Chord Diagrams and Coxeter Links

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Abstract

This paper presents a construction of fibered links (K, Σ) out of chord diagrams $\mathcal{L}$. Let Γ be the incidence graph of $\mathcal{L}$. Under certain conditions on $\mathcal{L}$ the symmetrized Seifert matrix of (K, Σ) equals the bilinear form of the simply-laced Coxeter system (W, S) associated to Γ; and the monodromy of (K, Σ) equals minus the Coxeter element of (W, S). Lehmer's problem is solved for the monodromy of these Coxeter links. [1]

1 Introduction

A *chord diagram* $\mathcal{L}$ is a collection of straight arcs, or *chords*, on the unit disk $D \subset \mathbb{R}^2$ connecting mutually disjoint pairs of points on the boundary of D. A *chord system* is a chord diagram such that the chords are ordered and oriented. Given two distinct oriented chords ℓ_1 and ℓ_2 define their linking number $\mathrm{link}(\ell_1, \ell_2)$ to be the linking number of their endpoints considered as oriented 0-spheres on S^1.

To any chord system $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ one can associate an n-dimensional vector space $\mathbb{R}^{\mathcal{L}}$ together with a skew-symmetric *link form* F defined by

$$F(\ell_i, \ell_j) = \begin{cases} 0 & \text{if } i = j \\ \mathrm{link}(\ell_i, \ell_j) & \text{if } i \neq j \end{cases}$$

The first result of this paper is a construction of a fibered link whose Seifert form is equivalent to the link form of a given chord system.

A fibered link (K, Σ) is a pair consisting of a link $K \subset S^3$ and an oriented surface Σ so that the complement $S^3 \setminus K$ fibers locally trivially over S^1 with fiber Σ. If (K, Σ) is fibered, then $S^3 \setminus K$ can be obtained by gluing $\Sigma \times [0, 1]$ to itself by the identification

$$(x, 0) = (h(x), 1) \qquad x \in \Sigma,$$

where $h : \Sigma \to \Sigma$ is an orientation preserving homeomorphism, called the *monodromy* of the fibered link. A fibered link has an associated skew-symmetric *Seifert form* T defined on $\mathrm{H}_1(\Sigma; \mathbb{R})$ (see, for example, [B-Z], Chapter 13, or section 3 of this paper).

[1] 2000 Mathematics Subject Classification: 57M27, 51F15

Theorem 1.1 *Given a chord system $\mathcal{L}$ there is a fibered link (K, Σ) together with an isomorphism*

$$\phi : (\mathbb{R}^{\mathcal{L}}, F) \to (\mathrm{H}_1(\Sigma; \mathbb{R}), T),$$

which preserves forms.

Given a matrix A let A^+ be the upper-triangular part of A. That is, if $A = [a_{i,j}]$, then $A^+ = [a'_{i,j}]$ where

$$a'_{i,j} = \begin{cases} a_{i,j} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

Define

$$A^{\text{symm}} = A^+ + (A^+)^t$$

to be the *symmetric matrix* associated to A.

We will say a chord system $\mathcal{L}$ is of *Coxeter-type* if the off-diagonal entries of F^+ are non-positive, that is, they are either 0 or -1. A pair (K, Σ) is a *Coxeter link* if it is a fibered link associated, in the manner described in Theorem 1.1, to a chord system $\mathcal{L}$ of Coxeter-type.

Our next theorem shows that if $\mathcal{L}$ is of *Coxeter-type*, and Γ is the incidence matrix of $\mathcal{L}$, then the simply-laced Coxeter system associated to Γ is naturally related to any Coxeter link (K, Σ) for $\mathcal{L}$.

Let (W, S) be the simply-laced Coxeter system associated to the incidence graph Γ of $\mathcal{L}$. Then W is generated by $S = \{s_1, \ldots, s_n\}$, where each s_i is the reflection on $\mathbb{R}^{\mathcal{L}}$ defined by

$$s_i(s_j) = s_j - B(s_i, s_j)s_i,$$

where B is a symmetric form defined by Γ (see Section 2). The *Coxeter element* associated to (W, S) is the product $c = s_1 \cdots s_n \in \mathrm{GL}(\mathbb{R}^{\mathcal{L}})$.

Theorem 1.2 *Let $\mathcal{L}$ be a chord system of Coxeter-type, and let (K, Σ) be an associated Coxeter link. Let (W, S) be the simply-laced Coxeter system associated to the incidence graph of $\mathcal{L}$. Then $B = F^{symm}$ and the restriction homomorphism*

$$h_* : \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}_1(\Sigma; \mathbb{R})$$

of the monodromy of (K, Σ) satisfies

$$\phi \circ h_* = -c \circ \phi.$$

For the convenience of the reader, we review definitions and properties of Coxeter systems in Section 2 and properties of the monodromy of fibered links in Section 3. We give some examples and obstructions for graphs to be incidence graphs of chord diagrams in Section 4. In Section 5 we prove Theorem 1.1 and Theorem 1.2. Our construction generalizes arborescent links [Con] and slalom links [A'C2] which apply to the case when Γ is a tree. In Section 6 we give some examples of Coxeter links. Finally, in Section 7, we settle Lehmer's problem for Coxeter links.

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2 Coxeter Systems

In this section we recall some properties of simply-laced Coxeter systems. See also [Bour] or [Hum] for more complete expositions.

Let Γ be a finite graph with no self-loops or multiple edges. Let $S = \{s_1, \ldots, s_n\}$ be an ordering on the set of vertices. The *adjacency matrix* A of Γ is the matrix $A = [a_{i,j}]$, where

$$a_{i,j} = \begin{cases} 1 & \text{if there is an edge between } s_i \text{ and } s_j \\ 0 & \text{otherwise} \end{cases}$$

Given a graph Γ, and an ordering on the edges $S = \{s_1, \ldots, s_n\}$, there is an associated group W with distinguished set of generators S given by

$$W = \langle\ S\ :\ (s_i s_j)^{m_{i,j}} \quad i,j = 1, \ldots, n\ \rangle,$$

where

$$m_{i,j} = \begin{cases} 1 & \text{if } i = j \\ a_{i,j} + 2 & \text{if } i \neq j \end{cases}$$

The pair (W,S) is called the *simply-laced Coxeter system* associated to Γ, and S is the set of *Coxeter generators.* More general Coxeter groups are obtained by letting $m_{i,j}$ vary among integers greater than 3 or ∞ whenever s_i and s_j are connected by an edge in Γ. In the general setting, the graph Γ together with edges labeled $m_{i,j}$ determines the Coxeter system.

Coxeter systems have naturally associated representations as groups acting on a vector space V preserving a particular bilinear form B. Here $V = \mathbb{R}^S$ be the n-dimensional vector space over $\mathbb{R}$ with basis $e_1, \ldots, e_n$ and B is the bilinear form defined by

$$B(e_i, e_j) = -2\cos\left(\frac{\pi}{m_{i,j}}\right).$$

The *Coxeter representation* of W in $\mathrm{GL}(V)$ is defined by

$$s_i(e_j) = e_j - B(e_i, e_j)e_i.$$

If (W,S) is simply-laced, then we have

$$s_i(e_j) = \begin{cases} -e_i & \text{if } i = j \\ e_j & \text{if } i \neq j \text{ and } a_{i,j} = 0 \\ e_i + e_j & \text{if } a_{i,j} = 1 \end{cases}$$

We will use the following two Lemmas in our proofs of the main theorems.

Lemma 2.1 *The simply-laced Coxeter system associated to a graph Γ has associated bilinear form $B = 2I - A$, where I is the $n \times n$ identity matrix.*

Lemma 2.2 *If $\mathcal{L}$ is a chord diagram with incidence graph Γ, and F is its link form, then the bilinear form B associated to the simply-laced Coxeter system associated to Γ satisfies $B = 2I + F^{symm}$ if and only if $\mathcal{L}$ is of Coxeter-type.*

A Coxeter system is called *spherical* if its Coxeter group is a finite reflection group on Euclidean space. It is called *affine* if its Coxeter group is isomorphic to a group of affine reflections. The Coxeter group W is finite, and therefore (W, S) is spherical, if and only if B is positive definite; and (W, S) is affine if and only if B is positive semi-definite (see, for example, [Hum] Section 4.7, and Theorem 6.4).

The *Coxeter element* c of (W, S) is given by

$$c = s_1 \cdots s_n.$$

Thus c depends on the choice of ordering on S. If Γ is a tree, then c is determined up to conjugacy ([Bour] Chapter 5, Lemma 1) and hence its spectrum is determined by the Coxeter system. This is not the case if Γ contains circuits.

The geometry of the Coxeter system is visible in the spectrum of the Coxeter element (cf. [A'C1].)

Theorem 2.3 **([How] Theorem 4.1)** *Let c be a Coxeter element for a Coxeter system (W, S).*

(1) (W, S) is spherical if and only if all the eigenvalues of c are roots of unity other than 1.

(2) (W, S) is affine if and only if c has an eigenvalue equal to 1 and all eigenvalues c have modulus one.

For any matrix $M = [a_{i,j}]$, let M^{u} be the strictly upper triangular part of M, that is, $M^{\mathrm{u}} = [M_{i,j}]$, where

$$M_{i,j} = \begin{cases} a_{i,j} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}$$

Theorem 2.4 **([How] Theorem 2.1)** *Let (W, S) be a Coxeter system with bilinear form B. Then $c = -U^{-1}U^t$, where $U = I + B^u$.*

Corollary 2.5 *If (W, S) is a simply-laced Coxeter system associated to the graph Γ, and A is its adjacency matrix, then $c = -U^{-1}U^t$, where $U = I - A^+$.*

3 Monodromy of fibered links

Let K be a fibered link, with fibering surface Σ. Then we have the following.

(1) $\Sigma \subset S^3$ is an oriented surface with boundary equal to K; and

(2) there is an associated homeomorphism

$$\tau : S^3 \setminus \Sigma \to \Sigma \times I,$$

where I is the open interval $(0,1)$.

Let $\Sigma^+ = \Sigma \times \{0\}$ and $\Sigma^- = \Sigma \times \{1\}$. Then $S^3 \setminus K$ is homeomorphic to $\Sigma \times I$ with Σ^- glued to Σ^+ by a homeomorphism

$$h : \Sigma \to \Sigma$$

called the monodromy of the fibration. Here Σ^- and Σ^+ are identified with Σ in the obvious way. The induced map

$$h_* : \mathrm{H}_1(\Sigma; \mathbb{Z}) \to \mathrm{H}_1(\Sigma; \mathbb{Z})$$

is called the *monodromy* of K, and doesn't depend on the choice of trivialization τ.

For any loop γ on Σ, the inclusion of Σ in $\Sigma \times I$ induces a map

$$\iota : \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}_1(\Sigma^+; \mathbb{R})$$

which we will denote by $\iota(\gamma) = \gamma^+$.

Alexander duality gives a non-degenerate pairing

$$\mathrm{H}_1(\Sigma; \mathbb{R}) \times \mathrm{H}_1(S^3 \setminus \Sigma; \mathbb{R}) \to \mathbb{R}$$

by linking number in S^3:

$$\big(\gamma, \gamma'\big) \mapsto \mathrm{link}(\gamma, \gamma').$$

Let $\gamma_1, \ldots, \gamma_n$ be a basis for $\mathrm{H}_1(\Sigma; \mathbb{R})$. The *Seifert matrix* M of K with respect to Σ is given by $M = [\mathrm{link}(\gamma_i^+, \gamma_j)]$, and $T = M - M^t$ is called the *Seifert form* of (K, Σ).

The following theorem is well-known in knot theory (see, for example, [Rolf].)

Theorem 3.1 *The restriction of the monodromy of a fibered link K*

$$h_* : \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}(\Sigma; \mathbb{R})$$

written with respect to the basis $\gamma_1, \ldots, \gamma_n$ equals

$$M^{-1} M^t.$$

By Theorem 3.1 and Corollary 2.5 to find a Coxeter link associated to a simply-laced Coxeter graph Γ with adjacency matrix A it suffices to find a fibered link whose Seifert matrix M is $I - A^+$.

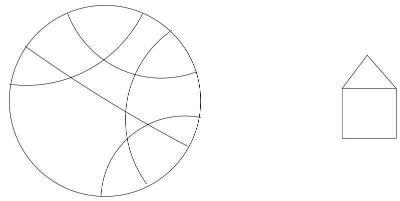

Figure 1. Realization of a graph.

4 Admissible Graphs

A chord diagram $\mathcal{L}$ is a collection of straight paths on the unit 2-disk D joining pairs of points on the boundary of D. The *incidence graph* Γ of a chord diagram $\mathcal{L}$ is the graph with vertices corresponding to chords and an edge between two vertices if and only if the chords meet in the interior of the disk. We will call $\mathcal{L}$ a *realization* of Γ. Figure 1 gives an example. (For ease of illustration, we will draw the chords as circular arcs.)

A finite graph Γ is *realizable* if it is the incidence graph of a chord diagram. An ordered graph Γ is *admissible* if there is a chord system of Coxeter-type for which Γ is the incidence graph with induced ordering. We will call two chord diagrams equivalent if they are the same up to isotopy of the pair $(D, \mathcal{L})$.

Lemma 4.1 *Given any realizable graphs Γ_1 and Γ_2, the join $\Gamma_1 \vee \Gamma_2$ of the graphs at one vertex is realizable.*

Proof. A realization of any graph Γ is equivalent to an embedding of a union of S^0's in S^1 one for each line in $\mathcal{L}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be realizations of Γ_1 and Γ_2, respectively. We can assume that Γ_1 and Γ_2 correspond to a common line ℓ in $\mathcal{L}_1$ and $\mathcal{L}_2$ passing through the center of D, say horizontally as in Figure 2. Furthermore, we can assume the endpoints of the arcs other than ℓ in $\mathcal{L}_1$ lie to the left of the vertical line through the center of D, and similarly the endpoints of the arcs in $\mathcal{L}_2$ other than ℓ lie to the right of the vertical line through the center of D. The union of the arcs in $\mathcal{L}_1$ and $\mathcal{L}_2$ form a 2-embedding for $\Gamma_1 \vee \Gamma_2$.

■

Corollary 4.2 *All finite trees are realizable.*

It is not hard to see that cyclic graphs, complete graphs, and complete bipartite graphs are realizable. A cyclic graph has a realization as shown in Figure 3.

Realizations of cyclic graphs have the following property.

Lemma 4.3 *Up to isotopy of $(D, \mathcal{L})$ realizations of cyclic graphs are uniquely determined.*

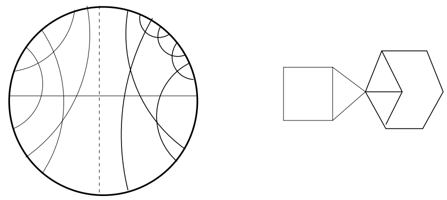

Figure 2. Join of two realizations.

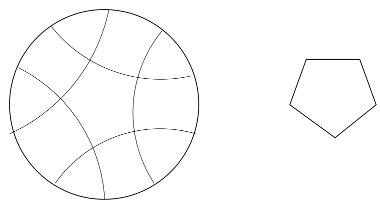

Figure 3. 5-Cycle.

Let Γ be a graph with vertices S. A subgraph $\Gamma' \subset \Gamma$ is an *induced subgraph* if for some $S' \subset S$, Γ' is the subgraph containing all edges in Γ whose endpoints are in S'.

If $S' \subset S$ is such that the induced subgraph has no edges then we say that S' is an *independent* set of vertices in Γ. In order for there to exist a line in D intersecting all arcs in an independent set S', the lines in S' must be parallel. Thus, we have the following.

Lemma 4.4 *A graph Γ is not realizable if there is a subset $S' \subset S$ such that*

(1) S' contains three vertices;

(2) S' is independent;

(3) there is an $s \in S$ so that for every $s' \in S'$ there is an edge in Γ joining s and s'; and

(4) there is an induced cyclic subgraph in Γ containing S'.

Figure 4 gives an example of a non-realizable graph.

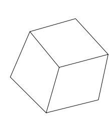

Figure 4. Non-realizable graph.

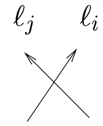

Figure 5. Orientation on a chord diagram $(i < j)$.

A chord system $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ is of *Coxeter-type* if whenever ℓ_i and ℓ_j intersect in D, for $i < j$, the intersection looks locally as in Figure 5.

Not all orderings on a realizable graph are admissible. For example, given an n-cyclic graph, the cyclic ordering has no Coxeter-type embedding for $n > 3$. Given a realizable graph, however, one can choose an ordering which is admissible, as shown in the following Lemma pointed out to the author by R. Vogeler.

Lemma 4.5 *Any chord diagram admits an orientation and ordering of Coxeter-type.*

Proof. Let $\mathcal{L}$ be any realization of a chord diagram on the unit disk D in $\mathbb{R}^2$. We may assume that none of the chords are vertical. Orient the chords so that the x-coordinate is increasing. Then order the chords from smallest to largest slope.

■

5 Construction

Given an oriented chord diagram, we will construct an associated fibered link. Let $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ be the realization of an oriented chord diagram in the unit disk $D \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. In $\mathbb{R}^3$ attach twice positively twisted bands $b_1, \ldots, b_n$ to D as in Figure 6, in the order given by the ordering of the arcs, i.e., so that b_i lies over b_j if $i > j$. Let Σ be the resulting surface, with orientation determined by the one on $D \subset \mathbb{R}^3$. Let K_Γ be the oriented boundary link.

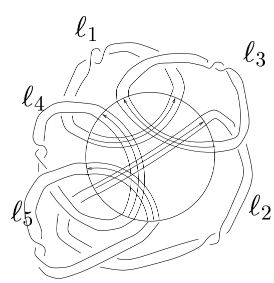

Figure 6. Murasugi sum.

Then Σ is obtained from the oriented disk D by a sequence of Murasugi sums of Hopf links. Hence K_Γ is a fibered link with fiber Σ ([Sta], cf. [Gab].)

Extend each oriented arc ℓ_i to an oriented closed loop γ_i going once around the corresponding attached band b_i. This gives a basis $\omega_1, \ldots, \omega_n$ for $\mathrm{H}_1(\Sigma; \mathbb{R})$. Let F be the link form for $\mathcal{L}$. By the construction we have

$$\mathrm{link}(\omega_i^+, \omega_j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ F^+(\ell_i, \ell_j) & \end{cases}$$

Thus, the Seifert form T for (K, Σ) is given by

$$\begin{aligned} T(\omega_i, \omega_j) &= \mathrm{link}(\omega_i^+, \omega_j) - \mathrm{link}(\omega_j^+, \omega_i) \\ &= F(\ell_i, \ell_j) \end{aligned}$$

This proves Theorem 1.1.

Suppose $\mathcal{L}$ is of Coxeter-type. Let A be its adjacency matrix. Then the Seifert matrix for (K, Σ) is given by $M = I - A^+$. By Theorem 3.1 the monodromy of the fibration is given by

$$h_* = M^{-1} M^t$$

The simply-laced Coxeter system (W, S) associated to Γ has bilinear form

$$B = 2I - A.$$

By Corollary 2.5, the Coxeter element of the simply-laced Coxeter system associated to Γ is given by

$$c = -U^{-1} U^T$$

where $U = I - A^+ = M$. Therefore $h_* = -c$, which proves Theorem 1.2.

Before concluding this section, we remark that the construction described above depends on less information than on the ordering of the chord diagram. The associated link is determined by the relative ordering of pairs of intersecting arcs in the chord diagram. We will call a chord diagram together with this information a *directed chord diagram*. Instead of an ordered incidence graph, we obtain a directed incidence graph.

As was pointed out in [Shi], the Coxeter element of a Coxeter system only depends up to conjugacy on the directed graph determined by ordered Coxeter graph. Similarly, we can see the following from the construction.

Proposition 5.1 *The fibered link associated to a chord system only depends on the directed chord diagram.*

A vertex v on a directed graph is called a *source* (resp. *sink*) if all edges with one endpoint equal to v point away from (resp. toward) v. It is not hard to see that the Coxeter element of a Coxeter system does not change its conjugacy class if a source node is changed to a sink. We have the following similar statement for links constructed from chord diagrams.

Proposition 5.2 *The link obtained from a directed chord diagram is equivalent to that obtained by reordering the chord diagram so that a source is replaced by a sink.*

Proof. Replacing a source by a sink amounts to the same as passing one of the twisted bands through the disk D from the negative to the positive side. Although this may change the isotopy type of the embedding of Σ in $S^3 \setminus K$ it does not change the link.

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6 Examples of Coxeter links.

This section contains some examples of Coxeter links.

Example 1. Trees

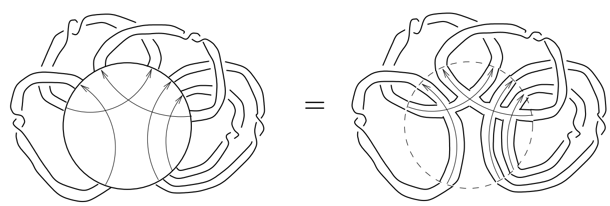

Figure 7. Coxeter links and plumbing.

When Γ is a tree, our construction gives arborescent links [Con]. This is easily seen by isotoping the disk to a neighborhood of the chord diagram as in Figure 7.

As stated earlier in Chapter 2, the Coxeter element of a simply laced Coxeter system (W, S) doesn't depend, up to conjugacy, on the ordering of S. Visualizing the Coxeter link of a tree as a plumbing link, one can see that the relative ordering of any two overlapping chords on a tree chord diagram can be switched by passing one of the bands through itself.

Proposition 6.1 *If $\mathcal{L}$ is a chord system whose incidence graph is a tree then the associated Coxeter link doesn't depend on the ordering on the chord diagram.*

On the other hand, there can be more than one embedding of a tree as a positive chord system giving rise to distinct links as shown in Figure 8. One sees that the link on the left has two knotted components, while the one on the right has a component which is the unknot.

Example 2. A_n

The A_n Coxeter graph where vertices are numbered consecutively gives rise to the $(n+1, 2)$ torus knot.

This can be seen inductively as follows. A cross shaped portion of the chord diagram, where the vertical chord has higher index in the ordering than the horizontal one, gives rise to the portion of a link shown in Figure 10.

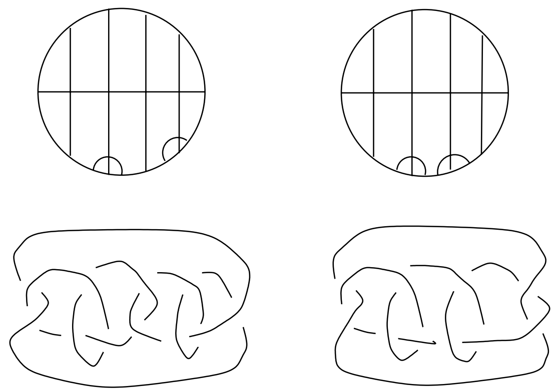

Figure 8. Two embeddings of the same tree and their associated links.

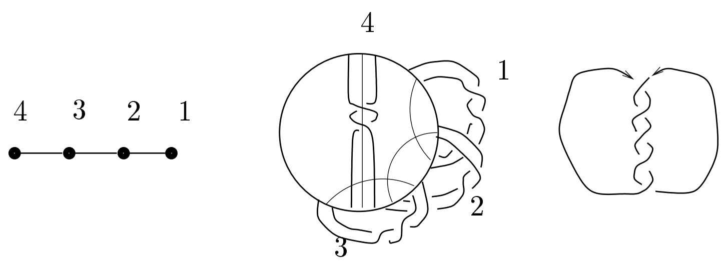

Figure 9. A_n gives rise to the $(n+1,2)$ torus knot.

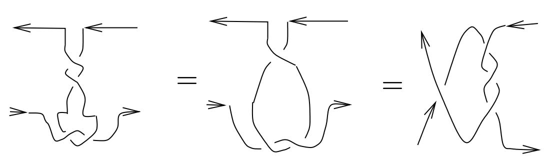

Figure 10. Basic transformation.

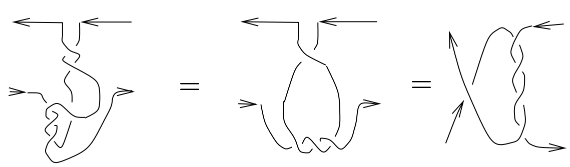

Figure 11. Induction step.

Thus, our claim follows by induction using the induction step illustrated in Figure 11.

Example 3. Star graphs

Let $p_1, \ldots, p_k$ be positive integers. Consider the graph $\mathrm{Star}(p_1, \ldots, p_k)$ obtained by taking the union of $A_{p_1}, \ldots, A_{p_k}$ attached at an end vertex as in Figure 12.

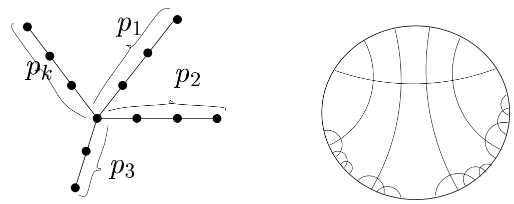

Figure 12. Star graph and its realization.

Directing the graph so that all edges point to the multiple vertex and using intermediate steps shown in Figure 10 and Figure 11 the reader can verify that the corresponding Coxeter link $L_{p_1,\ldots,p_k}$ is a $(p_1, \ldots, p_k, -1, \ldots, -1)$-pretzel link, where there are $k-2$ twists of order -1 (see Figure 13).

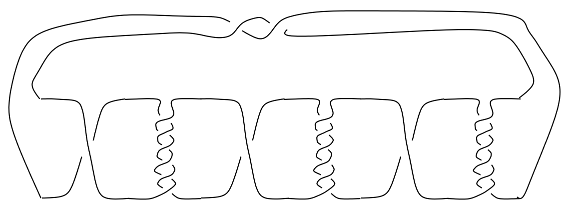

Figure 13. Coxeter link for a star graph.

Since D_n is a star graph, we see that an associated Coxeter link is the iterated torus link given by taking a Hopf link and replacing one component by the $(n-1, 2)$ torus link. The groups E_6, E_7 and E_8 give rise to the $(-2, 3, 3)$-pretzel, the $(-2, 3, 4)$-pretzel and the $(-2, 3, 5)$-pretzel knots, respectively. The affine Coxeter system E_9 gives rise to the $(-2, 3, 6)$-pretzel knot, and the hyperbolic Coxeter system E_{10} gives rise to the $(-2, 3, 7)$-pretzel knot.

Example 4. $\tilde{A}_n$

Cyclic graphs correspond to the affine Coxeter systems $\tilde{A}_n$, hence any Coxeter element has eigenvalue one (see Theorem 2.3), and the rest of the eigenvalues lie on the unit circle.

For cyclic graphs, subtleties are already exhibited for small n.

For $n = 3$, there is only one possible ordering on the 3-cycle, but there are two possible embeddings in the disk. One of these embeddings (Figure 14 a)) is of Coxeter-type and has characteristic polynomial

$$\Delta(t) = 1 + t - t^2 - t^3 = (1 + t)^2(1 - t).$$

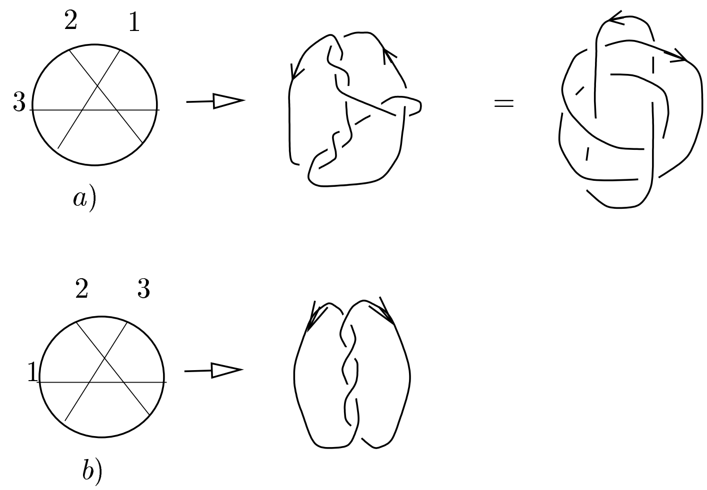

Figure 14. Triangle systems.

As can be seen in the figure, this is an iterated torus link. The other is not of Coxeter-type (Figure 14 b)) and gives rise to the $(4, 2)$ torus link $T_{4,2}$ with characteristic polynomial

$$\Delta(t) = 1 - t + t^2 - t^3 = (1 - t)(1 + t^2).$$

Note that the link in Figure 14 b) is the Coxeter link for the A_3 diagram, once we choose a different basis for $\mathrm{H}_1(\Sigma; \mathbb{R})$.

For $n = 4$, there are two ordered embeddings of the cycle which are of Coxeter-type (see Figure 15 a) and b)). Since the two orderings in a) and b) differ by changing a sink (vertex 4 in a)) to a source (vertex 1 in b)) the corresponding links are equivalent by Proposition 5.2. They equal the (8^3_10 links in Rolfsen's table [Rolf]). The characteristic polynomial for both fibered links is given by:

$$\Delta(t) = 1 - 2t^2 + t^4 = (t + 1)^2(t - 1)^2.$$

The clockwise and counter-clockwise embeddings of the 4-cycle with cyclic ordering are not of Coxeter-type. The corresponding knots are shown in Figure 15 c). Both have Alexander polynomial

$$\Delta(t) = 1 - t - t^3 + t^4 = (1 - t)^2(1 + t + t^2)$$

with respect to the fibration.

For $n = 5$, there are two ordered embeddings of the cycle of Coxeter-type as shown in Figure 16. The distinct orderings give rise to the characteristic polynomials

$$\begin{aligned} \Delta_1(-t) &= 1 - t - t^4 + t^5; \quad \text{and} \\ \Delta_2(-t) &= 1 - t^2 - t^3 + t^5. \end{aligned}$$

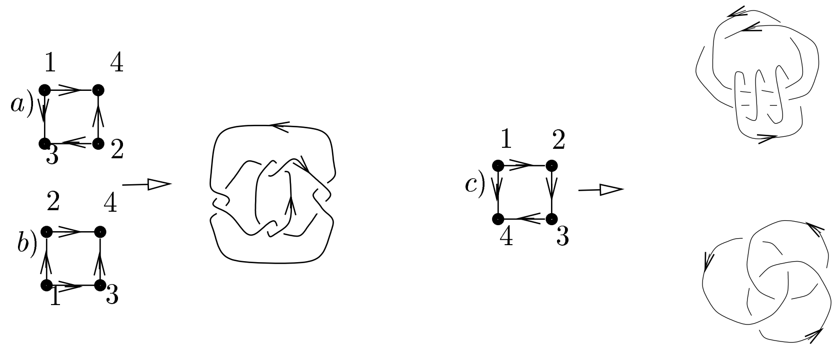

Figure 15. Square systems.

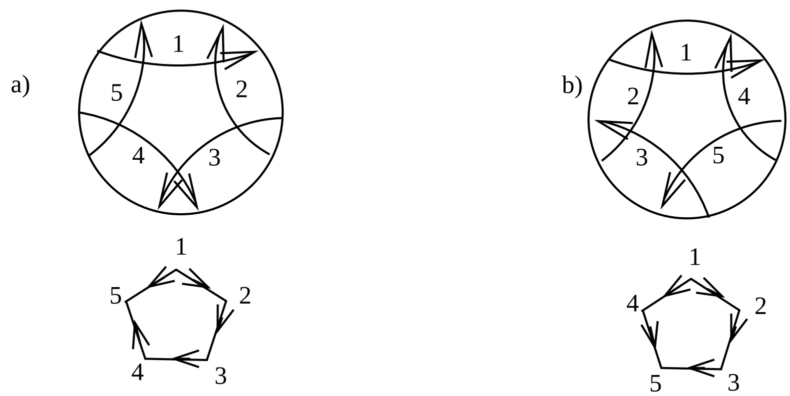

Figure 16. Two Coxeter-type orderings for the 5-cycle

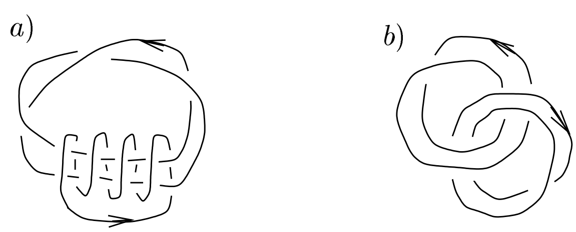

Figure 17. Two Coxeter links for the 5-cycle

The orderings give rise to the distinct links shown in Figure 17.

5. Smallest hyperbolic Coxeter link.

The simply-laced minimal hyperbolic Coxeter system of smallest dimension is a triangle with a tail, which has only one realization as a chord diagram . There are three positive chord systems obtained by adding a chord to the positive triangle chord diagram in Example 6 in three different ways. By exchanging sinks and sources, however, it is possible to go from any one of these chord systems to any other (see Proposition 5.2.) Thus, the Coxeter link is uniquely determined and equals the knot shown in Figure 18, which is the mirror of the 10_{145}-knot in Rolfsen's table [Rolf]. The Alexander polynomial of this knot is

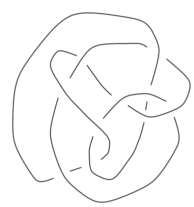

Figure 18. Smallest hyperbolic Coxeter link.

$$\Delta(t) = 1 + t - 3t^2 + t^3 + t^4$$

and its Mahler measure is: $2.36921\ldots$.

7 Application to Lehmer's problem

Given a polynomial $p(x) \in \mathbb{Z}[x]$, the *Mahler measure* $\|p\|_M$ of $p(x)$ is the product of the roots of $p(x)$ outside the unit circle. Lehmer's problem [Leh] asks: For any $\delta > 0$ does there exist a monic integer polynomial $p(x)$ whose Mahler measure satisfies $1 < \|p\|_M < 1 + \delta$? For degrees up to 40 (see [Boyd],[Mos]), the polynomial with smallest Mahler measure is Lehmer's polynomial [Leh]

$$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Lehmer's problem reduces to a study of monic reciprocal polynomials [Smy]. The characteristic polynomial $\Delta(x)$ of the monodromy of a fibered link is necessarily monic, integer, and reciprocal. Conversely, if $\Delta(x)$ is monic, integer, reciprocal and $\Delta(1) = \pm 1$ then it is the characteristic (Alexander) polynomial of a fibered knot [Bur]. For general fibered links, the restriction on $\Delta(1)$ does not hold. Thus, fibered links are a natural source of examples to study Lehmer's problem. Kirby's Problem 5.12 in [Kir] mentions the connection between Alexander polynomials of knots and Lehmer's problem. Lehmer's problem translates to a question about multi-variable Alexander polynomials which is studied, for example, in [S-W].

Lehmer's polynomial appears as the Alexander polynomial of the $(2, 3, 7, -1)$-pretzel knot [Reid]. The (-2,3,7)-pretzel knot is equivalent to the $(2, 3, 7, -1)$-pretzel knot which is one of the family of pretzel links $K_{p_1,\ldots,p_k}$ (see Example 6). The characteristic polynomials of the monodromy $\Delta_{p_1,\ldots,p_k}(x)$ have Mahler measure greater than or equal to that of $p_L(x)$ [Hir].

The pretzel link $K_{2,3,7}$ comes from the $2, 3, 7$ star graph, which is also known as E_{10}, one of the minimal hyperbolic Coxeter systems. McMullen proved the following result [McM].

Theorem 7.1 *Let c be the Coxeter element of a Coxeter system, and let*

$$\lambda(c) = \max\{|\alpha| \ : \ \alpha \text{ is an eigenvalue of } c\}.$$

Then either $\lambda(c) = 1$ *or* $\lambda(c) \geq \lambda(c_0)$ *where* c_0 *is the Coxeter element for* E_{10} *Thus, if* q_c *is the characteristic polynomial of c, then*

$$\|q_c\|_M \geq \|q_{c_0}\|_M = \|p_L\|_M$$

for all Coxeter elements c.

Thus, we have the following Corollary.

Corollary 7.2 *If* $p(x)$ *is the characteristic polynomial for the monodromy of a Coxeter link, then* $\|p\|_M \geq \|p_L\|_M$.

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