POLYNOMIAL PERIODICITY FOR BETTI NUMBERS OF COVERING SURFACES

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1. Introduction.

Fix a smooth complex projective surface Y and a finite union of irreducible curves $\mathcal{B} \subset Y$. To each finite group G and surjective group homomorphism

$$\phi : \pi_1(Y - \mathcal{B}) \to G,$$

one can uniquely associate a normal surface X_ϕ and a finite surjective morphism

$$\rho_\phi : X_\phi \to Y,$$

with the following property (see [G-R]): the restriction, $\rho_\phi^u = \rho_\phi|_{X_\phi^u}$, where X_ϕ^u is the preimage $\rho_\phi^{-1}(Y - \mathcal{B})$, is, topologically, the regular unbranched covering of $Y - \mathcal{B}$ determined by ϕ. We call $\rho_\phi : X_\phi \to Y$ the *branched covering* associated to ϕ. (The definition generalizes to other dimensions, but we will be mainly concerned with the surface case.)

For each positive integer n, let

$$\phi_n : \pi_1(Y - \mathcal{B}) \to \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$$

be the natural homomorphism and let $\rho_n : X_n \to Y$ be the corresponding branched covering. For any desingularization $\sigma : \widetilde{X}_n \to X_n$, let $\widetilde{\rho}_n : \widetilde{X}_n \to Y$ be the composition $\rho_n \circ \sigma$. In this paper we show that, with certain restrictions on $\mathcal{B}$, the first Betti number $b_1(\widetilde{X}_n)$, a birational invariant, is "polynomial periodic" as a function of n.

The evidence from classical as well as recent computations shows that the number $b_1(\widetilde{X}_n)$ exhibits a combination of polynomial and periodic behavior as a function of n. If one replaces the surface Y by a smooth curve and does the same construction, the Riemann-Hurwitz formula immediately implies that the first Betti number of the coverings is a polynomial in n. Zariski [Za1], [Za2] showed that in the case Y is $\mathbb{P}^2$ and $\mathcal{B}$ is the union of an irreducible curve with nodes and cusps and a line in general position ($\mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$ in this case) that $b_1(\widetilde{X}_n)$ is periodic. Libgober [Lb] and Vacquié [V] have extended this result, Libgober to more general irreducible curves, and Vacquié to the case when the curve is not irreducible (in this case the map ϕ_n is defined slightly differently.)

For Hirzebruch coverings (see [Hz]), where Y is $\mathbb{P}^2$ and $\mathcal{B}$ is a finite union of lines, there has been various work (for example, [Is], [G], [Ho] and [Zu]) on computing the invariant. Polynomial or quasi-polynomial behavior was found by Gläser [Gl] and Zuo [Zu] for certain examples. In [Zu] Zuo computes an example for which $b_1(\widetilde{X}_n) = q(n) + c$, where q is a polynomial and c varies periodically with n.

Zuo's main result (in [Zu]) goes further in describing $b_1(\widetilde{X}_n)$.

Theorem 1.1. *Let Y be any smooth surface and let $\mathcal{B}$ be any finite union of smooth curves on Y with the following properties: they are linearly equivalent as divisors, intersect one another transversally and their self intersection is positive. Then there is a polynomial $q(n)$ and an integer n_0 so that $b_1(\widetilde{X}_n) = q(n)$ unless n_0 divides n. The degree of $q(n)$ equals one less than the maximum number of curves in $\mathcal{B}$ intersecting in a point. For any n, $b_1(\widetilde{X}_n)$ differs from $q(n)$ by a bounded constant depending only on Y and $\mathcal{B}$.*

The aim in this paper is to show that under certain conditions the sequence $b_1(\widetilde{X}_n)$ is polynomial periodic, a concept we will now define.

Definition 1.2. An integer valued function $f : \mathbb{N} \to \mathbb{Z}$ is *polynomial periodic* if there is an integer N and a finite sequence of polynomials $p_0(x), \ldots, p_{N-1}(x)$ so that if $n \equiv i (\mod N)$ then

$$f(n) = p_i(n).$$

In [S] (Theorem 1.2) Sarnak proves the following result. (See also [Lr].)

Theorem 1.3. *Let Y^u be a topological space homotopy equivalent to a finite CW complex and, for each $n > 0$, let*

$$\rho_n^u : X_n^u \to Y^u$$

be the unbranched covering associated to the homomorphism

$$\pi_1(Y^u) \to H_1(Y^u; \mathbb{Z}/n\mathbb{Z}).$$

Then $b_1(X_n^u)$ is polynomial periodic.

Since the case where Y^u is the complement of a finite union of irreducible curves $\mathcal{B}$ in a smooth surface Y is included in this statement, this leads Sarnak to ask whether $b_1(\widetilde{X}_n)$, for the desingularized branched coverings $\widetilde{X}_n$, is also polynomial periodic.

Remark 1.4. In [Lb], Libgober proves that if Y is $\mathbb{P}^2$ and $\mathcal{B}$ is the union of any irreducible curve C and line L in general position, then the difference $b_1(\widetilde{X}_n) - b_1(X_n^u)$ equals one. Thus, Theorem 1.3 proves polynomial periodicity for this situation.

The following simple observation will be useful in our analysis.

Remark 1.5. Given Y and $\mathcal{B}$, if Y' is another smooth surface, $\sigma : Y' \to Y$ is an isomorphism outside of $\mathcal{B}$ and $\mathcal{B}' = \sigma^{-1}(\mathcal{B})$, then $\pi_1(Y - \mathcal{B}) = \pi_1(Y' - B')$ and the corresponding branched coverings X_n and X'_n are birationally equivalent.

We will also need the following definition.

Definition 1.6. Define the *global inertia subgroup* $I_C \subset \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$ associated to an irreducible curve C on Y to be the subgroup generated by a positively oriented meridinal loop around C in $Y - \mathcal{B}$. (The orientation of the loop is given by the complex structure.) Thus, if $C \not\subset \mathcal{B}$ then $I_C = (0)$.

The main result is the following.

Theorem 1.7. *Let Y and $\mathcal{B}$ be such that there exists a smooth surface Y' and a morphism $\sigma : Y' \to Y$ which is an isomorphism outside the preimage $\mathcal{B}'$ of $\mathcal{B}$ with the properties that*

(1) *$\mathcal{B}'$ consists of smooth curves with normal crossings,*
(2) *I_C is infinite for all curves $C \subset \mathcal{B}'$ and*
(3) *for all distinct curves $C, D \subset \mathcal{B}'$ with $C \cap D \neq \emptyset$, we have $I_C \cap I_D = (0)$.*

Then $b_1(\widetilde{X}_n)$ is polynomial periodic.

Theorem 1.7 gives us the following corollary (see Remark 5.2.)

Corollary 1.8. *Let Y be $\mathbb{P}^2$ and let $\mathcal{B}$ be a union of smooth curves intersecting transversally. Then $b_1(\widetilde{X}_n)$ is polynomial periodic.*

To prove Theorem 1.7, we study an algorithm (described in [Ho]) for computing the first Betti number and break up the problem into showing that two numbers, the first Betti number and the nullity of the intersection matrix for irreducible curves above the branch locus are polynomial periodic. (Note that although the number of curves above the branch locus depends on the smooth model $\widetilde{X}_n$ which is chosen, the nullity is independent.) The difference between these two numbers, as we show in section 4, is the first Betti number.

Here is a rough outline of the algorithm. By Remark 1.5 we can assume $\mathcal{B}$ consists of smooth curves with normal crossings.

(1) Find a presentation for the fundamental group of $Y - \mathcal{B}$.
(2) Using Fox Calculus, find a presentation matrix $\mathcal{M}$ (related to the *Alexander matrix*) for $\mathrm{H}_1(\widetilde{Y - \mathcal{B}}; \mathbb{Z})$ as a $\mathbb{Z}[G]$-module, where $G = \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$ and $\widetilde{Y - \mathcal{B}}$ is the maximal abelian unbranched covering of $Y - \mathcal{B}$ and from this find $b_1(X_n^u)$.
(3) Let $\mathcal{C}$ be the union of curves $\mathcal{B}$ and, if $\mathcal{B}$ doesn't support an ample divisor, also include an ample curve in general position. Choose a suitable smooth model $\widetilde{X}_n$ for X_n, and find the nullity, $\mathrm{Null}(\widetilde{\rho}_n^{-1}(\mathcal{C}))$, of the intersection matrix for the curves $\widetilde{\rho}_n^{-1}(\mathcal{C})$ in $\widetilde{X}_n$.

The numbers obtained in parts (2) (as shown by Sarnak) and (3) (the new result of this paper) are polynomial periodic functions in n. The proofs of both these facts are similar because they come down to transforming geometric problems into essentially combinatorial ones involving finitely generated groups and matrices with entries in group rings. We review the proof of Theorem 1.3 from this point of view in section 2. Fox in [F] shows how to find the first Betti number of unbranched coverings by a formal manipulation of combinatorial data such as matrices with entries in a group ring. Sarnak uses Fox's result and the following key proposition ([S], Proposition 1.7) to prove Theorem 1.3.

Proposition 1.9. *Let $f_1, \dots, f_\ell$ be polynomials in k variables. Then the number of k-tuples of nth roots of unity satisfying $f_1, \dots, f_\ell = 0$ is polynomial periodic as a function of n.*

We generalize the results of section 2 in section 3, where again the crucial step is Proposition 1.9. In sections 5 and 6, we show how finding the number in part (3) reduces to the situation described in section 3.

Here are some remarks about the other numerical invariants of $\widetilde{X}_n$. These are, however, not birational invariants even for smooth models, so we must designate $\widetilde{X}_n$ to be the minimal smooth model which admits a birational morphism to X_n and is an isomorphism outside the singular points of X_n.

Iversen [Iv] has a formula for the topological Euler characteristic of branched coverings, which shows that $\chi_{\text{top}}(\widetilde{X}_n)$ is polynomial periodic in n. Since the first and second Betti numbers are related by an equation depending on $\chi_{\text{top}}(\widetilde{X}_n)$, given the same hypotheses as in Theorem 1.7 all the Betti numbers of $\widetilde{X}_n$ are polynomial periodic.

The Chern numbers for branched coverings and their desingularizations can be computed fairly easily and can be shown to be polynomial periodic for the minimal smooth models $\widetilde{X}_n$. For surfaces, the Chern numbers in combination with the first Betti number allow one to compute the arithmetic and topological genera and the Hodge numbers. Thus, it follows that all these invariants also are polynomial periodic under the assumptions of Theorem 1.7.

To get a more complete picture, it would be nice to know whether the growth of the Picard number for $\widetilde{X}_n$ has a similar description. So far we only get a polynomial periodic lower bound given by the rank of the intersection matrix of $\widetilde{\rho}_n^{-1}(\mathcal{C})$.

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2. Basic results on unbranched coverings and polynomial periodicity.

In this section we sketch Sarnak's proof of Theorem 1.3.

Fox Calculus provides a way of transforming a hard to handle geometric problem to a formal algebraic one. Given a presentation of the fundamental group of a space Y^u, one can construct a more manageable matrix from which one can compute the first Betti numbers of finite abelian unbranched coverings.

Assuming Proposition 1.9, we review his proof of polynomial periodicity for the ranks of such matrices. One of our aims is to set up notation that will be useful for the new results in section 3.

We begin with a brief review of Fox calculus.

Let Y^u be homotopy equivalent to a finite CW-complex. Let $\phi : \pi_1(Y^u) \to G$ be any surjective group homomorphism. Let $\rho^u_\phi : X^u_\phi \to Y^u$ be the corresponding regular unbranched covering. The group G acts on the homology groups of X^u_ϕ and there is a chain complex for X^u_ϕ:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_2(X^u_\phi) & \xrightarrow{\partial} & C_1(X^u_\phi) & \xrightarrow{\partial} & C_0(X^u_\phi) \\
& \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\cdots \longrightarrow & \mathbb{C}[G]^s & \xrightarrow{\mathcal{R}_\phi} & \mathbb{C}[G]^r & \xrightarrow{\mathcal{A}_\phi} & \mathbb{C}[G].
\end{array}$$

Now, let $G = \mathrm{H}_1(Y^u;\mathbb{Z})$ and let $\phi : \pi_1(Y^u) \to \mathrm{H}_1(Y^u;\mathbb{Z})$ be the natural map. There is a matrix $\mathcal{M}$ representing the map $\mathcal{R}_\phi$, which can be obtained from any finite presentation of $\pi_1(Y^u)$ using Fox Calculus. This matrix, called the *Alexander matrix*, has the following universal property. For any abelian group H and surjection $\phi_H : \pi_1(Y^u) \to H$ there is a factorization $\phi_H = \psi_H \circ \phi$ where $\psi_H : G \to H$. Let $\mathcal{M}_H$ be $\mathcal{M}$ with all entries replaced by their images under the map $\mathbb{Z}[G] \to \mathbb{Z}[H]$ induced by ψ_H. Then $\mathcal{M}_H$ represents the map $\mathcal{R}_{\phi_H}$.

Let $G(n)$ be the group $\mathrm{H}_1(Y^u;\mathbb{Z}/n\mathbb{Z})$. (So $G(n) = G \otimes \mathbb{Z}/n\mathbb{Z}$.) For each n, let X^u_n be the unbranched covering associated to the natural map

$$\pi_1(Y^u) \to \mathrm{H}_1(Y^u;\mathbb{Z}/n\mathbb{Z}).$$

Let $\mathcal{M}_n$ be the matrix representing $\mathcal{R}_{G(n)}$. Since $\mathrm{Rank}_\mathbb{C}(\mathcal{A}_{G(n)})$ must equal $|G(n)|-1$, we have

$$\mathrm{Null}_\mathbb{C}(\mathcal{A}_{G(n)}) = (r-1)|G(n)| + 1.$$

Therefore, we have

$$\begin{aligned}
b_1(X^u_n) &= (r-1)|G(n)| + 1 - \mathrm{Rank}_\mathbb{C}(\mathcal{M}_n) \\
&= (r-1)|G(n)| + 1 - (s|G(n)| - \mathrm{Null}_\mathbb{C}(\mathcal{M}_n)) \\
&= (r-s-1)|G(n)| + 1 + \mathrm{Null}_\mathbb{C}(\mathcal{M}_n).
\end{aligned}$$

The problem of finding the first Betti number of the unbranched coverings X^u_n as a function of n comes down to a problem in a purely formal combinatorial setting which we will now describe.

Let G be any finitely generated abelian group. For each positive integer n, let $G(n) = G \otimes \mathbb{Z}/n\mathbb{Z}$ and let

$$\psi_n : G \to G(n)$$

be reduction modulo n. Let $A = \mathbb{C}[G]$ and $A(n) = \mathbb{C}[G(n)]$ and let

$$\Psi_n : A \to A(n)$$

be the $\mathbb{C}$-algebra homomorphism induced by ψ_n.

Definition 2.1. A sequence of elements $p_n \in A(n)$ is *stable* if there is a $p \in A$ so that $\Psi_n(p) = p_n$ for all n. A sequence of $r \times s$ matrices M_n with entries in $A(n)$ is *stable* if all its entries are stable.

An $r \times s$ matrix M_n with entries in $A(n)$ defines a map

$$M_n : A(n)^s \to A(n)^r$$

between $\mathbb{C}$-vector spaces. Thus, it has a well-defined $\mathbb{C}$-nullity, the dimension of its kernel, and $\mathbb{C}$-rank, the dimension of its image. We will denote these by $\mathrm{Null}_\mathbb{C}(M_n)$ and $\mathrm{Rank}_\mathbb{C}(M_n)$, respectively.

Proposition 2.2. *If M_n is a stable sequence of $r \times s$ matrices with entries in $A(n)$, then the sequence $Null_{\mathbb{C}}(M_n)$ is polynomial periodic.*

Returning to the geometric problem, let Y^u be any topological space which is homotopy equivalent to a finite CW-complex. Tensor the underlying spaces of the matrices $\mathcal{M}$ and $\mathcal{M}_n$ with $\mathbb{C}$. By the universality of the Alexander matrix, explained above, $\mathcal{M}_n$ is a stable sequence of matrices. Thus, Proposition 2.2 implies Theorem 1.3.

Remark 2.3. As Sarnak points out ([S], Corollary 1.4), Proposition 2.2 can be applied directly to show that the first Betti number of certain sequences of *branched* coverings is polynomial periodic, as long as one can find stable sequences of matrices M_n presenting the first homology group of the branched coverings. This was done by Murasugi and Mayberry [M-M] in the case of branched coverings over the three-sphere branched along a link. They showed how to augment the Alexander matrices to give presentation matrices for the branched coverings. These augmented matrices are stable, so Proposition 2.2 implies that the first Betti numbers are polynomial periodic. To the author's knowledge, such augmented matrices for presenting homology groups for general branched coverings have not been found.

For any ideal $\mathcal{J}(n) \subset A(n)$, since $\mathcal{J}(n)$ is closed under multiplication by elements of $A(n)$, M_n induces a linear map on the direct sum $\mathcal{J}(n)^s$:

$$M_n|_{\mathcal{J}(n)^s} : \mathcal{J}(n)^s \to \mathcal{J}(n)^r.$$

We will denote the $\mathbb{C}$-nullity and $\mathbb{C}$-rank of this map by $\mathrm{Null}_{\mathbb{C}}(M_n, \mathcal{J}(n))$ and $\mathrm{Rank}_{\mathbb{C}}(M_n, \mathcal{J}(n))$, respectively. Proposition 2.2 can be extended as follows.

Proposition 2.4. *If M_n is a stable sequence of $r \times s$ matrices with entries in $A(n)$, $\mathcal{J}(n) \subset A(n)$ is a sequence of ideals, and there exists an ideal $\mathcal{J} \subset A$ so that $\Psi_n(\mathcal{J}) = \mathcal{J}(n)$, then $Null_{\mathbb{C}}(M_n, \mathcal{J}(n))$ is polynomial periodic.*

The proofs of Proposition 2.2 and Proposition 2.4 reduce to a study of zero sets of certain ideals in polynomial rings.

First we fix some new notation. Let k be any positive integer, and let R be the polynomial ring $\mathbb{C}[t_1, \ldots, t_k]$. Let $\Omega_n \subset R$ be the ideal generated by

$$t_1^n - 1, \ldots, t_k^n - 1.$$

Let $R(n)$ be the quotient algebra

$$R(n) = R/\Omega_n.$$

(We will later write $A(n)$ in terms of $R(n)$.) For any ideal $\mathcal{J} \subset R$, let $V(\mathcal{J}) \subset \mathbb{C}^k$ be the set of zeros of the polynomials in $\mathcal{J}$. Let $\mathcal{J}(n) \subset R$ be the ideal generated by $\mathcal{J}$ and Ω_n, and let $V_n(\mathcal{J}) = V(\mathcal{J}(n))$. For a finite set S, let $|S|$ denote the number of points in S and let 1_S be the indicator function for S.

The following is a restatement of Proposition 1.9.

Proposition 2.5. *For any ideal $\mathcal{J} \subset R$, the sequence $|V_n(\mathcal{J})|$ is polynomial periodic.*

Now fix n and let T be any $r \times s$ matrix with coefficients in $R(n)$. For $m = 1, \ldots, s$, let F_m be the ideal of $m \times m$ minors of T, called the mth *fitting ideal* for T. Let $F'_m \subset R$ be any ideal which maps onto F_m under the quotient map $R \to R(n)$. Note that $|V_n(F'_m)|$ doesn't depend on this choice.

Lemma 2.6. *The $\mathbb{C}$-nullity of T is given by*

$$Null_{\mathbb{C}}(T) = \sum_{m=1}^{s} |V_n(F'_m)|.$$

Proof. The set of monomials in $t_1, \ldots, t_k$ of the form

$$t_1^{q_1} \ldots t_k^{q_k} \qquad \text{where } 0 \le q_i \le n-1, \text{ for } i = 1, \ldots, k,$$

forms a basis for $R(n)$ as a vector space over $\mathbb{C}$.

Consider the action of $(\mathbb{Z}/n\mathbb{Z})^k$ on $R(n)$, where $(a_1, \ldots, a_k) \in (\mathbb{Z}/n\mathbb{Z})^k$ acts by

$$t_i \mapsto t_i^{a_i}.$$

We will construct a new basis for R which diagonalizes this action. Let

$$q_{\omega_0}(t) = 1 + \omega_0 t + {\omega_0}^2 t^2 + \cdots + {\omega_0}^{n-1} t^{n-1}$$

for ω_0 any nth root of unity. Let $W_n = V(\Omega_n)$. Then the set of

$$p_\omega = \prod_{i=1}^{k} q_{\omega_i}(t_i),$$

where ω ranges over all elements of W_n, spans $R(n)$. Furthermore, for each $\omega \in W_n$ and $j = 1, \ldots, k$,

$$\begin{aligned} t_j p_\omega &= t_j \prod_{i=1}^{k} q_{\omega_i}(t_i) \\ &= \omega_j^{-1} \prod_{i=1}^{k} q_{\omega_i}(t_i) \\ &= \omega_j^{-1} p_\omega. \end{aligned}$$

Thus, for any $f \in R(n)$, $f p_\omega = f(\omega^{-1}) p_\omega$, where $\omega^{-1} = (\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_k^{-1})$.

Let $R_\omega(n) \subset R(n)$ be the subspace spanned by $\{p_\omega\}$. Then $R_\omega(n)$ is an ideal in $R(n)$, since it is closed under multiplication by elements of $R(n)$, so T restricts to a linear map on the direct sum $R_\omega(n)^s$:

$$T|_{R_\omega(n)^s} : R_\omega(n)^s \to R_\omega(n)^r.$$

Since $R(n)^s$ decomposes as

$$R(n)^s = \bigoplus_{\omega \in W_n} R_\omega(n)^s,$$

we have

$$\mathrm{Null}_{\mathbb{C}}(T) = \sum_{\omega \in W_n} \mathrm{Null}_{\mathbb{C}}(T, R_\omega(n)).$$

Now the restriction of T to $R_\omega(n)^s$, acts in the same way as the matrix $T(\omega^{-1})$ obtained from T by replacing all entries f of T by their values $f(\omega^{-1})$. Thus, $\mathrm{Null}_{\mathbb{C}}(T, R_\omega(n))$ is the nullity of the matrix $T(\omega^{-1})$ and

$$\begin{aligned}\mathrm{Null}_{\mathbb{C}}(T) &= \sum_{\omega \in W_n} \sum_{m=1}^{s} 1_V(F_m)(\omega^{-1}) \\ &= \sum_{m=1}^{s} |V_n(F'_m)|.\end{aligned}$$

□

Let $\mathcal{J} \subset R(n)$ be an ideal. For any transformation

$$T : R(n)^s \to R(n)^r,$$

let $T|_{\mathcal{J}^s}$ be the restriction to the direct sum $\mathcal{J}^s$ and let $\overline{T} : (R(n)/\mathcal{J})^s \to (R(n)/\mathcal{J})^r$ be the induced transformation on quotient algebras. Let $\mathcal{J}' \subset R$ be any ideal so that $\mathcal{J}$ is the image of $\mathcal{J}'$ under the quotient map $R \to R(n)$.

Lemma 2.7. *The $\mathbb{C}$-nullity of the restriction $T|_{\mathcal{J}^s}$ is given by*

$$\mathit{Null}_{\mathbb{C}}(T, \mathcal{J}) = \sum_{m=1}^{s} (|V_n(F'_m)| - |V_n(F'_m + \mathcal{J}')|),$$

and the $\mathbb{C}$-nullity of the induced transformation $\overline{T}$ is given by

$$\mathit{Null}_{\mathbb{C}}(\overline{T}) = \sum_{m=1}^{s} |V_n(F'_m + \mathcal{J}')|.$$

Proof. Let p_ω be as constructed in the proof of Lemma 2.6. The set of p_ω where ω lies in $W_n \setminus V_n(\mathcal{J}')$ forms a basis for $\mathcal{J}$. Thus, to find $\mathrm{Null}_{\mathbb{C}}(T, \mathcal{J})$ we need to sum over ω in $W_n \setminus V_n(\mathcal{J}')$. That is,

$$\begin{aligned}\mathrm{Null}_{\mathbb{C}}(T, \mathcal{J}) &= \sum_{\omega \in W_n \setminus V_n(\mathcal{J}')} \sum_{m=1}^{s} 1_V(F_m) \\ &= \sum_{m=1}^{s} (|V_n(F'_m)| - |V_n(F'_m) \cap V_n(\mathcal{J}')|).\end{aligned}$$

Since $V_n(F'_m) \cap V_n(\mathcal{J}') = V_n(F'_m + \mathcal{J}')$ this completes the first part of the claim.

The quotient space $R(n)/\mathcal{J}$ has as basis the image of the elements p_ω under the quotient map, where ω ranges in $V_n(\mathcal{J}')$. It follows that the nullity is found by taking the sum where ω ranges over elements of $V_n(\mathcal{J}')$. The rest of the proof proceeds as in the previous paragraph. $\square$

Proof of Proposition 2.2. Choose k generators for G. Let $R = \mathbb{C}[t_1, \ldots, t_k]$ as before. Then there is a natural $\mathbb{C}$-algebra homomorphism

$$\alpha : R \to A$$

which sends the elements $t_1, \ldots, t_k$ to these generators. Let $\alpha_n = \Psi_n \circ \alpha$.

Since all elements of $G(n)$ have finite order they can be represented as the image of a product of positive powers of generators of G. Thus, $\alpha_n : R \to A(n)$ is onto and $A(n)$ is isomorphic to the quotient ring

$$Q(n) = R/\mathcal{J}'(n)$$

where $\mathcal{J}'(n) \subset R$ is the kernel of the map α_n. Let $\mathcal{J}' \subset R$ be the kernel of the map α. Then $\mathcal{J}'(n) = \mathcal{J}' + \Omega_n$ for all n.

Let M'_n be the map on $Q(n)^s$ corresponding to M_n:

$$M'_n : Q(n)^s \to Q(n)^r.$$

The sequence of these maps are also stable and have the same rank as M_n. By the second part of Lemma 2.7, the sequence of $\mathbb{C}$-nullities is polynomial periodic. $\square$

Proof of Proposition 2.4. Let $\mathcal{J}(n) \subset A(n)$ be ideals and let $\mathcal{J} \subset A$ be an ideal so that $\Psi_n(\mathcal{J}) = \mathcal{J}(n)$. Let $B(n)$ be the quotient of $A(n)$ by the ideal $\mathcal{J}(n)$. Let $\overline{M}_n$ be the transformation

$$\overline{M}_n : B(n)^s \to B(n)^r$$

induced by M_n. Then $\mathrm{Null}_{\mathbb{C}}(M_n, \mathcal{J}(n))$ can be written as

$$\mathrm{Null}_{\mathbb{C}}(M_n, \mathcal{J}(n)) = \mathrm{Null}_{\mathbb{C}}(M_n) - \mathrm{Null}_{\mathbb{C}}(\overline{M}_n)$$

which by Proposition 2.2 is polynomial periodic. $\square$

3. Polynomial periodicity of intersection matrices.

This section is concerned with bilinear forms on certain group algebras.

We begin with some basic notation for general group algebras. Let S be any group and let $\mathbb{C}[S]$ be the corresponding group algebra. The elements of S form a natural basis for $\mathbb{C}[S]$. There is also a natural inner product on $\mathbb{C}[S]$ given by

$$\langle s_1, s_2 \rangle = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise.} \end{cases}$$

Our main concern is with abelian groups. Let G be a finitely generated abelian group and let $H_1, \ldots, H_k$ be subgroups of G. Let $C(i) = G/H_i$. For $g \in G$ let $g(i)$ be the corresponding coset in $C(i)$.

For each positive integer n, let $G(n) = G \otimes \mathbb{Z}/n\mathbb{Z}$ and let $\psi_n : G \to G(n)$ be the quotient map. Let $H_1(n), \ldots, H_k(n)$ be the images of $H_1, \ldots, H_k$ under the map ψ_n.

Let $C(i,n)$ denote the set of cosets $G(n)/H_i(n)$. For any element $g \in G(n)$, let $g(i,n)$ be the corresponding coset in $C(i,n)$. We will simultaneously consider $g(i,n) \in C(i,n)$ as a subset of $G(n)$ and also as an element of the coset group with the multiplicative law

$$g(i,n)h(i,n) = gh(i,n).$$

Let $\mathbb{C}[C(i,n)]$ be the corresponding group algebra.

Let $V(n)$ be the vector space

$$\bigoplus_{i=1}^{k} \mathbb{C}[C(i,n)].$$

This has a canonical basis given by the union

$$\bigcup_{i=1}^{k} C(i,n)$$

and elements of $V(n)$ can be written uniquely as

$$\sum_{i=1}^{k} \sum_{g(i,n) \in C(i,n)} \alpha_{g(i,n)} g(i,n),$$

where $\alpha_{g(i,n)} \in \mathbb{C}$.

We extend the inner product $\langle , \rangle$ to all of $V(n)$ so that $\langle g(i,n), h(j,n) \rangle = 0$ whenever $i \neq j$.

There are two other kinds of bilinear forms on $V(n)$ which we will be concerned with: the *plain* intersection form and the *twisted* intersection form. The plain intersection form

$$\mathrm{I} : V(n) \times V(n) \to \mathbb{C}$$

is defined by

$$\mathrm{I}(g(i,n), h(j,n)) = |g(i,n) \cap h(j,n)|$$

or, in other words, the number of elements in the intersection of the $g(i,n)$ and $h(j,n)$ considered as sets.

A twisted intersection form will depend on some parameters. For each $i,j = 1, \ldots, k$, let $r(i,j)$ be positive integers, and, for $p = 1, \ldots, r(i,j)$, let $c(i,j,p)$ be complex numbers and let $tw(i,j,p,n)$ be elements of $G(n)$. The latter will be called the *twisting* elements. The *twisted* intersection form associated to these parameters,

$$\mathrm{T} : V(n) \times V(n) \to \mathbb{C},$$

is defined by

$$\mathrm{T}(g(i,n), h(j,n)) = \sum_{p=1}^{r(i,j)} \mathrm{I}(g(i,n), tw(i,j,p,n)^{-1} h(j,n)) c(i,j,p).$$

Note that T need not be symmetric, since we do not assume that $tw(i,j,p,n) = tw(i,j,p,n)^{-1}$.

The main goal of this section is to prove the following Proposition.

Proposition 3.1. *If the twisting elements $tw(i,j,p,n)$ are stable as elements of $\mathbb{C}[G(n)]$, then the nullity of the associated twisted intersection form is polynomial periodic.*

Note that, since the plain intersection form I is a special case of the twisted intersection form, Proposition 3.1 applies to these as well.

For any subspace $W \subset V(n)$, define $\mathrm{Null}(\mathrm{T}, W)$ to be the nullity of T restricted to W.

Here is an outline of our proof. We define an index set $\mathcal{T}$ and for each $\tau \in \mathcal{T}$ define groups R_τ. We let $R_\tau(n) = R_\tau \otimes \mathbb{Z}/n\mathbb{Z}$ and let $\Phi_{\tau,n} : R_\tau \to R_\tau(n)$ be the quotient maps. We define subspaces $\mathcal{U}_\tau(n)$ of $V(n)$ which naturally imbed into a group algebra of the form $\mathbb{C}[R_\tau(n)]^s$. The images of $\mathcal{U}_\tau(n)$ are of the form $\mathcal{J}_\tau(n)^s$, where $\mathcal{J}_\tau(n) \subset \mathbb{C}[R_\tau(n)]$ is an ideal and there is a $\mathcal{J}_\tau \subset \mathbb{C}[R_\tau]$ so that $\Phi_{\tau,n}(\mathcal{J}_\tau) = \mathcal{J}_\tau(n)$. We show that, for all $v \in \mathcal{U}_\tau(n)$ and $w \in \mathcal{U}_{\tau'}(n)$, if $\tau \neq \tau'$ then $\mathrm{T}(v,w) = 0$. We also show that $V(n)$ decomposes into a direct sum of the $\mathcal{U}_\tau(n)$. Thus, we have

$$\mathrm{Null}(\mathrm{T}, V(n)) = \sum_{\tau \in \mathcal{T}} \mathrm{Null}(\mathrm{T}, \mathcal{U}_\tau(n)).$$

In the final step we show that the intersection form T on $\mathcal{U}_\tau(n)$ corresponds to the restriction of one on $\mathbb{C}[R_\tau(n)]^s$ defined by a stable sequence of $s \times s$ matrices. By Proposition 2.4, the nullities of the restrictions to $\mathcal{J}_\tau(n)^s$ are polynomial periodic.

Our index set will be the set $\mathcal{T}$ of subsets of $\{1, \ldots, k\}$. Let $e_i \in \mathcal{T}$ be the element $e_i = \{i\}$. Define a partial ordering on $\mathcal{T}$ by $\tau < \tau'$ if $\tau \subset \tau'$. Define addition on $\mathcal{T}$ by $\tau + \tau' = \tau \cup \tau'$. Let $|\tau|$ be the number of elements in τ.

For each $\tau \in \mathcal{T}$, let

$$S_\tau = \sum_{i \in \tau} H_i,$$

and let $R_\tau = G/S_\tau$. For $\tau' > \tau$, let

$$\eta_{\tau,\tau'} : \mathbb{C}[R_\tau] \to \mathbb{C}[R_{\tau'}]$$

be the quotient map. Let $\mathcal{J}_\tau \subset \mathbb{C}[R_\tau]$ be the ideal given by

$$\mathcal{J}_\tau = \bigcap_{\tau' > \tau} \ker(\eta_{\tau,\tau'}).$$

For each integer $n > 0$, let $R_\tau(n) = R_\tau \otimes \mathbb{Z}/n\mathbb{Z}$ and let $\Phi_{\tau,n} : \mathbb{C}[R_\tau] \to \mathbb{C}[R_\tau(n)]$ be the quotient map. Let $\mathcal{J}_\tau(n) = \Phi_{\tau,n}(\mathcal{J})$. Let $\overline{R_\tau(n)}$ be a lift of the quotient map $G(n) \to R_\tau(n)$. For each $i \in \tau$, let $S_\tau(i,n)$ be the image of S_τ in $C(i,n)$.

Any element of $\mathbb{C}[C(i,n)]$ can be written uniquely as

$$\sum_{g \in \overline{R_\tau(n)}} \; \sum_{h(i,n) \in S_\tau(i,n)} \alpha_{gh(i,n)} gh(i,n),$$

where $\alpha_{gh(i,n)} gh(i,n) \in \mathbb{C}$. Thus, there is a natural map

$$\sigma(\tau,i,n) : \mathbb{C}[C(i,n)] \to \mathbb{C}[R_\tau(n)]$$

given by

$$\sum_{g(n)\in\overline{R_\tau(n)}}\ \sum_{h(i,n)\in S_\tau(i,n)} \alpha_{gh(i,n)} gh(i,n) \mapsto \sum_{g(n)\in R_\tau(n)} \overline{\alpha_{g(n)}} g(n),$$

where

$$\overline{\alpha_{g(n)}} = \sum_{h(i,n)\in S_\tau(i,n)} \alpha_{gh(i,n)}.$$

The choice of lifting $\overline{R_\tau(n)}$ doesn't affect this map since as long as $i \in \tau$ any $g(n) \in \overline{R_\tau(n)}$ determines a unique element $g(i,n) \in C(i,n)$.

If $i \in \tau$, define $\mathcal{K}_\tau(i,n)$ to be the kernel of $\sigma(\tau,i,n)$, i.e.,

$$\mathcal{K}_\tau(i,n) = \left\{ \sum_{g(i,n)\in C(i,n)} \alpha_{g(i,n)} g(i,n) : \sum_{h(i,n)\in S_\tau(i,n)} \alpha_{gh(i,n)} = 0 \quad \forall g \in G(n) \right\},$$

and define

$$\mathcal{L}_\tau(i,n) = \left\{ \sum_{g(i,n)\in C(i,n)} \alpha_{g(i,n)} g(i,n) : \alpha_{hg(i,n)} = \alpha_{g(i,n)} \quad \forall h(i,n) \in S_\tau(i,n) \right\}.$$

If $i \notin \tau$, define $\mathcal{K}_\tau(i,n) = \mathcal{L}_\tau(i,n) = 0$. Then, for $i \in \tau$, $\sigma_\tau(i,n)$ restricts to an isomorphism on $\mathcal{L}_\tau(i,n)$.

Let

$$\mathcal{U}_\tau(i,n) = (\bigcap_{\tau'>\tau} \mathcal{K}_{\tau'}(i,n)) \cap \mathcal{L}_\tau(i,n)$$

Then $\sigma_\tau(i,n)(\mathcal{U}_\tau(i,n)) = \mathcal{J}_\tau(n)$ for all i such that $i \in \tau$.

Thus, setting $s = |\tau|$ and

$$\mathcal{U}_\tau(n) = \bigoplus_{i\in\tau} \mathcal{U}_\tau(i,n) \subset V(n)$$

we have maps

$$\sigma_\tau(n) : \mathcal{U}_\tau(n) \xrightarrow{\cong} \mathcal{J}_\tau(n)^s \subset \mathbb{C}[R_\tau(n)]^s.$$

We claim that

(1) If $v \in \mathcal{U}_\tau(n)$, $w \in \mathcal{U}_{\tau'}(n)$ and $\tau \neq \tau'$ then $T(v,w) = 0$. (I.e., the $\mathcal{U}_\tau(n)$ are *orthogonal* with respect to the bilinear form T.)

(2)
$$V(n) = \bigoplus_{\tau\in\mathcal{T}} \mathcal{U}_\tau(n).$$

We will show that in fact the $\mathcal{U}_\tau(n)$ are orthogonal with respect to any bilinear form

$$\mathrm{B} : V(n) \times V(n) \to \mathbb{C}$$

satisfying

$$\mathrm{B}(sg(i,n), h(j,n)) = \mathrm{B}(g(i,n), s^{-1}h(j,n)),$$

for $s \in G(n)$. Note that $\langle , \rangle$, I and T all have this property.

For any $g_1 \in G(n)$, define $[g_1](\tau, i, n)$ to be the element of $\mathbb{C}[C(i,n)]$ given by

$$[g_1](\tau, i, n) = \sum_{g_2 \in S_\tau(i,n)} g_1 g_2(i, n).$$

We can write any element of $\mathcal{L}_\tau(i, n)$ uniquely as

$$\sum_{g_1 \in \overline{R_\tau(n)}} \alpha_{g_1} [g_1](\tau, i, n).$$

Lemma 3.2. *For any $\tau \in \mathcal{T}$, with $i \in \tau$, $s \in S_\tau(n)$, $g, h \in G(n)$ and for any $j = 1, \ldots, k$, we have*

$$\mathrm{B}([g](\tau, i, n), sh(j, n)) = \mathrm{B}([g](\tau, i, n), h(j, n)).$$

Proof. We have

$$\begin{aligned} [g](\tau, i, n) &= \sum_{h \in S_\tau(i,n)} gh(i, n) \\ &= s^{-1} \sum_{h' \in S_\tau(i,n)} gh'(i, n) \\ &= s^{-1} [g](\tau, i, n), \end{aligned}$$

where the sum in the second row is taken over $h' = s'h$, where s' is the image of s in $S_\tau(i, n)$. $\square$

It is easy to see that $\mathbb{C}[C(i,n)]$ breaks up into the direct sum

$$\mathbb{C}[C(i, n)] = \mathcal{K}_\tau(i, n) \oplus \mathcal{L}_\tau(i, n).$$

Furthermore, $\mathcal{K}_\tau(i, n)$ and $\mathcal{L}_\tau(i, n)$ are orthogonal with respect to the natural inner product $\langle , \rangle$ on $\mathbb{C}[C(i, n)]$. We will now show that $\mathcal{K}_\tau(i, n)$ and $\mathcal{L}_\tau(i, n)$ are also orthogonal with respect to the form B.

Lemma 3.3. *Take any τ with $i, j \in \tau$. Then the spaces $\mathcal{K}_\tau(i, n)$ and $\mathcal{L}_\tau(j, n)$ are orthogonal in $V(n)$ with respect to* B.

Proof. For all $h_1, g_1 \in \overline{R_\tau(n)}$, Lemma 3.2 implies

$$\begin{aligned} &\sum_{h_2 \in S_\tau(i,n)} \alpha_{h_1 h_2(i,n)} \mathrm{B}(h_1 h_2(i, n), [g_1](\tau, j, n)) \\ &= \left(\sum_{h_2 \in S_\tau(i,n)} \alpha_{h_1 h_2(i,n)} \right) \mathrm{B}(h_1(i, n), [g_1](\tau, j, n)) \end{aligned}$$

Take any $v \in \mathcal{K}_\tau(i, n)$, $w \in \mathcal{L}_\tau(i, n)$. Then w is a linear combination of $[g_1](\tau, i, n)$, where g_1 ranges in $\overline{R_\tau(n)}$ and v is a linear combination of

$$\sum_{h_2 \in S_\tau(i,n)} \alpha_{h_1 h_2(i,n)} h_1 h_2(i, n),$$

where h_1 ranges in $\overline{R_\tau(n)}$ and

$$\sum_{h_2 \in S_\tau(i,n)} \alpha_{h_1 h_2(i,n)} = 0.$$

Thus, we have $\mathrm{B}(v, w) = 0$. $\square$

Lemma 3.4. *For any τ and $i \in \tau$*

$$\mathcal{U}_\tau(i,n) = \left(\bigcap_{j \notin \tau} \mathcal{K}_{e_i+e_j}(i,n) \right) \cap \mathcal{L}_\tau(i,n).$$

Proof. It is enough to show that if $j \notin \tau$ then

$$\mathcal{K}_{\tau+e_j}(i,n) \cap \mathcal{L}_\tau(i,n) = \mathcal{K}_{e_i+e_j}(i,n) \cap \mathcal{L}_\tau(i,n).$$

Since $\mathcal{K}_{e_i+e_j}(i,n) \subset \mathcal{K}_{\tau+e_j}(i,n)$, we have one obvious inclusion. Now take any $v \in \mathcal{K}_{\tau+e_j}(i,n) \cap \mathcal{L}_\tau(i,n)$. We can write v as

$$v = \sum_{g(i,n) \in C(i,n)} \alpha_{g(i,n)} g(i,n).$$

Let $h_1, \ldots, h_r \in H_j(n)$ be coset representatives for

$$(H_i(n) + H_j(n))/H_i(n)$$

and let $g_1, \ldots, g_t \in \sum_{i \in \tau} H_i(n)$ be coset representatives for

$$(\sum_{i \in \tau} H_i(n))/(H_i(n) + H_j(n)).$$

Then, since $v \in \mathcal{K}_{\tau+e_j}(i,n)$,

$$\sum_{\ell=1}^{r} \sum_{m=1}^{t} \alpha_{h_\ell g_m(i,n)} = 0$$

and, since $v \in \mathcal{L}_\tau(i,n)$,

$$\alpha_{h_\ell g_m(i,n)} = \alpha_{h_\ell g_1(i,n)}$$

for all ℓ and m. Therefore,

$$\sum_{\ell=1}^{r} \alpha_{h_\ell g_m(i,n)} = 0,$$

for all $m = 1, \ldots, t$. So $v \in K_{e_i+e_j}(i,n)$. $\square$

Lemma 3.5. *If $\tau \neq \tau'$, $v \in \mathcal{U}_\tau(i,n)$ and $w \in \mathcal{U}_{\tau'}(j,n)$, then* $\mathrm{B}(v,w) = 0$.

Proof. Let d be such that $d \notin \tau$ and $d \in \tau'$. We can assume $i \in \tau$ and $j \in \tau'$, since otherwise $\mathcal{U}_\tau(i,n)$ or $\mathcal{U}_{\tau'}(j,n)$ would be trivial.

We will do the case where $d = j$ and the case where d is not equal to i or j separately.

Suppose $d = j$, in other words, $j \notin \tau$. Set $\tau_0 = e_i + e_j$. Then by Lemma 3.4 we have

$$v \in K_{\tau_0}(i,n).$$

Thus, we can write v as

$$v = \sum_{g_1 \in \overline{R_{\tau_0}(n)}} \sum_{g_2 \in S_{\tau_0}(i,n)} \alpha_{g_1 g_2(i,n)} g_1 g_2(i,n),$$

where, for all $g_1 \in \overline{R_{\tau_0}(n)}$,

$$\sum_{g_2 \in S_{\tau_0}(i,n)} \alpha_{g_1 g_2(i,n)} = 0.$$

For all $g_2 \in S_{\tau_0}(i,n)$, $g_1 \in \overline{R_{\tau_0}(n)}$ and $h \in C(j,n)$

$$\mathrm{B}(g_1 g_2(i,n), h(j,n)) = \mathrm{B}(g_1(i,n), h(j,n)).$$

Thus, for any $h(j,n) \in C(j,n)$,

$$\begin{aligned}
\mathrm{B}(v, h(j,n)) &= \sum_{g_1 \in \overline{R_{\tau_0}(n)}} \sum_{g_2 \in S_{\tau_0}(i,n)} \alpha_{g_1 g_2(i,n)} \, \mathrm{B}(g_1 g_2(i,n), h(j,n)) \\
&= \sum_{g_1 \in \overline{R_{\tau_0}(n)}} \left(\sum_{g_2 \in S_{\tau_0}(i,n)} \alpha_{g_1 g_2(i,n)} \right) \mathrm{B}(g_1(i,n), h(j,n)) \\
&= 0.
\end{aligned}$$

Therefore, since w is a linear combination of such $h(j,n)$, we have $\mathrm{B}(v,w) = 0$.

Now suppose $i,j \in \tau$ and $i,j \in \tau'$, $d \notin \tau$ and $d \in \tau'$. Then $v \in \mathcal{K}_{e_j+e_d}(i,n) \subset \mathcal{K}_\tau(i,n)$ and $w \in \mathcal{L}_\tau(j,n)$. Thus, by Lemma 3.3, we have $\mathrm{B}(v,w) = 0$. $\square$

We have thus proven (1). To prove (2), we will show that $\mathbb{C}[C(i,n)]$ decomposes into the direct sum of the $\mathcal{U}_\tau(i,n)$, as τ ranges in $\mathcal{T}$. Since, by Lemma 3.5, the $\mathcal{U}_\tau(i,n)$ are orthogonal with respect to the usual inner product $\langle , \rangle$ on $\mathbb{C}[C(i,n)]$, it suffices to show the following.

Lemma 3.6. *The space $\mathbb{C}[C(i,n)]$ is spanned by $\mathcal{U}_\tau(i,n)$ as τ ranges among elements of $\mathcal{T}$ containing i.*

Proof. We will show inductively that

$$\mathbb{C}[C(i,n)] = \bigoplus_{\ell=1}^{L} \bigoplus_{|\tau|=\ell, i\in\tau} \mathcal{U}_\tau(i,n) \oplus \Big(\bigvee_{|\tau'|=L+1} \mathcal{L}_{\tau'}(i,n) \Big)$$

for all $L = 1, \ldots, k-1$, where $\bigvee_\ell S_\ell$ denotes the *join* or span of the subspaces S_ℓ. Since $\mathcal{L}_\tau(i,n) = \mathcal{U}_\tau(i,n)$ for $\tau = \{1, \ldots, k\}$, this will conclude the proof.

The proof is by induction on L. If $|\tau| = 1$ then $\mathcal{U}_\tau(i,n)$ is either trivial (if $i \notin \tau$) or $\tau = e_i$. In the latter case

$$\mathcal{L}_\tau(i,n) = \mathbb{C}[C(i,n)]$$

and hence

$$\mathcal{U}_\tau(i,n) = \bigcap_{j \neq i} \mathcal{K}_{e_i+e_j}(i,n).$$

Now the orthogonal complement of $\mathcal{K}_{e_i+e_j}(i,n)$ in $\mathbb{C}[C(i,n)]$ is $\mathcal{L}_{e_i+e_j}(i,n)$. Therefore, the orthogonal complement of $\mathcal{U}_\tau(i,n)$ is

$$\bigvee_{j \neq i} \mathcal{L}_{e_i+e_j}(i,n) = \bigvee_{|\tau|=2} \mathcal{L}_\tau(i,n).$$

Thus,

$$\mathbb{C}[C(i,n)] = \mathcal{U}_\tau(i,n) \oplus (\bigvee_{|\tau|=2} \mathcal{L}_\tau(i,n)).$$

Now assume the hypothesis for $L-1$. Then we have

$$\mathbb{C}[C(i,n)] = \bigoplus_{\ell=1}^{L-1} \bigoplus_{|\tau|=\ell} \mathcal{U}_\tau(i,n) \oplus (\bigvee_{|\tau|=L} \mathcal{L}_\tau(i,n)).$$

For each τ and each $\tau' > \tau$, the orthogonal complement of $\mathcal{L}_{\tau'}(i,n)$ in $\mathcal{L}_\tau(i,n)$ is $K_{\tau'}(i,n) \cap \mathcal{L}_\tau(i,n)$. Therefore,

$$\begin{aligned}\mathcal{L}_\tau(i,n) &= \left(\bigcap_{\tau<\tau'} K_{\tau'}(i,n) \cap \mathcal{L}_\tau(i,n)\right) \oplus \left(\bigvee_{\tau<\tau'} \mathcal{L}_{\tau'}(i,n)\right) \\ &= \mathcal{U}_\tau(i,n) \oplus \left(\bigvee_{\tau<\tau'} \mathcal{L}_{\tau'}(i,n)\right).\end{aligned}$$

Since $\mathcal{L}_{\tau''}(i,n) \subset \mathcal{L}_{\tau'}(i,n)$ for $\tau'' > \tau'$, we have

$$\bigvee_{|\tau|=L} \mathcal{L}_\tau(i,n) = \bigoplus_{|\tau|=L} \mathcal{U}_\tau(i,n) \oplus \left(\bigvee_{|\tau'|=L+1} \mathcal{L}_{\tau'}(i,n)\right).$$

□

Now we will restrict to the particular bilinear forms I and T. The following lemma can be shown from elementary facts about cosets.

Lemma 3.7. *The bilinear form* I *has the following properties. If* $g, h \in G(n)$, *then*

$$\mathrm{I}(g(i,n), h(j,n)) = \begin{cases} |H_i(n) \cap H_j(n)| & \text{if } gh^{-1} \in H_i(n) + H_j(n) \\ 0 & \text{otherwise.} \end{cases}$$

If $g, h \in \overline{R_{e_i+e_j}(n)}$, *then, since* $(H_i(n) + H_j(n)) \cap \overline{R_{e_i+e_j}(n)} = (1)$,

$$\mathrm{I}(g(i,n), h(j,n)) = \langle g, h\rangle |H_i(n) \cap H_j(n)|.$$

Let $tw(\tau, i, j, p, n)$ be the image of $tw(i,j,p,n)$ in $\mathbb{C}[R_\tau(n)]$.

Lemma 3.8. *If τ is such that $i, j \in \tau$, and $g, h \in \overline{R_\tau(n)} \subset G(n)$, then*

$$T([g](\tau, i, n), [h](\tau, j, n)) = |S_\tau(n)| \sum_{p=1}^{r(i,j)} \langle g, tw(\tau, i, j, p, n)^{-1} h \rangle c(i, j, p).$$

Proof. Let $M_2(i, n) \subset S_\tau(i, n)$ and $M_2(j, n) \subset S_\tau(j, n)$ be sets of coset representatives for

$$S_\tau(i, n)/S_{e_i+e_j}(i, n)$$

and

$$S_\tau(j, n)/S_{e_i+e_j}(j, n),$$

respectively. Both of these are isomorphic to $S_\tau(n)/(H_i(n) + H_j(n))$.

We have, for all $s \in S_{e_i+e_j}(i, n)$,

$$\mathrm{T}(g(i, n), sh(j, n)) = \mathrm{T}(g(i, n), h(j, n)).$$

Thus, for $g_1, h_1 \in \overline{R_\tau(n)}$, $\mathrm{T}([g_1](\tau, i, n), [h_1](\tau, j, n))$ equals

$$\sum_{g_2(i,n) \in M_2(i,n)} \sum_{h_2(j,n) \in M_2(j,n)} d_1(i, j)\, \mathrm{T}(g_1 g_2(i, n), h_1 h_2(j, n)),$$

where

$$\begin{aligned} d_1(i, j) &= |S_{e_i+e_j}(i, n)||S_{e_i+e_j}(j, n)| \\ &= \frac{|H_i(n) + H_j(n)|^2}{|H_i(n)||H_j(n)|}. \end{aligned}$$

Since $\mathrm{T}(g_1 g_2(i, n), h_1 h_2(j, n)) = \mathrm{T}(g_1(i, n), h_1 h_2 g_2^{-1}(j, n))$, we have

$$\mathrm{T}([g_1](i, n), [h_1](j, n)) = \sum_{h_2(j,n) \in M_2(j,n)} d_2(i, j)\, \mathrm{T}(g_1(i, n), h_1 h_2(j, n)),$$

where

$$d_2(i, j) = |M_2(i, n)| \frac{|H_i(n) + H_j(n)|^2}{|H_i(n)||H_j(n)|} = \frac{|S_\tau(n)|}{|H_i(n) \cap H_j(n)|}.$$

Recall that

$$\begin{aligned} &T(g_1(i, n), h_1 h_2(j, n)) \\ &\qquad = \sum_{p=1}^{r(i,j)} I(g_1(i, n), tw(i, j, p, n)^{-1} h_1 h_2(j, n)) c(i, j, p) \end{aligned}$$

for all $h_1, g_1 \in R_\tau(n)$ and $h_2 \in M_2(j, n)$. Now write

$$tw(i, j, p, n) = stw(i, j, p, n)_1$$

where $s \in S_{e_i+e_j}$.

Then we have

$$\begin{aligned}I(g_1(i,n), tw(i,j,p,n)^{-1}h_1h_2(j,n)) &= I(g_1(i,n), tw(i,j,p,n)_1^{-1}h_1h_2(j,n))\\ &= \langle g_1, tw(i,j,p,n)_1^{-1}h_1h_2\rangle |H_i(n)\cap H_j(n)|.\end{aligned}$$

If we now write

$$tw(i,j,p,n)_1 = tw(i,j,p,n)_2 tw(\tau,i,j,p,n),$$

where $tw(i,j,p,n)_2 \in M_2(j,n)$, then we have

$$\langle g_1, tw(i,j,p,n)_1^{-1}h_1h_2\rangle = \langle g_1, tw(\tau,i,j,p,n)^{-1}h_1\rangle\langle tw(i,j,p,n)_2, h_2\rangle.$$

As h_2 varies over $M_2(j,n)$, it will equal $tw(i,j,p,n)_2$ exactly once. Thus,

$$\begin{aligned}T([g_1](i,n),[h_1](j,n)) &= \frac{S_\tau(n)}{H_i(n)\cap H_j(n)|}T(g_1(i,n), h_1h_2(j,n))\\ &= |S_\tau(n)\sum_{p=1}^{r(i,j)}\langle g_1, tw(\tau,i,j,p,n)^{-1}h_2\rangle c(i,j,p).\end{aligned}$$

□

Proof of Proposition 3.1. Let T$'$ be the intersection form on $\mathcal{J}_\tau(n)^s$ induced by the restriction of T to $\mathcal{U}_\tau(n)$. For $\mathbb{C}[R_\tau(n)]$ considered as a $\mathbb{C}[R_\tau(n)]$-module, we can write the generators as $E_i(n) = [1](i,n)$ for all i with $i \in \tau$. If f_1 and f_2 are elements of $\mathbb{C}[R_\tau(n)]$ then, by Lemma 3.8,

$$\mathrm{T}'(f_1E_i(n), f_2E_j(n)) = |S_\tau(n)|\sum_{p=1}^{n}\langle f_1, c(i,j,p)tw(\tau,i,j,p,n)^{-1}f_2\rangle$$

defines an extension of the intersection form T$'$ to $\mathbb{C}[R_\tau(n)]^s$. One can write down an intersection matrix for T$'$ on $\mathbb{C}[R_\tau(n)]^s$ with respect to the generators $E_i(n)$, for $i\in\tau$, as

$$\left[|S_\tau(n)|\sum_{p=1}^{r(i,j)} c(i,j,p)tw(\tau,i,j,p,n)^{-1}\right]_{i,j\in\tau}.$$

Since $|S_\tau(n)|$ is a nonzero constant not depending on i the $\mathbb{C}$-rank and $\mathbb{C}$-nullity of this matrix doesn't change if we replace it by

$$M_n = \left[\sum_{p=1}^{r(i,j)} c(i,j,p)tw(\tau,i,j,p,n)^{-1}\right].$$

By the hypotheses of Proposition 3.1, $tw(i,j,p,n)$ are stable. Therefore, the images $tw(\tau,i,j,p,n)$ and hence the entries of M_n are also stable. Thus, by Proposition 2.4, the sequence of $\mathbb{C}$-nullities of M_n restricted to $\mathcal{J}_\tau(n)^s$, or $\mathrm{Null}_{\mathbb{C}}(\mathrm{T},\mathcal{U}_\tau(n))$, is polynomial periodic. Therefore,

$$\mathrm{Null}_{\mathbb{C}}(\mathrm{T},V(n)) = \sum_{\tau\in\mathcal{T}}\mathrm{Null}_{\mathbb{C}}(T,\mathcal{U}_\tau(n))$$

is polynomial periodic. □

4. Difference between the first Betti numbers of a smooth surface and the complement of curves.

Let X be a smooth complex projective surface and let $\mathcal{C}$ be a finite union of curves on X. This section concerns the relation between the difference of Betti numbers

$$b_1(X-\mathcal{C})-b_1(X)$$

and intersections of curves in $\mathcal{C}$.

We begin with some notation. By curve we will hereafter always mean irreducible curve. Let $\mathrm{Pic}_{\mathbb{Q}}(X)$ and $\mathrm{NS}_{\mathbb{Q}}(X)$ be the Picard group and Néron-Severi group of X tensored with $\mathbb{Q}$. Let $\mathrm{Pic}_{\mathbb{Q}}(X,\mathcal{C})$ and $\mathrm{NS}_{\mathbb{Q}}(X,\mathcal{C})$ be the subspaces generated by divisors supported on $\mathcal{C}$. Let $\#\mathcal{C}$ be the number of components in $\mathcal{C}$.

The following fact was originally communicated to me by A. Landman and A. Libgober.

Proposition 4.1. *Let X be a smooth complex projective surface and $\mathcal{C}$ a finite union of curves on X. Then*

$$b_1(X-\mathcal{C})-b_1(X)=\#\mathcal{C}-\dim_{\mathbb{Q}}\mathrm{NS}_{\mathbb{Q}}(X,\mathcal{C}).$$

The proof of this proposition can be found in [Ho], Proposition I.6.3, but we give a briefer version here.

Proof of Proposition 4.1. We look at the exact homology sequence for the pair $(X, X-\mathcal{C})$ with rational coefficients:

$$\begin{aligned}\mathrm{H}_2(X)\to\mathrm{H}_2(X,X-\mathcal{C})\to\\ \mathrm{H}_1(X-\mathcal{C})\to\mathrm{H}_1(X)\to\mathrm{H}_1(X,X-\mathcal{C}).\end{aligned}$$

There are non-degenerate pairings coming from intersections:

$$\mathrm{H}_i(X,X-\mathcal{C})\otimes\mathrm{H}_{4-i}(\mathcal{C})\to\mathbb{Q}.$$

This implies that $\mathrm{H}_1(X,X-\mathcal{C})$ is trivial, and the map

$$\mathrm{H}_2(X)\to\mathrm{H}_2(X,X-\mathcal{C})$$

is dual to the map

$$i:\mathrm{H}_2(\mathcal{C})\to\mathrm{H}_2(X)$$

induced by inclusion. The claim then follows since $\#\mathcal{C}$ equals $\dim_{\mathbb{Q}}\mathrm{H}_2(\mathcal{C})$ and $\mathrm{NS}_{\mathbb{Q}}(X,\mathcal{C})$ is isomorphic to the image of the map i. $\square$

Let $\mathrm{Null}(\mathcal{C})$ be the nullity of the intersection matrix for $\mathcal{C}$, that is, if we enumerate the curves $C_1,\ldots,C_k$ in $\mathcal{C}$, then $\mathrm{Null}(\mathcal{C})$ equals the nullity of the $k\times k$ matrix with entries $[a_{i,j}]$, where $a_{i,j}=C_i.C_j$. Of course, the nullity doesn't depend on the ordering of the curves.

Proposition 4.2. *The difference* $\#\mathcal{C} - \dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C})$ *can be computed in one of the following ways.*

(1) *If* $\mathcal{C}$ *supports an ample divisor, then*

$$Null(\mathcal{C}) = \#\mathcal{C} - \dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C}).$$

(2) *If* $\mathcal{C}$ *doesn't support an ample divisor, then for any ample curve* H *on* X

$$Null(\mathcal{C} \cup H) = \#\mathcal{C} - \dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C}).$$

Proof. If $\mathcal{C}$ supports an ample divisor then any divisor D supported on $\mathcal{C}$, with the property that $D.E = 0$ for all E supported on $\mathcal{C}$, has the property that $D.H = 0$ for some ample divisor H and that $D^2 = 0$. By the Hodge Index Theorem, such a divisor D is numerically equivalent to zero. Therefore, the kernel of the intersection matrix equals the kernel of the map from $\mathrm{Pic}_{\mathbb{Q}}(X, \mathcal{C})$ to $\mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C})$.

If $\mathcal{C}$ doesn't support an ample divisor and H is an ample curve, then, since $\mathcal{C} \cup H$ supports an ample divisor, by the same argument as above we have

$$\mathrm{Null}(\mathcal{C} \cup H) = \#\mathcal{C} \cup H - \dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C} \cup H).$$

On the other hand, $\#\mathcal{C} \cup H$ and $\dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C} \cup H)$ both equal one more than $\#\mathcal{C}$ and $\dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X, \mathcal{C})$. The latter holds because any divisor numerically equivalent to an ample divisor must be ample by the Nakai-Moishezon criterion. $\square$

Proposition 4.1 and 4.2 imply that given a smooth surface Y and a finite union of curves $\mathcal{C}$ we can compute the first Betti number of the associated desingularized branched coverings $\widetilde{X}_n$ in the following way. If $\mathcal{C}$ supports an ample divisor then so does the preimage $\widetilde{\rho}_n^{-1}(\mathcal{C})$ of $\mathcal{C}$ in $\widetilde{X}_n$. Hence

$$b_1(\widetilde{X}_n) = b_1(X_n^u) - \mathrm{Null}(\widetilde{\rho}_n^{-1}(\mathcal{C})).$$

If $\mathcal{C}$ doesn't support an ample divisor, then we find an ample curve H on Y and

$$b_1(\widetilde{X}_n) = b_1(X_n^u) - \mathrm{Null}(\widetilde{\rho}_n^{-1}(\mathcal{C} \cup H)).$$

Thus, to prove polynomial periodicity for $b_1(\widetilde{X}_n)$, in light of Theorem 1.3, it suffices to show that the nullity of the intersection matrix for curves above the branch locus and possibly an extra curve supporting an ample divisor is polynomial periodic.

5. Intersections on abelian coverings.

Let Y be a smooth surface and let $\mathcal{B}$ be a finite union of curves in Y. Let $\rho : X \to Y$ be an abelian covering with branch locus $\mathcal{B}$ and Galois group G. We assume (recall Remark 1.7) that $\mathcal{B}$ is a union of smooth curves with normal crossings. As in section 3, for a finite set S, $|S|$ denotes the number of elements in S. Recall that by curve we always mean an irreducible curve.

For any irreducible algebraic subset V on Y, there are associated *inertia* and *stabilizer* subgroups I_V and H_V of G defined as follows:

$$I_V = \{g \in G \quad | \quad \text{for all } x \in \rho^{-1}(V),\ gx = x\}$$
$$H_V = \{g \in G \quad | \quad \text{for all irreducible components } V' \subset \rho^{-1}(V),\ g(V') = V'\}.$$

Here are some elementary observations (see also [Ho], Chapter II.)

(1) The subvariety V is contained in the branch locus $\mathcal{B}$ if and only if $I_V \neq (0)$.

(2) Given an irreducible component $V' \subset \rho^{-1}(V)$ there is a canonical one-to-one correspondence between cosets G/H_V and irreducible components of $\rho^{-1}(V)$ by the map
$$gH_V \mapsto gV'.$$

(3) For a point $p \in Y$, we have
$$H_p = I_p = \sum_{p \in C \subset Y} I_C = \sum_{p \in C \subset \mathcal{B}} I_C.$$

(4) For a point $p \in C \subset Y$ and a component $C' \subset \rho^{-1}(C)$, we have
$$|C' \cap \rho^{-1}(p)| = \frac{|H_C|}{|I_p|}.$$

(5) Let $p \in C \cap D$ for two distinct curves $C, D \subset Y$. Let $C' \subset \rho^{-1}(C)$ and $D' \subset \rho^{-1}(D)$ be two curves so that $C' \cap D' \cap \rho^{-1}(p) \neq \emptyset$. Then
$$|C' \cap D' \cap \rho^{-1}(p)| = \frac{|H_C \cap H_D|}{|I_p|}.$$

Furthermore, for any $a, b \in G$, $aC' \cap bD' \cap \rho^{-1}(p)$ is nonempty if and only if $aH_C \cap bH_D \neq \emptyset$.

Note that if $aH_C \cap bH_D \neq \emptyset$ then
$$|aH_C \cap bH_D| = |H_C \cap H_D|.$$

Putting together these facts, we have the following.

Proposition 5.1. *Let $\mathcal{C}$ be a finite union of curves in Y, and for each curve $C \subset \mathcal{C}$ fix a curve $C' \subset \rho^{-1}(C)$ in the preimage. For each triple (C, D, p), where $C, D \subset \mathcal{C}$ are distinct curves and $p \in C \cap D$, let $tw(C, D, p) \in G$ be an element so that*
$$C' \cap tw(C, D, p)D' \cap \rho^{-1}(p) \neq \emptyset.$$
Then, for all elements $a, b \in G$,
$$|aC' \cap bD'| = \sum_{p \in C' \cap D'} \frac{|aH_C \cap tw(C, D, p)^{-1} bH_D|}{|I_p|}.$$

If X is smooth, $\mathcal{C}$ contains only smooth curves and has only normal crossings, then we can find the intersection matrix of curves in $\rho^{-1}(\mathcal{C})$ by using the fact that, for all distinct pairs of curves $C, D \in \mathcal{C}$, all curves $C' \subset \rho^{-1}(C)$ and $D' \subset \rho^{-1}(D)$ intersect transversally, so
$$C'.D' = |C' \cap D'|.$$

Also, for any single curve $C' \subset \rho^{-1}(C)$, the curves in $\rho^{-1}(C)$ are disjoint so we have

$$(\deg \rho)(\#\rho^{-1}(C))(mC')^2 = (\deg \rho)^2 C^2,$$

where the multiplicity m of C' equals $|I_C|$. This can be rewritten as

$$|G|\frac{|G|}{|H_C|}|I_C|^2 C'^2 = |G|^2 C^2,$$

so

$$C'^2 = \frac{|H_C|}{|I_C|^2} C^2.$$

In the case that X is not smooth we need to study resolutions of surface singularities. Our analysis follows that of Laufer ([Lf], Chapter II.) By the hypothesis, ρ restricts locally to branched coverings of a complex disk branched along two distinct lines through the origin.

Any singular point on X clearly must lie above a crossing of two curves $C, D \in \mathcal{B}$. Now take any point q lying above such a crossing and let $p = \rho(q)$. Since p is a smooth point on Y, one can find an analytic neighborhood $U \subset Y$ containing p, isomorphic to a complex disk, and local coordinates (ζ, η) on U so that p is at the origin and C and D are given by the local equations $\zeta = 0$ and $\eta = 0$, respectively.

In general, for any finite branched covering $\rho_p : V \to U$ branched along $\zeta = 0$ and $\eta = 0$, there is an analytic isomorphism

$$\alpha : \mathcal{D} - \{x = 0\} \cup \{y = 0\} \to V - \rho_p^{-1}(\{\zeta = 0\} \cup \{\eta = 0\}),$$

where $\mathcal{D}$ is a complex disk with coordinates x and y and the composition $\rho_p \circ \alpha$, after some possible additional change of the coordinates ζ and η, is the same as the map

$$(x, y) \mapsto (x^r, x^s y^t)$$

where r, s, t are integers and $0 \leq s < t$, $0 < r$.

Returning to our original situation, the integers r, s, t are closely related to the inertia subgroups I_C and I_D. To show this we start by defining meridinal loops around curves.

For any curve $C \subset Y$ and $p \in C$ a smooth point on C not contained in any $D \subset \mathcal{B}$, for $D \neq C$. Let B_p be a small ball around p and let ∂B_p be its boundary. Then ∂B_p is a three-sphere, and $\partial B_p \cap C$ is a circle canonically oriented by the complex structure of C. The positive generator of $\mathrm{H}_1(\partial B_p - C; \mathbb{Z})$ with respect to the given orientation determines an element $\mu_C \in \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$. Since the set of smooth points on C, which are not contained in $D \subset \mathcal{B}$ for any $D \neq C$, is path-connected, μ_C is a well-defined element. We call μ_C the *meridinal loop* around C in $\mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$ (see Definition 1.6.)

Remark 5.2. (*Proof of Corollary 1.8*) Let us consider the case where Y is $\mathbb{P}^2$ and $\mathcal{B}$ is a union of curves not all going through one point. Then by the Lefschetz hyperplane theorem and Van Kampen's theorem it is not hard to see that $\mathrm{H}_1(\mathbb{P}^2 - \mathcal{B}; \mathbb{Z})$ is generated by the μ_C where $C \subset \mathcal{B}$ and its only relation is

$$\sum_{C \subset \mathcal{B}} \deg(C)\mu_C = 0.$$

Let $\mathcal{P}$ be the set of points in $\mathcal{B}$ lying in the intersection of three or more curves in $\mathcal{B}$. By blowing up all the points in $\mathcal{P}$ we obtain a new branch locus $\mathcal{B}'$ as above where all the curves are smooth and have normal crossings. Since $Y - \mathcal{B}$ and $Y' - \mathcal{B}'$ are canonically isomorphic there is a canonical identification between $\mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$ and $\mathrm{H}_1(Y' - \mathcal{B}'; \mathbb{Z})$. If $\widehat{C} \subset \mathcal{B}'$ is the proper transform of a curve $C \subset \mathcal{B}$ then $\mu_C = \mu_{\widehat{C}}$ and if E is the exceptional curve above a point $p \in \mathcal{P}$ then

$$\mu_E = \sum_{p \in C \subset \mathcal{B}} \mu_C.$$

Thus, $\mathbb{P}^2$ and $\mathcal{B}$ satisfy condition (1) of Theorem 1.7. $\square$

It is not hard to see that if we let $g_C \in G$ be the image of the meridinal loop μ_C under the map

$$\mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}) \to G$$

determined by the covering, then I_C is generated by g_C in G (see, for example, [Ho], section I.4.)

Putting together the definitions, the integers r, s, t are such that

$$\begin{cases} rg_C + sg_D = 0 \\ tg_D = 0 \end{cases}$$

generate the relations in $I_C + I_D$.

If $s = 0$ the map α defined above extends to all of $\mathcal{D}$ and thus q has a neighborhood isomorphic to a complex disk and q is a smooth point. If $s \neq 0$ then one can desingularize X at q by adding exceptional curves with the following properties. Let

$$\frac{t}{s} = k_1 - \frac{1}{k_2 - \frac{1}{k_3 - \dots}}$$

be a continued fraction expansion for $\frac{t}{s}$, where $k_1, k_2, \dots$ are positive integers. This process terminates and we get a sequence of ℓ integers $k_1, \dots, k_\ell$ all greater than or equal to 2. Then X can be desingularized at q by adding ℓ exceptional curves $E_1, \dots, E_\ell$ with

$$E_i^2 = -k_i,$$

$$E_i . E_j = \begin{cases} 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and for all curves $F \subset \rho^{-1}(\mathcal{C})$, we have

$$F . E_i = \begin{cases} 1 & \text{if } i = 1 \text{ and } F = C' \\ 1 & \text{if } i = \ell \text{ and } F = D' \\ 0 & \text{otherwise.} \end{cases}$$

The resulting surface $\widetilde{X}$ is the minimal smooth model lying over X.

We now restate the results of this section in terms of our sequences of coverings. Let

$$\rho_n : X_n \to Y$$

be branched coverings, branched along a finite union of smooth curves $\mathcal{B}$ with normal crossings and defined by the map

$$\pi_1(Y - \mathcal{B}) \to \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z}).$$

Let $\widetilde{\rho}_n : \widetilde{X}_n \to Y$ be the composition of ρ_n with the desingularization $\widetilde{X}_n \to X_n$ described above. Let $G = \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$, $G(n) = \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$ and $\psi_n : G \to G(n)$ the quotient map. For each curve $C \subset \mathcal{B}$ let $\mu_C \in G$ be the meridinal curve around C and let $\mu_C(n) \in G(n)$ be its image under ψ_n. Let $\mathcal{C} \supset \mathcal{B}$ be a union of smooth curves with normal intersections. Let $\mathcal{P}$ be the set of points of intersection on $\mathcal{C}$.

For each curve $C \subset Y$, let $H_C \subset G$ be the subgroup given by the image of the map

$$\mathrm{H}_1(C - \overline{C - \mathcal{B}}; \mathbb{Z}) \to \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$$

induced by inclusion. Let $H_C(n) = \psi_n(H_C)$. It is not hard to see that $H_C(n)$ is the stabilizer subgroup for C for the branched covering ρ_n. Let $I_C \subset G$ be the subgroup generated by μ_C and let $I_C(n) = \rho_n(I_C)$. Then $I_C(n)$ is generated by $\mu_C(n)$ and is the inertia subgroup for C.

For any curve $C \subset \mathcal{C}$, let $t(C, n) = |I_C(n)|$. For each triple C, D, p where $C, D \subset \mathcal{C}$ are distinct curves and $p \in C \cap D$, define

$$r(C, D, n) = \min\{r \in \mathbb{Z}_{>0} : \exists s \in \mathbb{Z}_{\geq 0} \quad \text{such that} \quad r\mu_C(n) + s\mu_D(n) = 0.\}$$
$$s(C, D, n) = \min\{s \in \mathbb{Z}_{\geq 0} : r(C, D, n)\mu_C(n) + s\mu_D(n) = 0.\}$$

Note that $r(C, D, n)$ and $s(C, D, n)$ depend on the order in which you take C and D.

To describe intersections above $\mathcal{C}$ there are two main cases to consider. The simplest case is the following.

Case 1. Assume that for all distinct curves $C, D \in \mathcal{B}$ with $C \cap D \neq \emptyset$ and for all $n > 1$, we have $s(C, D, n) = 0$. Then we have

(1) X_n is nonsingular for all $n > 1$,
(2) for all distinct curves $C, D \in \mathcal{C}$ with $p \in C \cap D$, $I_C(n) \cap I_D(n) = (0)$ and
(3) if for each $C \subset \mathcal{C}$ we fix a curve $C' \subset \rho^{-1}(C)$ and if, for each triple C, D, p of distinct curves $C, D \in \mathcal{B}$ and a point $p \in C \cap D$, we find $tw(C, D, p, n) \in G(n)$ so that

$$C' \cap D' \cap \rho^{-1}(p) \neq \emptyset,$$

then intersections for curves above $\mathcal{C}$ are given by

$$aC'.bD' = \frac{1}{|I_C(n)||I_D(n)|} \sum_{p \in C \cap D} |aH_C(n) \cap tw(C, D, p, n)^{-1} bH_D(n)|.$$

The rank and nullity of the resulting matrix doesn't change if we delete the leading constant

$$\frac{1}{|I_C(n)||I_D(n)|}$$

(this corresponds to multiplying rows and columns of the intersection matrix by nonzero integers.) Let

$$c(C, D, p) = \begin{cases} 1 & \text{if } C \neq D \\ \dfrac{C^2}{|\mathcal{P} \cap C|} & \text{if } C = D. \end{cases}$$

Then, $\mathrm{Null}(\rho^{-1}(\mathcal{C}))$ equals the nullity of the matrix with rows and columns corresponding to

$$aC',$$

where C ranges in $\mathcal{C}$ and a ranges in $G(n)/H_C(n)$, with the entry in the (aC', bD') place given by

$$\sum_{p \in C \cap D \cap \mathcal{P}} |aH_C(n) \cap tw(C, D, p, n)^{-1} bH_D(n)| c(C, D, p).$$

If we can arrange so that the $tw(C, D, p, n)$ are stable as functions of n then Proposition 3.1 shows that the nullities of the intersection matrices are polynomial periodic. We show how this can be done in section 6.

Remark 5.3. In general, however, Case 1 doesn't apply. Furthermore, it won't apply even after a succession of blowing up the branch locus and pulling back the covering. For example, let $\mathcal{B}$ be the union of a line L and any nodal curve C in $\mathbb{P}^2$ so that L and C are in general position.

Then

$$\mathrm{H}_1(\mathbb{P}^2 \setminus \mathcal{B}; \mathbb{Z}) \simeq \mathbb{Z}$$

and the meridinal loop μ_C is a generator for this group.

Let p be a nodal singularity on C. For any sequence of blowups over p, meridinal loops around the exceptional curves will be positive multiples of μ_C (see [Ho], Proposition I.4.11.) Therefore, there will always be an exceptional curve E intersecting the proper transform $\widehat{C}$ of C so that $\mu_E = m\mu_C$ for some m. If n is relatively prime to m then $I_C(n) \cap I_E(n)$ is non-trivial.

Case 2. We now consider the more general case.

Lemma 5.4. *Suppose that one of the following is true*

(1) *both I_C and I_D are infinite and $I_C \cap I_D = (0)$; or*
(2) *both I_C and I_D finite.*

(These are the conditions set up in Theorem 1.7.) Then

(1) *$\ell(n), k_1(n), \ldots, k_{\ell(n)}(n)$ are periodic*
(2) *and one of the following holds: either*

(a) $$\frac{|I_C(n)|}{|I_p(n)|} \quad \text{and} \quad \frac{1}{|I_p(n)|}$$

are both periodic or

(b)
$$\frac{|I_C(n)|}{|I_p(n)|} = \frac{\alpha(n)}{n}$$
$$\frac{1}{|I_p(n)|} = \frac{\beta(n)}{n^2}$$

for periodic functions $\alpha(n)$ and $\beta(n)$.

We will first show that the proof of Theorem 1.7 reduces to proving Lemma 5.4 and proving that the twisting elements are stable.

Fix a such that $0 \leq a \leq N-1$. We will show that for each a there is an intersection matrix of the form described in section 3.

Take any positive integer $n \equiv a(\mod N)$. Enumerate the curves $C_1, \ldots, C_k \in \mathcal{C}$. Let $C_i' \subset \widetilde{\rho}_n^{-1}(C_i)$ be any choice of curve for each curve $C_i \subset \mathcal{C}$. For $p \in \mathcal{P}$ and $i = 1, \ldots, \ell(p, n)$, let $E(p, i)$ be the curves mapping to p under the map $\widetilde{\rho}_n$.

Intersections for curves in $\widetilde{\rho}_n^{-1}(\mathcal{C}) \subset \widetilde{X}_n$ can then be described as follows. The rows and columns of the intersection matrix can be made to correspond to the union of the set of

$$\alpha C_i',$$

where $i = 1, \ldots, k$ and $\alpha \in G(n)/H_C(n)$, and the set of

$$\alpha E(p, j)$$

where $p \in \mathcal{P}$, $j = 1, \ldots, \ell(p, a)$ and $\alpha \in G(n)/I_p(n)$. Define for each $p \in \mathcal{P} \cap C_i \cap C_j$ and $f = 1, \ldots, \ell(p, a)$

$$\begin{aligned} H_{E(p,f)(n)} &= I_p(n) = I_{C_i}(n) + I_{C_j}(n) = (I_{C_i} + I_{C_j})(n) \\ I_{E(p,f)(n)} &= (0). \end{aligned}$$

For each $i = 1, \ldots, k$ let

$$c(C_i, C_i, p) = \frac{C_i^2}{|\mathcal{P} \cap C_i|}.$$

For each triple C_i, C_j, p with $i < j$ and $p \in C_i \cap C_j$ let

$$c(C_i, C_j, p) = \begin{cases} 1 & \text{if } p \notin \mathcal{P} \\ 0 & \text{otherwise.} \end{cases}$$

If $p \in \mathcal{P}$ define

$$c(C_i, E(p, f), p) = \begin{cases} \dfrac{|I_{C_i}(n)|}{|I_p(n)|} & \text{if } f = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$c(C_j, E(p, f), p) = \begin{cases} \dfrac{|I_{C_j}(n)|}{|I_p(n)|} & \text{if } f = \ell(p, a) \\ 0 & \text{otherwise} \end{cases}$$
$$c(E(p, f), E(p, f), p) = -\frac{k_f}{|I_p(n)|}.$$

Then for any $C, D \subset \widetilde{\rho}_n(\mathcal{C})$, we have

$$\alpha C.\beta D = \sum_{p \in \widetilde{\rho}_n(C) \cap \widetilde{\rho}_n(D) \cap \mathcal{P}} \frac{|\alpha H_C(n) \cap tw(C, D, p, n)^{-1} \beta H_D(n)|}{|I_C(n)||I_D(n)|} c(C, D, p).$$

If, in Lemma 5.4, (2)(a) holds, we multiply rows and columns corresponding to αC by $|I_C(n)|$ to get the intersection matrix for a twisted intersection of the form studied in section 3.

In the case of (2)(b), we do the same as above, but also multiply the rows and columns corresponding to $\alpha E(p, f)$ by n to get the intersection matrix in the desired form.

Thus, if we can find stable sequences of twisting elements $tw(C, D, p, n)$ (as we do in the next section), then $\mathrm{Null}(\widetilde{\rho}^{-1}(\mathcal{C}))$ is polynomial periodic for positive integers n with $n \equiv a(\mod N)$. This shows that $\mathrm{Null}(\widetilde{\rho}^{-1}(C))$ is polynomial periodic as a function of n and hence so is $b_1(\widetilde{X}_n)$.

Proof of Lemma 5.4. For (1) it suffices to show that the rational number

$$\frac{t(D, n)}{s(C, D, n)}$$

is periodic for all distinct curves $C, D \subset \mathcal{B}$ with $C \cap D \neq \emptyset$.

To show (2), since

$$|I_p(n)| = |I_C(n) + I_D(n)| = \frac{|I_C(n)||I_D(n)|}{|I_C(n) \cap I_D(n)|} = \frac{t(C, n)t(D, n)}{|I_C(n) \cap I_D(n)|}$$

and

$$|I_C(n) \cap I_D(n)| = \frac{t(C, n)}{\gcd(t(C, n), r(C, D, n))},$$

it suffices to show that either

$$t(C, n) \quad \text{and} \quad \gcd(t(C, n), r(C, D, n))$$

are both periodic or they are both of the form $n\alpha(n)$ where $\alpha(n)$ is periodic.

From the above discussion one sees that Lemma 5.4 is implied by the following two lemmas.

Lemma 5.5. *Take any pair of distinct curves $C, D \subset \mathcal{B}$ with $C \cap D \neq \emptyset$. If I_C is infinite, then $t(C, n) = n\alpha(n)$, where $\alpha(n)$ is periodic. If, in addition, $I_C \cap I_D = (0)$ then $r(C, D, n) = n\beta(n)$ and $s(C, D, n) = n\gamma(n)$ where $\beta(n)$ and $\gamma(n)$ are periodic.*

Lemma 5.6. *If I_C and I_D are finite, then $t(C, n)$, $r(C, D, n)$ and $s(C, D, n)$ are all periodic.*

To prove these lemmas we make the following definitions. Since G is a finitely generated group, G/I_C, G/I_D and $G/(I_C + I_D)$ are also finitely generated. Thus, their torsion parts are finite groups. Let $A(C)$ (respectively, $A(D)$, $B(C, D)$) be the set of possible finite orders of elements in G/I_C (respectively, G/I_D, $G/(I_C + I_D)$).

For $a \in A(D)$, define

$$t_a(D) = \min\{t \in \mathbb{Z}_{>0} : \exists g \in G, ag = t\mu_D\}$$

and, for $b \in B(C, D)$, define

$$\begin{aligned} r_b(C,D) &= \min\{r \in \mathbb{Z}_{>0} : \exists g \in G, \quad \exists s \in \mathbb{Z}_{\geq 0}, \quad r\mu_C + s\mu_D = bg\} \\ s_b(C,D) &= \min\{s \in \mathbb{Z}_{\geq 0} : \exists g \in G, \quad r_b(C,D)\mu_C + s\mu_D = bg\}. \end{aligned}$$

If $|I_D| < \infty$ set

$$t_0(D) = \min\{t \in \mathbb{Z}_{>0} : t\mu_D = 0\}.$$

Otherwise set $t_0(D) = \infty$. If $|I_C| < \infty$ or $I_C \cap I_D \neq (0)$ set $r_0(C, D)$ and $s_0(C, D)$ so that $r_0(C, D)$ is the minimal positive integer so that

$$r_0(C,D)\mu_C + s\mu_D = 0,$$

for some non-negative integer s and $s_0(C, D)$ is the minimal non-negative integer such that

$$r_0(C,D)\mu_C + s_0(C,D)\mu_D = 0.$$

Otherwise set $r_0(C, D)$ equal to ∞ and $s_0(C, D)$ equal to 0.

Proof of Lemma 5.5. Suppose I_C is infinite and t is the minimal positive solution to

$$t\mu_C = ng$$

for some $g \in G$. The order a of the image of g in G/I_C must lie in $A(C)$. There is also a positive integer t' dividing a so that

$$t'\mu_C = ag.$$

Since I_C is infinite t must equal $\frac{n}{a}t'$. By the definition of t_a we have $t_a \leq t'$, so by the minimality of t we must have $t' = t_a$ and, in fact,

$$\begin{aligned} t &= \min_{a \in A(C), a|n} \{\frac{nt_a}{a}\} \\ &= n \min_{a \in A(C), a|n} \{\frac{t_a}{a}\}. \end{aligned}$$

Dividing by n gives a number which is clearly periodic.

Now suppose that I_D is infinite and $I_C \cap I_D = (0)$. Suppose r is a minimal positive solution to

$$r\mu_C + s\mu_D = ng$$

for some $g \in G$ and non-negative integer s. Let b be the order of the image of g in $G/(I_C + I_D)$. There are positive integers r', s', with r' minimal, so that

$$r'\mu_C + s'\mu_D = bg.$$

By the hypothesis $r_0(C,D)=\infty$. So

$$r=\frac{n}{b}r' \qquad \text{and} \qquad s=\frac{n}{b}s'.$$

By the minimality conditions on r and s, $r'=r_b$ and $s'=s_b$. Thus

$$\begin{aligned} r &= \min_{b\in B(C,D),b|n}\{\frac{nr_b}{b}\} \\ &= n\min_{b\in B(C,D),b|n}\{\frac{r_b}{b}\}. \end{aligned}$$

and again dividing by n gives a periodic function. We can write this as $r_{b(n)}/b(n)$, where $b(n)$ is periodic and s is equal to

$$\frac{ns_{b(n)}}{b(n)}$$

which is n times a periodic function. □

Proof of Lemma 5.6. Suppose $|I_C|=t_0(C)$ is finite. Keeping the notation in the proof of Lemma 5.5 we have $\frac{n}{a}t'-t$ is divisible by $t_0(C)$. Furthermore, since t_a is minimal, we have $t_a|t'$ (we could replace t_a by $\gcd(t_a,t')$.) So

$$\begin{aligned} t &= \frac{n}{a}t'(\mod t_0(D)) \\ &= m\frac{n}{a}t_a(\mod t_0(D)) \end{aligned}$$

for some $m\in\{1,\dots,t_0(D)-1\}$.

On the other hand, take any $g'\in G$ so that

$$t_a(C)\mu_C=ag'.$$

Then, for any $m=1,\dots,t_0(D)-1$ we have

$$mt_a(C)\mu_C=a(mg').$$

This implies that

$$t=\min_{a\in A(C),m=1,\dots,t_0(C)-1,a|n}\{m\frac{n}{a}t_a(C)(\mod t_0(C))\}.$$

Take any positive integers a,α and N, with $0\le\alpha\le N-1$. For all n divisible by a, the value of $\frac{n}{a}(\mod N)$ is α if and only if $n(\mod N)=a\alpha(\mod N)$. This imposes a requirement on the modulus of n. It follows that $t(C,n)$ is periodic in n.

If I_D is also finite, then $r_0(C,D)$ and $s_0(C,D)$ are both finite. As with t

$$r=m\frac{n}{b}r'(\mod r_0(C,D))$$

for some m. Since $r_b(C,D)$ is minimal one can easily check that $r_b(C,D)$ must divide r'. Thus,

$$r = \min_{b\in B(C,D), m=1,\dots,r_0(C,D), b|n} \{\frac{mn}{b} r_b(C,D) (\mod r_0(C,D))\}$$

which is periodic. We can write r as

$$\begin{aligned} r &= \frac{m(n)n}{b(n)} r_{b(n)} (\mod r_0(C,D)) \\ &= \frac{m(n)n}{b(n)} r_{b(n)} - L r_0(C,D) \end{aligned}$$

where $m(n)$ and $b(n)$ are periodic functions in n and L equals

$$\frac{\frac{m(n)}{b(n)} r_{b(n)} n - \frac{m(n)}{b(n)} r_{b(n)} n (\mod r_0(C,D))}{r_0(C,D)}.$$

Note that for any positive integer α, $L(\mod \alpha)$ is periodic. Since

$$r_{b(n)}\mu_C + s_{b(n)}\mu_D = b(n)g',$$

we have

$$\begin{aligned} ng &= r\mu_C + s\mu_D \\ &= \frac{m(n)n}{b(n)}[b(n)g' - s_{b(n)}\mu_D] + s\mu_D + Ls_0(C,D)\mu_D \\ &= m(n)ng' + (s + Ls_0(C,D) - \frac{m(n)n}{b(n)} s_{b(n)})\mu_D, \end{aligned}$$

so

$$s = (Ls_0(C,D) - \frac{m(n)n}{b(n)} s_{b(n)}) (\mod t_0(D))$$

which is periodic. □

6. Liftings of the intersection graph and twisting elements.

Throughout this section, fix a smooth surface Y and two finite unions of smooth curves $\mathcal{B} \subset \mathcal{C} \subset Y$ with normal crossings. Let $\rho_n : X_n \to Y$ be the branched coverings corresponding to the composition ϕ_n of the maps

$$\phi : \pi_1(Y - \mathcal{B}) \to \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$$

and

$$\psi_n : \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}) \to \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z}).$$

As before, let $G = \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$, $G_n = \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{P}$ be the set of intersections on $\mathcal{C}$. In Section 5 we reduced the proof of Theorem 1.7 to proving the following lemma.

Lemma 6.1. *There are choices of (irreducible) curves C' in $\rho_n^{-1}(C)$ one for each curve $C \subset \mathcal{C}$ and elements $tw(C, D, p, n)$ in the covering groups $G(n)$ so that*

$$C' \cap tw(C, D, p, n) D' \cap \rho_n^{-1}(p)$$

is nonempty and so that the sequence of elements $tw(C, D, p, n)$ is stable for each triple C, D, p.

Proof. We will find the $tw(C, D, p, n)$ as images of a single element in $\pi_1(Y \setminus \mathcal{C})$ under the map ϕ_n and use the fact that $\psi_n \circ \phi = \phi_n$ to show that there exists $tw(C, D, p) \in G$ so that $\psi_n(tw(C, D, p)) = tw(C, D, p, n)$.

One way to make the choices of curves $C' \subset \rho_n^{-1}(C)$ and to find the twisting elements $tw(C, D, p, n)$ goes as follows. First, for each $p \in \mathcal{P}$, let $\mathcal{B}(p)$ be a small analytic neighborhood of p so that

(1) $\mathcal{B}(p) \cap \mathcal{B}(q) = \emptyset$ for $p \neq q$;
(2) each connected component of $\rho_n^{-1}(\mathcal{B}(p))$ contains a single point of $\rho_n^{-1}(p)$.

For each curve $C \subset \mathcal{C}$, order the points $p_0, \ldots, p_{s_C}$ in $\mathcal{P} \cap C$ and let Γ_C be the graph homeomorphic to $[0, s_C]$, with vertices $\mathcal{V}_C$ the integers in $[0, s_C]$. We will write the vertices as $v_C(0), \ldots, v_C(s_C)$. For $a = 0, \ldots, s$, let $I_{C,p_a} = (a - \frac{1}{4}, a + \frac{1}{4}) \subset \Gamma_C$. Let

$$f_C : \Gamma_C \to \mathcal{C} - \mathcal{P}$$

be an immersion so that

(1) $f_C(I_{C,p}) \subset \mathcal{B}(p)$, for all $p \in \mathcal{P} \cap C$;
(2) $f_C^{-1}(\mathcal{B}(p)) = I_{C,p}$.

Let Γ be the graph

$$\coprod_{C \subset \mathcal{C}} \Gamma_C \cup \Sigma$$

where Σ is the set of edges defined as follows: Enumerate the curves $C_1, \ldots, C_k \subset \mathcal{C}$. Let Σ be defined so that there is an edge in Σ connecting two vertices $v_1 \in \mathcal{V}_{C_i}$ and $v_2 \in \mathcal{V}_{C_j}$ where $i \neq j$. Assume $i < j$, $v_1 = v_{C_i}(a)$ and $v_2 = v_{C_j}(b)$. Suppose

(1) $f_{C_i}(v_1), f_{C_j}(v_2) \in \mathcal{B}(p)$ for some $p \in C_i \cap C_j$;
(2) b is the least integer among $1, \ldots, s_{C_j}$ so that (1) holds;
(3) there is no index ℓ with $i < \ell < j$ so that $C_i \cap C_\ell \neq \emptyset$ and $C_\ell \cap C_j \neq \emptyset$.

Then, since Γ can be defined inductively, starting with Γ_{C_1} and at each step attaching the connected, simply connected graph Γ_{C_j} to an already existing connected, simply connected graph by a single edge, we can conclude that Γ is connected and simply connected.

Now let

$$F : \Gamma \times [0, 1] \to Y$$

be an immersion so that

(1) $F(\Sigma, t) \subset \coprod_{p \in \mathcal{P}} \mathcal{B}(p)$, for $t \in [0, 1]$;
(2) $F(I_{C,p}, t) \subset \mathcal{B}(p)$, for $p \in \mathcal{P}$ and $t \in [0, 1]$;
(3) $F(\gamma, 0) = f_C(\gamma)$, for $C \subset \mathcal{C}$ and $\gamma \in \Gamma_C$;
(4) $F(\gamma, t) \in Y - \mathcal{C}$, for $\gamma \in \Gamma$ and $t > 0$.

Since $f(\Gamma_C) \subset \mathcal{C} - \mathcal{P}$, F can be made so that $F(I_{C,p} \times [0,1])$ intersects $\mathcal{C}$ transversally at smooth points. Since $\Gamma \times (0,1]$ is simply connected and

$$F(\Gamma \times (0,1]) \subset Y - \mathcal{C},$$

for each n there exists a lifting

$$F_n : \Gamma \times (0,1] \to X_n$$

and F_n is also an immersion.

Lemma 6.2. *For a given choice of lifting $F_n : \Gamma_C \times (0,1] \to X_n$, there is a unique curve $C' \subset \rho_n^{-1}(C)$ which intersects the closure $\overline{F_n(\Gamma_C \times (0,1])}$.*

Proof. Since ρ_n is an open mapping, we have

$$\rho_n(\overline{F_n(\Gamma_C \times (0,1])}) = F(\Gamma_C \times [0,1]).$$

Thus,

$$\rho_n(\overline{F_n(\Gamma_C \times (0,1])}) \cap C \subset C - \mathcal{P}.$$

Since for all points $q \in C - \mathcal{P}$, $I_q = I_C$, ρ_n restricts to an unbranched covering over $C - \mathcal{P}$ (the covering having several connected components one contained in each irreducible component of $\rho_n^{-1}(C)$.) Since F_n is an immersion

$$\overline{F_n(\Gamma_C \times (0,1]))} - F_n(\Gamma_C \times (0,1])$$

is connected and thus must lie in a single curve $C' \subset \rho_n^{-1}(C)$. $\square$

We can also define twisting elements for the above choices of C' using the same information. Take any triple (C, D, p) with $C, D \subset \mathcal{C}$, $C \neq D$ and $p \in C \cap D$. We want to find $tw(C, D, p) \in \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$ so that

$$C' \cap \phi_n(tw(C, D, p))D' \cap \rho_n^{-1}(p) \neq \emptyset.$$

If the vertices $v_C(p) \in \Gamma_C$ and $v_D(p) \in \Gamma_D$ are connected by an edge e in Σ, then we automatically have

$$C' \cap D' \cap \rho_n^{-1}(p) \neq \emptyset$$

since $F(e) \subset \mathcal{B}(p)$, so both C' and D' must contain the unique point in the intersection of the component of $\rho_n^{-1}(\mathcal{B}(p))$ containing $F_n(e)$ and the fiber $\rho_n^{-1}(p)$.

Otherwise, let γ_1 be a path on $F(\Gamma \times \{1\})$ from $F(v_D(p), 1)$ to $F(v_C(p), 1)$. Let γ_2 be a path on $\mathcal{B}(p) - \mathcal{C}$ from $F(v_C(p, 1))$ to $F(v_D(p, 1))$. The composition $\gamma_1\gamma_2$ defines a closed path γ contained in $Y - \mathcal{C}$ from $F(v_D(p), 1)$ to itself and hence an element $tw(C, D, p) \in \mathrm{H}_1(Y - \mathcal{B}; \mathbb{Z})$.

Lemma 6.3. *With $tw(C, D, p)$ defined as above,*

$$C' \cap \phi_n(tw(C, D, p))D' \cap \rho_n^{-1}(p)$$

is nonempty.

Proof. Fix n and let $g_n = \phi_n(tw(C, D, p))$. Let $\mathcal{B}(p)_C$ (resp., $\mathcal{B}(p)_D$) be the connected component of $\rho_n^{-1}(\mathcal{B}(p))$ containing $F_n(v_C(p), 1)$ (resp., $F_n(v_D(p), 1)$.) Let q_C (resp., q_D) be the point in $\mathcal{B}(p)_C \cap \rho_n^{-1}(p)$ (resp., $\mathcal{B}(p)_D \cap \rho_n^{-1}(p)$).) Then it suffices to show that $g_n(q_D) = q_C$.

Let γ' be the lift of γ, defined above, with basepoint $F_n(v_D(p), 1) \in \mathcal{B}(p)_D$. Then since $\gamma_1 \subset F(\Gamma \times \{1\})$, its endpoint lies in $F(\Gamma_C \times \{1\})$. Thus, the endpoint of the lift of γ_1' with basepoint $F_n(v_D(p), 1)$ equals $F_n(v_C(p), 1) \in \mathcal{B}(p)_C$. Since $\gamma_2 \in \mathcal{B}(p)$, the lift γ_2' with basepoint $F_n(v_C(p), 1)$ must lie in $\mathcal{B}(p)_C$. By uniqueness of liftings $\gamma' = \gamma_1'\gamma_2'$, so the endpoint of γ' lies in $\mathcal{B}(p)_C$. This implies that $g_n(\mathcal{B}(p)_D) = \mathcal{B}(p)_C$ and therefore $g(q_D) = q_C$. $\square$

This completes the proof of Theorem 1.7.

There are still many questions to be answered about polynomial periodicity of numerical invariants for coverings. It seems likely that the Betti numbers are polynomial periodic in much higher generality: with no restrictions on the branch locus or on the dimension of the base variety. Theorem 1.5 might also be generalized in the realm of general topological spaces. Another direction of further research is to find actual formulas for the polynomials and periodicities which occur. These could provide interesting isotopy invariants for the imbedding of the branch locus in the base space.

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