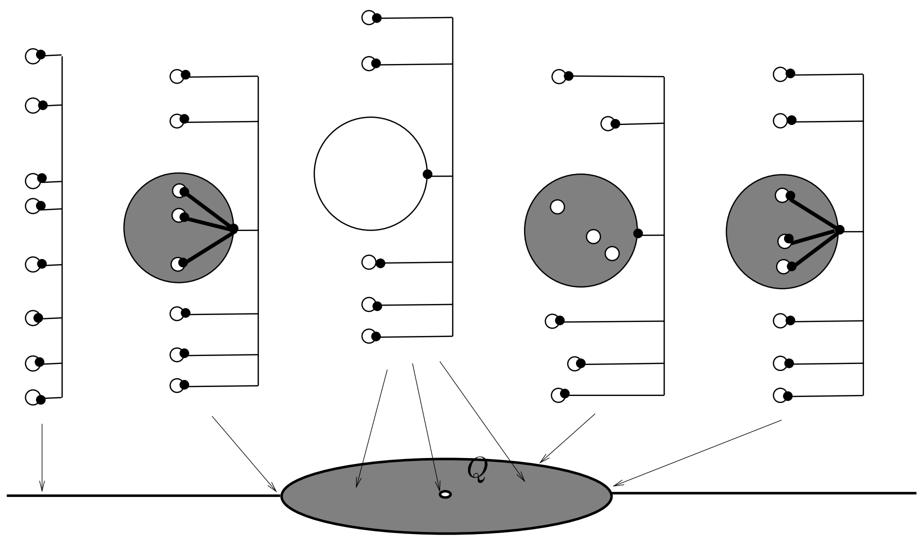

Figure 3. Fibers over $V_{\mathcal{Q}}$.

Fix $P \in \mathcal{P}$. By deforming S_P, we can think of it as the union of a solid torus lying over the T_Q, and a solid torus $D_Q \times S^1$. Thus, the fibers of $\rho_{\mathcal{L}}$ over $V_{\mathcal{Q}}$ retracts to the union of $M_B(\mathcal{L}) \cap \rho_{\mathcal{L}}^{-1}(V_Q)$, and the line segments shown in Figure 3.

The union of the right-most points of the circles and the diagonal line segments over the real line, shown in in Figure 3, is the image of σ on $M_B(\mathcal{L})$ (after the deformation of S_P). The union of the vertical line segments in the fibers over the real line can be made to lie in a plane Π. Together with the union of the horizontal line segments drawn in Figure 3, these line segments define a suspension over $\sigma(\Sigma_{\mathcal{L}})$. This proves (i). For the fibers over points in V_Q not on the real line, the union of the right most points on the large circles drawn in Figure 3 lies on a disk in the complementary solid torus on S_P. Thus, these points, and the horizontal and vertical line segments of Figure 3 retract to $\sigma(\Sigma_{\mathcal{L}})$ and to Π, respectively. This completes the proof of (ii).

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To see that the map α exists, one needs now only verify that there is a continuous map from the incidence graph onto the skeleton which is a homotopy equivalence. This can be verified by Figure 4.

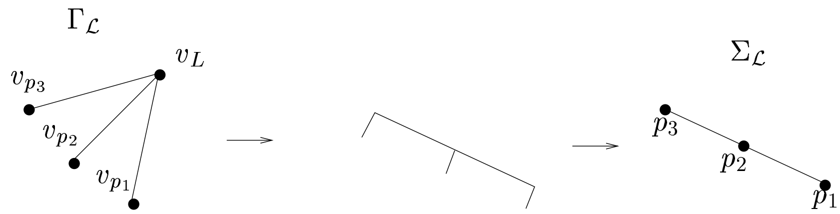

Figure 4. Incidence graph and skeleton.

Step 2: Explicit description in terms of ordered incidence graph. To describe α explicitly, we need to fix generators of the fundamental group of $M(\mathcal{L})$ using its structure as a graph of groups over $\Gamma_{\mathcal{L}}$. This can be achieved by presenting generators and relations for the fundamental groups of the vertex manifolds, and by fixing a path connecting the basepoints of $M_{i(y)}$ and $M_{t(y)}$, for each edge $y \in \mathcal{Y}(\Gamma_{\mathcal{L}})$ (recall Section 2, for basic properties of graphs of groups.)

For $L \in \mathcal{A}$, M_L is homotopy equivalent to

$$S^1 \times D \setminus \{\deg(v_L)\text{-points}\},$$

and its fundamental group is isomorphic to

$$\mathbb{Z} \times F_{\deg(v_L)},$$

where F_n is the free group on n generators.

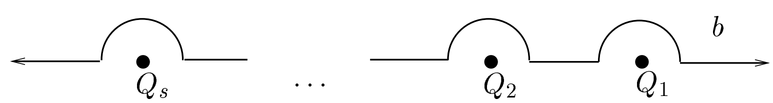

Figure 5. Image of basesets under the projection ρ.

It will be helpful to fix a basepoint for $\pi_1(M_L)$. This is the same as choosing a contractable subset of M_L which we will call the *baseset.* To do this we fix a (real 1-dimensional) curve b in the image of ρ as shown in Figure 5, which avoids the singular values. The semicircles can be taken to be contained in the boundaries of the disks D_{Q_i}. The straight line segments lie on the real line. For any $L \in \mathcal{A}$, since ρ restricted to L is an isomorphism, the baseset b lifts to a curve on $L \setminus B(\mathcal{P})$. By adding $i\epsilon$ to the y-coordinate of every point on the lift of b we get a base set b_L in M_L. Any element of $\pi_1(M_L)$ may be represented as a path whose endpoints lie on b_L.

Given $P \in \mathcal{P}$, let $k = \deg v_P$. The space M_P is homeomorphic to

$$S^3 \setminus H_k,$$

where H_k is a k-component positive Hopf link in S^3.

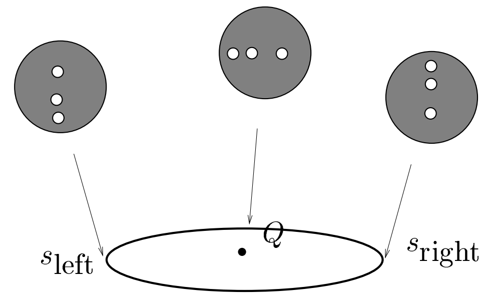

Figure 6. The point vertex manifold M_P.

We can decompose M_P into the union of a solid torus and a solid torus minus k closed tubes. It will be useful to think of the latter as a fiber bundle over the boundary of a disk T_Q centered around $Q = \rho(P)$ as in Figure 6.

For $P \in \mathcal{P}$, define a baseset b_P for $\pi_1(M_P)$ as follows. Let b_P be the right most point on the fibers over T_Q. This set is a contractable disk in M_P. The fundamental group of M_P can be presented as

$$\langle\ \gamma_1, \ldots, \gamma_k : \gamma_1 \ldots \gamma_k = \gamma_{c(1)} \ldots \gamma_{c(k)}, c \in Z_k\ \rangle,$$

where Z_k is the group of cyclic permutations of $1, \ldots, k$. The generators $\gamma_1, \ldots, \gamma_k$ can be realized as

$$\gamma_i = \tau_i \mu_i \tau_i^{-1}, \qquad \text{for } i = 1, \ldots, k,$$

where τ_i and μ_i are the paths drawn in Figure 7.

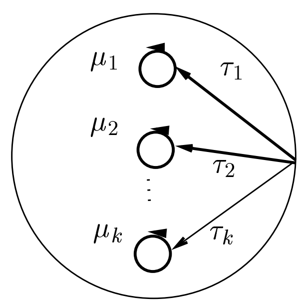

Figure 7. Generators on the Fiber over s_{right}.

Finally, we define the connecting paths associated to the edges of $\Gamma_{\mathcal{L}}$. Take any $y \in \mathcal{Y}(\Gamma_{\mathcal{L}})$ with $i(y) = v_P$ and $t(y) = v_L$. For some i, the path τ_i on M_P goes from the basepoint on M_P to the baseset on M_L. We take this to be the connecting path $\tau(y)$ associated to y. The connecting path $\tau(\overline{y})$ for $\overline{y}$ is just the connecting path for y with opposite orientation. This completes our description of the fundamental group.

Now to define the map α explicitly, we first define a homotopy equivalence $\beta : \Gamma_{\mathcal{L}} \to \Sigma_{\mathcal{L}}$ as follows:

(1) send the point vertices v_P, for $P \in \mathcal{P}$, to the point corresponding to P on $\Sigma_{\mathcal{L}}$;

(2) send the line vertices v_L, $L \in \mathcal{A}$, to the rightmost point on $L \cap \Sigma_{\mathcal{L}}$ (in B);

(3) send an edge y of $\Gamma_{\mathcal{L}}$ onto the unique straight line segment on $\Sigma_{\mathcal{L}}$ connecting the images of $i(y)$ and $t(y)$.

This defines a continuous map $\beta : \Gamma_{\mathcal{L}} \to \Sigma_{\mathcal{L}}$ which is a homotopy equivalence.

To describe $\alpha = \beta \circ \sigma$ explicitly, it suffices to define $\alpha(y)$ for all $y \in \mathcal{Y}(\Gamma_{\mathcal{L}})$, and since $\alpha(\overline{y}) = \alpha(y)^{-1}$ as paths, we need only define $\alpha(y)$ for y of the form $y = y(L, P)$.

Note that $\alpha(v_L)$ lies on b_L. Define elements $A(L,P) \in \pi_1(M(\mathcal{L}))$ associated to pairs $L \in \mathcal{A}$, $P \in \mathcal{P}$, with $P \in L$ as follows. Let j be such that $y(P, L)$ is the jth edge emanating from v_P. If $j = k$, define $A(L,P)$ to be the trivial loop, otherwise define

$$A(L,P) = \tau_j^{-1} \mu_k^{-1} \mu_{k-1}^{-1} \dots \mu_{j+1}^{-1} \tau_j,$$

These give well defined elements of $\pi_1(M(\mathcal{L}))$ with respect to the maximal tree.

Lemma 3.2 *For $L \in \mathcal{A}$, let $P_1, \dots, P_k$ be the ordering on the points in $\mathcal{P} \cap L$, corresponding to the ordering of the edges emanating from v_L. For $P \in \mathcal{P}$, let ℓ be such that $y(L,P)$ is the ℓth edge emanating from v_L. Then the path $\alpha(y(L,P))$ is homotopy equivalent to to*

$$A(L,P_1)A(L,P_2)\dots A(L,P_{\ell-1})\tau(L,P)$$

relative to b_L.

Proof. The path $\beta(y(L,P))$ can be broken up into segments

$$\beta(y(L,P)) = a(L,P_1)a(L,P_2)\dots a(L,P_{\ell-1})a'(L,P_\ell),$$

where $a(L,P_j)$ is a path on $L \cap \Sigma_{\mathcal{L}}$ which passes through P_j, but no other points in $\mathcal{P} \cap L$, and $a'(L,P_\ell)$ is a path on $L \cap \Sigma_{\mathcal{L}}$ whose endpoint is P_ℓ. This is illustrated in Figure 8. By

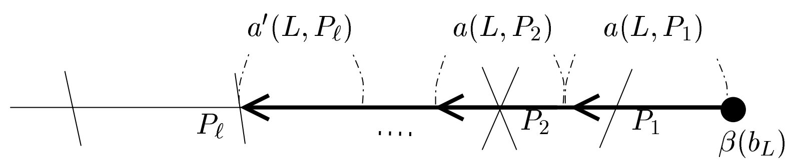

Figure 8. The path $\beta(y(L,P))$.

comparing Figure 3 and Figure 7, we see that $a'(L,P_\ell) = \tau(L,P)$. Thus, to complete the

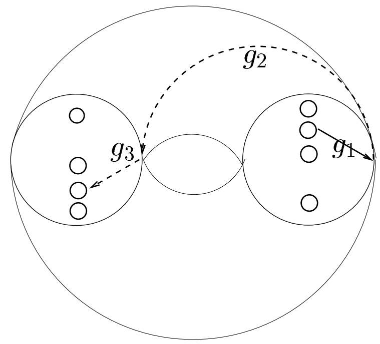

Figure 9. Path through a point-vertex manifold.

proof, it suffices to find each $\sigma(a(L,P_j))$ in terms of $\tau(L,P_j)$ and the standard generators for $\pi_1(M_{P_j})$.

The path $\sigma(a(L,P))$ breaks up into three curve segments $g_1g_2g_3$ contained in M_{P_j}, for $j=1,\ldots,\ell$. The first segment g_1 goes from b_L to b_P and equals $\tau(L,P)=\tau_j^{-1}$, the second g_2 runs along b_P, and the third g_3 goes from b_P back to b_L on the fiber over the point to the left of $\rho(P)$. Figure 9 illustrates how the curve segments g_1,g_2 and g_3 could look for the case that, $y(P,L)$ is the second from highest edge emanating from v_P.

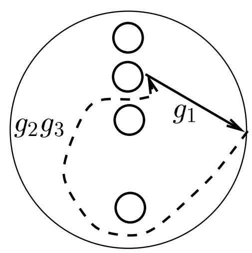

Figure 10. Lifts of $\Sigma_{\mathcal{L}}$ nearby a point of intersection

To find the homotopy class of g_2g_3 in terms of the standard generators of M_P given in Figure 7, one simply retracts g_2 to the base point in the fiber over s_r, and simultaneously rotates the fiber over s_ℓ. For example, the g_2g_3 in Figure 9 is homotopy equivalent to the path drawn in dashes in Figure 10. Comparing this with Figure 7 shows that $\sigma(a(L,P_j)) = A(L,P_j)$.

■

4 Unknotted line arrangements

A crucial ingredient in the proof of Theorem 1.1 given in Section 3 is the existence of a continuous map

$$\sigma : \Sigma_{\mathcal{L}} \to M(\mathcal{L})$$

from the skeleton $\Sigma_{\mathcal{L}} = \mathcal{L} \cap \mathbb{R}^2$ of $\mathcal{L}$ to the boundary 3-manifold $M_{\mathcal{L}}$, so that the image of σ lies on a plane embedded in $E(\mathcal{L})$ (see Lemma 3.1). In this section we define a generalization of skeleta, so that such a map σ exists for a larger class of "unknotted" line arrangements. For unknotted line arrangements Theorem 1.1 can be mimicked directly, that is, the homotopy type of $E(\mathcal{L})$ can be recovered from $M(\mathcal{L})$ by attaching a plane to $M(\mathcal{L})$ along the translated skeleton. For knotted arrangements, one obtains $E(\mathcal{L})$ from $M(\mathcal{L})$ by attaching a 3-manifold along the boundary of a tubular neighborhood of the skeleton.

Skeleta for arbitrary line arrangements. Let $\mathcal{L} \subset \mathbb{C}^2$ be a line arrangement with singular set $\mathcal{P} = \mathrm{Sing}(\mathcal{L})$. We will call a homeomorphism

$$f : E(\mathcal{L}) \to \mathbb{C}^2$$

admissible if the composition ρ of f with projection onto the first coordinate of $\mathbb{C}^2$ is one-to-one for all lines in $\mathcal{L}$, and is a topological fiber bundle outside of $\mathcal{Q} = \rho(\mathcal{P})$.

If ρ come from an admissible h, and

$$I \subset \mathbb{C}$$

is the image of a smooth proper map from $\mathbb{R}$ to $\mathbb{C}$ so that $\mathcal{Q}$ is in the image of I, we will call (ρ, I) an *admissible pair*. If (ρ, I) is admissible, we define

$$\Sigma_{\mathcal{L}}(\rho, I) = \rho^{-1}(I) \cap \mathcal{L},$$

to be the *skeleton* of $\mathcal{L}$ associated to (ρ, I).

For real line arrangements $\mathcal{L}$, the standard skeleton $\Sigma_{\mathcal{L}}$ defined in Section 3 is obtained by taking f to be the identity, and letting I to be the natural inclusion of $\mathbb{R}$ in $\mathbb{C}$.

For any admissible (ρ, I), the skeleton $\Sigma_{\mathcal{L}}(\rho, I)$ sits naturally in

$$\rho^{-1}(I) \cong \mathbb{R}^3.$$

We will say that $\mathcal{L}$ is *unknotted* if there is an admissible pair (ρ, I) so that the skeleton $\Sigma_{\mathcal{L}}(\rho, I)$ lies on a properly embedded plane Π in $\rho^{-1}(I)$ such that the restriction of ρ to Π is a line bundle over I. We will call Π the *spanning plane* for $\Sigma_{\mathcal{L}}(\rho, I)$. For real line arrangements, the spanning plane for the standard skeleton is just the set of real points of $\mathbb{C}^2$.

Unknotted line arrangements are generalizations of pseudo-real arrangements. A line arrangement $\mathcal{L} \subset \mathbb{C}^2$ is called *pseudo-real* if it can be changed to a real line arrangement by a combination of changes of coordinates or isotopies preserving incidences. For example, while the dual arrangement to the lines through pairs of flexes of a smooth cubic is not pseudo-real (by Sylvester's theorem), it is not known to the author whether it is unknotted.

For an arbitrary line arrangement $\mathcal{L}$, the general admissible pair (ρ, I) will yield a skeleton $\Sigma_{\mathcal{L}}(\rho, I)$, which is a non-trivial *braided wiring diagram* (see [C-S], for definitions and examples.) Let $N(\Sigma_{\mathcal{L}}(\rho, I)$ be a regular neighborhood of $\Sigma_{\mathcal{L}}(\rho, I)$ such that its boundary $M(\Sigma_{\mathcal{L}}(\rho, I))$ equals $M(\mathcal{L}) \cap \rho^{-1}(I)$. Let

$$E(\Sigma_{\mathcal{L}}(\rho, I)) = \rho^{-1}(I) \setminus N(\Sigma_{\mathcal{L}}(\rho, I)),$$

The following theorem generalizes Theorem 1.1, and describes the homotopy type of $E(\mathcal{L})$ in terms of $M(\mathcal{L})$ and the 3-manifold with boundary $E(\Sigma_{\mathcal{L}}(\rho, I))$.

Theorem 4.1 *Let $\mathcal{L} \subset \mathbb{C}^2$ be any line arrangement, and let (ρ, I) be any admissible pair. Then $E(\mathcal{L})$ is homotopy equivalent to*

$$M(\mathcal{L}) \cup E(\Sigma_{\mathcal{L}}(\rho, I)).$$

Proof. As in Theorem 1.1, we retract $\mathbb{C}$ to I union disks $N(\mathcal{Q})$ around $\mathcal{Q}$, and simultaneously retract $E(\mathcal{L})$ to

$$E' = E(\mathcal{L}) \cap \rho^{-1}(I \cup N(\mathcal{Q})).$$

Let

$$M' = M(\mathcal{L}) \cap \rho^{-1}(I \cup N(\mathcal{Q})).$$

Let $B(\mathcal{L})$ be the regular neighborhood of $\mathcal{L}$, whose boundary is $M(\mathcal{L})$. Locally over the connected components of $N(\mathcal{Q})$, the situation is as shown in Figure 3, and the complements of $B(\mathcal{L})$ on the fibers over each connected component of $N(\mathcal{Q})$ are isomorphic. Thus, they can be retracted to the complements over I. This yields

$$E' \simeq M' \cup E(\Sigma_{\mathcal{L}}(\rho, I)) \simeq M(\mathcal{L}) \cup E(\Sigma_{\mathcal{L}}(\rho, I)).$$

■

When $\Sigma_{\mathcal{L}}(\rho, I)$ is unknotted, the spanning plane Π induces a partial ordering on directed edges of the incidence graph $\Gamma_{\mathcal{L}}$, and we have the following generalization of Theorem 1.1.

Theorem 4.2 *Let $\mathcal{L} \subset \mathbb{C}^2$ be unknotted with respect to an admissible pair (ρ, I), and let $\Gamma^o_{\mathcal{L}}$ be the associated ordered incidence graph. Then there is a continuous map*

$$\alpha : \Gamma_{\mathcal{L}} \to M(\mathcal{L}),$$

depending only on $\Gamma^o_{\mathcal{L}}$, such that the complement $E(\mathcal{L})$ of $\mathcal{L}$ in $\mathbb{C}^2$ is homotopy equivalent to

$$M(\mathcal{L})/\alpha(\Gamma_{\mathcal{L}}).$$

Proof. First note that $E(\Sigma_{\mathcal{L}}(\rho, I))$ retracts to

$$M(\Sigma_{\mathcal{L}}(\rho, I)) \cup \Pi'$$

where Π' is a translation the spanning plane Π, intersecting $M(\Sigma_{\mathcal{L}}(\rho, I))$ in a translate of $\Sigma_{\mathcal{L}}(\rho, I)$ to $M(\Sigma_{\mathcal{L}}(\rho, I))$. Theorem 4.1 thus implies

$$E(\mathcal{L}) \simeq M(\mathcal{L}) \cup \Pi'.$$

This gives an analog of Lemma 3.1. The rest follows as in the proof of Theorem 1.1.

■

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