The Lehmer Polynomial and Pretzel Links

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Abstract

In this paper we find a formula for the Alexander polynomial $\Delta_{p_1,\ldots,p_k}(x)$ of pretzel knots and links with $(p_1,\ldots,p_k,$-1$,\ldots,$-1$)$ twists, where $p_1,\ldots,p_k$ are positive integers, and -1's appear $k-2$ times. The polynomial $\Delta_{2,3,7}(x)$ is the well-known Lehmer polynomial, which is conjectured to have the smallest Mahler measure among all monic integer polynomials. We confirm that $\Delta_{2,3,7}(x)$ has the smallest Mahler measure among the polynomials arising as $\Delta_{p_1,\ldots,p_k}(x)$.

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1 Introduction

The Lehmer polynomial, which is the monic, integer polynomial with smallest known Mahler measure, appears in geometry in two seemingly different guises. One is as the Alexander polynomial $\Delta_{2,3,7}(x)$ for a (-2, 3, 7)-pretzel knot ([Reid] p.34), and another is as the denominator of the growth function of the $(2, 3, 7)$-Coxeter reflection group ([Floy], p.483). We verify that $\Delta_{2,3,7}(x)$ is the polynomial with smallest Mahler measure which arises among all polynomials $\Delta_{p_1,\ldots,p_k}(x)$.

Throughout this paper, all polynomials will have integer coefficients. Given a monic polynomial $p(x)$, the product of the norms of the roots of $p(x)$ outside the unit circle is called the *Mahler measure* of $p(x)$ (hence cyclotomic polynomials have Mahler measure 1.)

A *palindromic* polynomial is a polynomial $p(x)$ whose coefficients are the same read from the left or from the right. Thus, $p(x)$ is palindromic if and only if it satisfies

$$p(x) = x^d p(\frac{1}{x}),$$

where d is the degree of $p(x)$, or equivalently the roots of $p(x)$ are closed under reciprocals.

A long standing open question, posed by Lehmer ([Leh], pp. 476) is whether the Mahler measure of an irreducible monic polynomial which is not cyclotomic can be made arbitrarily close to 1. For non-palindromic irreducible monic polynomials, the problem is solved: the polynomial

$$S(x) = x^3 - x - 1$$

has the smallest Mahler measure [Smy], but palindromic polynomials can have smaller Mahler measure.

In [Leh], Lehmer made an extensive search finding the best (smallest Mahler measure) irreducible monic palindromic polynomials of degrees 2,4,6 and 8. (An odd degree palindromic polynomial is necessarily reducible.) The 10th degree polynomial

$$L(x) = 1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}$$

found by Lehmer in 1933 ([Leh], pp. 477) is still the best known for arbitrary degree.

Not surprisingly, $L(x)$ and $S(x)$ both have only one root outside the unit circle. Let $\alpha > 1$ be any real algebraic integer with all conjugates on or within the unit circle. If at least one conjugate is on the unit circle, making the minimal polynomial necessarily palindromic, then α is called a *Salem number*. Otherwise, α is called a *PV number*. Thus, in addition to having smallest known Mahler measure, the Lehmer polynomial is the minimal polynomial for the smallest known Salem number $\alpha_L = 1.17628\ldots$ [Boyd], and the smallest PV-number, $\alpha_S = 1.32472\ldots$, is a root of $S(x)$.

The first appearance of $L(x)$ (actually $L(-x)$) in the literature may be in K. Reidemeister's book *Knot Theory* ([Reid] p.34), where $L(-x)$ is given as the Alexander polynomial for the $(-2, 3, 7)$-pretzel knot. In his list of open problems ([Kir], p. 340, problem 5.12), R. Kirby also draws attention to the connection between the minimality question for Mahler measure and knot theory. Alexander polynomials of knots and links are a natural place to look for examples pertaining to Lehmer's question. It is well known (see [Seif], p. 589, Satz 6, and [Lev]), that a polynomial $\Delta(x)$ is the Alexander polynomial of a knot if and only if $\Delta(x)$ is palindromic and $\Delta(1) = \pm 1$. By choosing an orientation on the connected components of a link, one can also define a single variable Alexander polynomial for a link with more than one connected component. This is the same as taking the usual multi-variable Alexander polynomial for the link, and identifying all variables.

Instead of looking at all knot and link polynomials we will look at a particular family. Consider the rational function

$$\mathrm{R}_{p_1,\ldots,p_k}(x) = (1-k) + x + (1-x)\left(\frac{1}{1-x^{p_1}} + \cdots + \frac{1}{1-x^{p_k}}\right),$$

depending on $k \geq 1$ positive integers $p_1, \ldots, p_k$. It is not hard to check that this function satisfies:

$$\mathrm{R}_{p_1,\ldots,p_k}(x) = x\mathrm{R}_{p_1,\ldots,p_k}(\frac{1}{x}).$$

Thus, multiplying by $[p_1]\ldots[p_k]$ gives the palindromic polynomial

$$Q_{p_1,\ldots,p_k}(x) = (x-k+1)[p_1][p_2]\ldots[p_k] + \sum_{i=1}^{k}[p_1]\ldots\widehat{[p_i]}\ldots[p_k],$$

where for any positive integer n, we define $[n]$ to be

$$[n] = 1 + x + x^2 + \cdots + x^{n-1}.$$

One can observe the following by a simple calculation.

Lemma 1.1 *The Lehmer polynomial $L(x)$ is equal to $Q_{2,3,7}(-x)$.*

Let $K_{p_1,\ldots,p_k}$ be the pretzel link with k positive twists of orders $p_1, \ldots, p_k$ and $k-2$ negative twists of orders 1. We prove the following Theorem in Section 2.

Theorem 1.2 *For any positive integers $p_1, \ldots, p_k$, the Alexander polynomial of the $K_{p_1,\ldots,p_k}$ equals $Q_{p_1,\ldots,p_k}(-x)$.*

Note (see Section 2), that the (-2, 3, 7)-pretzel knot is equivalent to the (2, 3, 7, -1)-pretzel knot, i.e., $K_{2,3,7}$. Furthermore, all the polynomials of low degree with minimal Mahler measure found by Lehmer occur as irreducible factors of $Q_{p_1,\ldots,p_k}(x)$, for some $p_1, \ldots, p_k$ (see Section 3).

Theorem 1.3 *Among the polynomials of the form $Q_{p_1,\ldots,p_k}(x)$, with a Salem factor, $L(x)$ has the smallest Mahler measure.*

The family $Q_{p_1,\dots,p_k}(x)$ is related to the growth functions of planar Coxeter groups. Consider the group

$$\langle\, g_1,\dots,g_k \;:\; (g_1g_2)^{p_1},\dots,(g_kg_1)^{p_k}, g_1^2,\dots,g_k^2 \,\rangle.$$

The generators $g_1,\dots,g_k$ can be represented as the reflections through sides of a compact k-sided polygon with angles

$$\frac{\pi}{p_i}, \qquad i=1,\dots,k,$$

in either the hyperbolic plane, Euclidean plane, or sphere, according to whether

$$\chi(p_1,\dots,p_k) = \frac{1}{p_1}+\cdots+\frac{1}{p_k} - k + 2$$

the orbifold Euler characteristic of the quotient surface by the Coxeter group, is less than, equal to, or greater than zero.

Cannon and Wagreich ([C-W] Prop. 3.1), and Floyd and Plotnick ([F-P] Theorem 5.1) show that the growth function of the planar Coxeter groups have the following form (see also [Bour].)

Theorem 1.4 (Floyd-Plotnick [F-P]) *The growth function of the planar Coxeter group corresponding to the integers $p_1,\dots,p_k$ with respect to the standard generators equals*

$$\frac{x+1}{\mathrm{R}_{p_1,\dots,p_k}(x)},$$

and its denominator $Q_{p_1,\dots,p_k}(x)$ is a product of cyclotomic polynomials and at most one Salem polynomial. The Salem polynomial occurs if and only if $\chi(p_1,\dots,p_k)<0$.

Theorem 1.3 thus has the following corollary.

Corollary 1.5 *Lehmer's Salem number α_L is the smallest number arising as the growth rate of a hyperbolic polygonal reflection group.*

There is a natural relation between $(p_1,\dots,p_k)$-pretzel knots, where $p_1,\dots,p_k$ can be positive or negative, and the $|p_1|,\dots,|p_k|$-orbifold 2-sphere, which indicates a partial relation between Q and Δ. The double branched covering of the 3-sphere branched along the $(-1,p_1,\dots,p_k)$ pretzel knot, fibers over the $|p_1|,\dots,|p_k|$ orbifold 2-sphere (cf. [Kaw].) Thus, the fundamental group of the complement of the pretzel link and the fundamental group of the orbifold are closely related.

This only partially explains the relation between the polynomials, however, since calculations show that the Alexander polynomial, and in particular its Mahler measure, is not preserved when a positively twisted strand is exchanged for a negatively twisted one of the same order. The seeming coincidence suggests that there may be a bound on the growth rate of the fundamental group of a knot or link complement in terms of the Mahler measure of the Alexander polynomial – a topic for further research, which will not be treated in this paper.

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2 Pretzel Knots

A $(p_1, \dots, p_k)$-pretzel link is a union of k-pairs of strands twisted $p_1, \dots, p_k$ times and attached along the tops and bottoms as in Figure 1.

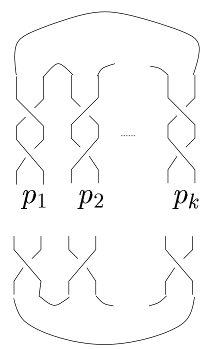

Figure 1. $(p_1, \dots, p_k)$-pretzel link.

The twists are oriented according to whether p_i is positive or negative. For example, in Figure 1, p_1 and p_k are positive integers, while p_2 is a negative integer.

The Alexander polynomial for a $p_1, \dots, p_k$-pretzel link, when $p_1, \dots, p_k$ are odd integers, is well known (see, for example, [Lic] p. 57.) When they are allowed to be even, the link may have several components: if the number of even twists p_i is $d > 2$, then the number of components of the knot is $d - 1$, otherwise the number of components is 1.

Proof of Theorem 1.2. We compute the Alexander polynomial by a standard method involving Seifert matrices (see, for example, [Rolf], Chapter 5.) Figure 2 is a (-2, 3, 7)-pretzel knot (drawn as the equivalent (2, 3, 7, -1)-pretzel knot) with an oriented Seifert surface shaded in.

Let $p_1, \dots, p_k$ be positive integers, and consider the link $K_{p_1,\dots,p_k}$, the pretzel link with k positive twists of orders $p_1, \dots, p_k$, and $k - 2$ negative twists of order 1 (as in the introduction). In order to define a single variable Alexander polynomial, we need to choose orientations on the components of the link. Our choice will be to orient the link so that the top strand connecting the twist p_i to p_{i+1} points right if i is even, left if i is odd, and the bottom strand points left if i is even, and right if i is odd. Thus, we get an oriented Seifert

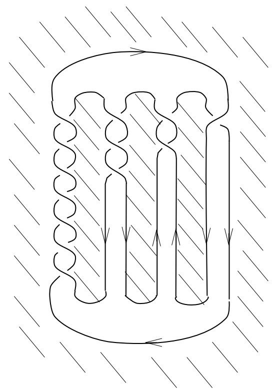

Figure 2. Seifert surface for $K_{2,3,7}$.

surface for the link with disks spanning pairs of twists and the strands that connect them as in Figure 2.

The Seifert matrix of an oriented link is given by choosing generating loops for the first homology of the Seifert surface, and seeing how their positive pushouts into the complement of the Seifert surface in the three sphere S^3 intersect with the original loops. This is seen in Figure 3, where the original loops are drawn with a dashed line, and the pushouts are drawn with a solid line.

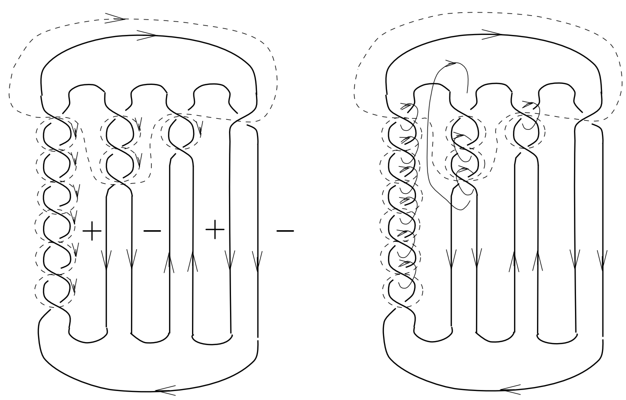

Figure 3. Generating loops for $K_{2,3,7}$.

In general, for $K_{p_1,\ldots,p_k}$, we obtain a $p_1 + \cdots + p_k - k + 1$ dimensional square Seifert matrix of the form

$$\mathcal{S}_K = \begin{bmatrix} & A_{p_1} & & & & & & & 0 \\ & & & A_{p_2} & & & & & \\ & & & & & \dots & & & \\ & & & & & & & A_{p_k} & 0 \\ 1\ 0\ \dots\ 0 & & 1\ 0\ \dots\ 0 & & & \dots & & 1\ 0\ \dots\ 0 & 1 \end{bmatrix}$$

where A_p is the $p-1 \times p-1$ matrix

$$A_p = \begin{bmatrix} 1 & \text{-1} & 0 & \dots & & & 0 \\ 0 & 1 & \text{-1} & & 0 & \dots & 0 \\ & & \dots & & & & \\ 0 & \dots & & & 0 & 1 & \text{-1} \\ 0 & \dots & & & & 0 & 1 \end{bmatrix}$$

If L is an oriented link and $\mathcal{S}_L$ is its Seifert matrix, then the Alexander polynomial $\Delta_(x)$ is the characteristic polynomial of the *Alexander matrix* $\mathcal{A}_L$ given by the matrix product

$$\mathcal{A}_L = \mathcal{S}_L \times \text{Transpose}(\mathcal{S}_L^{-1}).$$

The orientation on the link determines an infinite cyclic covering of the link complement in S^3. The matrix $\mathcal{A}_L$ represents the action of a generator of the covering group on the first homology considered as a module over the ring of Laurent polynomials.

In our situation, the matrix $\mathcal{A}_L$ is of the form

$$\mathcal{A}_L = \begin{bmatrix} & & & & & 0 \\ B_{p_1} & & & & & \\ & & & & & 0 \\ & & & & & \text{-1} \\ & & \dots & & & \dots \\ & & & & & 0 \\ & & & B_{p_k} & & \\ & & & & & 0 \\ & & & & & \text{-1} \\ 1\ 0 \dots\ 0 & 1\ 0\ \dots\ 0 & \dots & 1\ 0\ \dots\ 0 & 1\ 0\ \dots\ 0 & 1-k \end{bmatrix}$$

where B_p is the $p-1 \times p-1$ matrix

$$B_p = \begin{bmatrix} 0 & \text{-1} & 0 & \dots & & & 0 \\ 0 & 0 & \text{-1} & & 0 & \dots & 0 \\ & & \dots & & & & \\ 0 & \dots & & & & 0 & \text{-1} \\ 1 & \dots & & & & 1 & 1 \end{bmatrix}.$$

Let $N = p_1 + \cdots + p_k - k + 1$. If one identifies $\mathbb{C}^N$ with the linear subspace $V \subset \mathbb{C}^{N+k}$ given by the image of the map

$$\begin{aligned}
&(X_{1,1}, \dots, X_{1,p_1-1}, \dots, X_{k,1}, \dots, X_{k,p_k-1}, Y) \\
&\qquad\qquad \mapsto (X_{1,1}, \dots, X_{1,p_1-1}, -X_{1,1} - \cdots - X_{1,p_1-1}, \\
&\qquad\qquad\qquad\qquad \dots, \\
&\qquad\qquad\qquad\qquad X_{k,1}, \dots, X_{k,p_k-1}, -X_{k,1} - \cdots - X_{k,p_k-1}, Y),
\end{aligned}$$

then $\mathcal{A}_K$ is equivalent to the restriction to V of the matrix $-\mathcal{E}_K$, where $\mathcal{E}_K$ is given by:

$$\mathcal{E}_K = \left[\begin{array}{cccccc}
 & & & & & 0 \\
C_{p_1} & & & & & \\
 & & & & & 0 \\
 & & & & & 1 \\
 & & \dots & & & \dots \\
 & & & & & 0 \\
 & & & & C_{p_k} & 0 \\
 & & & & & 1 \\
1\ \text{-}1\ 0 \dots 0 & 1\ \text{-}1\ 0 \dots 0 & \dots & 1\ \text{-}1\ 0 \dots 0 & 1\ \text{-}1\ 0 \dots 0 & k-1
\end{array}\right]$$

where C_p is the $p \times p$ permutation matrix

$$C_p = \begin{bmatrix}
0 & 1 & 0 & \dots & & 0 \\
0 & 0 & 1 & 0 & \dots & 0 \\
 & & \dots & & & \\
0 & \dots & & & 0 & 1 \\
1 & 0 & \dots & & 0 & 0
\end{bmatrix}.$$

The eigenvalues of $\mathcal{E}_K$ are thus the negatives of the eigenvalues of $\mathcal{A}_K$, together with 1 counted with multiplicity k. The characteristic polynomial of $\mathcal{E}_K$ is

$$\mathrm{Ch}_{\mathcal{E}_K}(x) = (x^{p_1} - 1) \dots (x^{p_k} - 1)(x - k + 1) + (x - 1) \sum_{i=1}^{k} (x^{p_1} - 1) \dots (\widehat{x^{p_i} - 1}) \dots (x^{p_k} - 1)$$

(This can be seen, for example, by cofactor expansion with respect to the last column.) Dividing by $(x-1)^k$, gives

$$\begin{aligned}
\frac{\mathrm{Ch}_{\mathcal{E}_K}(x)}{(x-1)^k} &= [p_1] \dots [p_k](x - k + 1) + \sum_{i=1}^{k} [p_1] \dots \widehat{[p_i]} \dots [p_k] \\
&= Q_{p_1, \dots, p_k}(x)
\end{aligned}$$

The claim follows.

$\blacksquare$

By definition, a pretzel link depends only on the cyclic ordering of its twists. Theorem 1.2 implies the following stronger statement.

Corollary 2.1 *The Alexander polynomial for the $K_{p_1,\dots,p_k}$ does not depend on the ordering of $p_1,\dots,p_k$.*

Example: Theorem 1.2 implies that for the $(p,q,-2)$ pretzel knot, where p,q are odd integers, the Alexander polynomial is given by

$$\Delta_{p,q,2}(x) = \frac{1+2x+x^{1+p}+x^{1+q}-x^3-x^{p+q}+x^{p+2}+x^{q+2}+2x^{p+q+2}+x^{3+p+q}}{(1+x)^3}.$$

3 Minimality of $L(x)$

In this section, we give some properties of the functions $\mathrm{R}_{p_1,\dots,p_k}(x)$ and $Q_{p_1,\dots,p_k}(x)$, and prove Theorem 1.3.

First we verify that Lehmer's examples of degrees 2,4,6,8 (see [Leh]), are all factors of $Q_{p_1,\dots,p_k}(x)$, for some $p_1,\dots,p_k$:

$$\begin{aligned}
Q_{2,2,2,2,2}(x) &= (1-3x+x^2)(1+x)^3\\
Q_{4,4,4}(x) &= (1-x-x^2-x^3+x^4)(1+x^2)^2(1+x)^2\\
Q_{3,3,4}(x) &= (1-x^2-x^3-x^4+x^6)(1+x+x^2)\\
Q_{2,4,5}(x) &= 1-x^3-x^4-x^5+x^8.
\end{aligned}$$

The Lehmer polynomial $L(x)$ equals $Q_{2,3,7}(x)$.

The real roots of $Q_{p_1,\dots,p_k}(x)$ can be described in terms of the value

$$\begin{aligned}
\chi(p_1,\dots,p_k) &= \mathrm{R}_{p_1,\dots,p_k}(1)\\
&= \frac{1}{p_1}+\dots+\frac{1}{p_k}-k+2
\end{aligned}$$

as seen in the following proposition.

Lemma 3.1 *The triple $(2,3,7)$ gives the maximum negative value of*

$$\chi(p_1,\dots,p_k) = (2-k)+\frac{1}{p_1}+\dots+\frac{1}{p_k}$$

for any $p_1,\dots,p_k \geq 2$.

Proof. For $k \geq 5$, we have

$$2-k+\frac{1}{p_1}+\dots+\frac{1}{p_k} \leq 2-k+\frac{k}{2} = 2-\frac{k}{2} \leq -\frac{1}{2}.$$

For $k = 3$ and $k = 4$, the best possible are

$$\chi(2,3,7) = -\frac{1}{42}$$

and

$$\chi(2,2,2,3) = -\frac{1}{6},$$

respectively. Thus, $\chi(2,3,7)$ is the largest possible.

■

The following two Lemmas follow from Theorem 1.4, but since they can be simply verified, we include proofs here.

Lemma 3.2 *If*

$$\chi(p_1, \ldots, p_k) \geq 0,$$

then $Q_{p_1,\ldots,p_k}$ is a product of cyclotomic polynomials.

Proof. When $k = 1, 2$, we have

$$Q_p(x) = 1 + x + \cdots + x^{p-1}$$

and

$$Q_{p,q}(x) = 1 + x + \cdots + x^{p+q-1}$$

so the Mahler measure is always one.

For any k, we have

$$\chi(1, p_2, \ldots, p_k) = \chi(p_2, \ldots, p_k)$$

so we can assume that all the p_i are greater than 1. If $k \geq 4$, then we have

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} \leq \frac{k}{2} \leq k - 2,$$

with equality only for $(2,2,2,2)$. Let $\Phi_n(x)$ denote the nth cyclotomic polynomial, that is, the minimal polynomial for the nth root of unity. If $k = 3$ or 4, the only possibilities are

$$\begin{aligned}
Q_{2,2,n}(x) &= (1 + x^{n+1})(1 + x) \\
Q_{2,3,3}(x) &= (1 - x^2 + x^4)(1 + x + x^2) = \Phi_{12}(x)\Phi_(x) \\
Q_{2,3,4}(x) &= (1 - x^3 + x^6)(1 + x) = \Phi_{18}(x)\Phi_2(x) \\
Q_{2,3,5}(x) &= 1 + x - x^3 - x^4 - x^5 + x^7 + x^8 = \Phi_{30}(x) \\
Q_{2,3,6}(x) &= (1 + x)(1 - x)^2(1 + x + x^2)(1 + x + x^2 + x^3 + x^4) \\
Q_{2,4,4}(x) &= (x - 1)^2(1 + x)^2(1 + x^2)(1 + x + x^2) \\
Q_{3,3,3}(x) &= (x - 1)^2(1 + x)(1 + x + x^2)^2 \\
Q_{2,2,2,2}(x) &= (x - 1)^2(x + 1)^3.
\end{aligned}$$

In all the above examples, the Mahler measure is one.

■

Lemma 3.3 *If*

$$\chi(p_1, \ldots, p_k) < 0.$$

Then $Q_{p_1,\ldots,p_k}(x)$ has exactly one real root greater than 1.

Proof. Since

$$\mathrm{R}_{p_1,\ldots,p_k}(0) = 1$$

and

$$\mathrm{R}_{p_1,\ldots,p_k}(1) = \chi(p_1, \ldots, p_k) < 0,$$

the function $\mathrm{R}_{p_1,\ldots,p_k}$ must have a real root strictly between 0 and 1.

For all integers $p > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1-x}{1-x^p} = \frac{(1-p)x^p + px^{p-1} - 1}{(x-1)^{2p}}.$$

On the interval $[0, 1]$, the numerator is increasing and the denominator is decreasing:

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-p)x^p + px^{p-1} - 1 = p(p-1)x^{p-2}(1-x) > 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(x-1)^{2p} = 2p(x-1)^{2p-1} < 0.$$

Therefore, $\mathrm{R}_{p_1,\ldots,p_k}(x)$ is strictly decreasing and concave up on $[0, 1]$, and the root is unique.

Since $Q_{p_1,\ldots,p_k}(x)$ is palindromic, it follows that $\mathrm{R}_{p_1,\ldots,p_k}$, and hence $Q_{p_1,\ldots,p_k}$ has exactly one root greater than 1.

■

We are now ready to prove the main result of this section.

Proof of Theorem 1.3.

Observe that for $x > 0$ (and any k), the values of $\mathrm{R}_{p_1,\ldots,p_k}(x)$ strictly decrease if one increases any of the $p_1, \ldots, p_k$.

By Lemma 3.2 and 3.3, we know that $Q_{p_1 \ldots,p_k}(x)$ has a single root $\alpha_{p_1,\ldots,p_k}$ outside the unit circle if and only if $\chi(p_1, \ldots, p_k) < 0$. In the proof of Lemma 3.3 it was shown that the graph of $\mathrm{R}_{p_1,\ldots,p_k}(x)$ is concave up on $[0, 1]$. Thus, the zero $x_0 \in [0, 1]$ of $\mathrm{R}_{p_1,\ldots,p_k}(x)$ is strictly less than the x-intercept of the line joining $(0, 1)$ and $(1, \mathrm{R}_{p_1,\ldots,p_k}(1))$, giving

$$x_0 < \frac{1}{1 - \chi(p_1, \ldots, p_k)},$$

and

$$\alpha_{p_1,\ldots,p_k} = \frac{1}{x_0} > 1 - \chi(p_1, \ldots, p_k).$$

When $k \geq 4$, Lemma 3.1 implies that aside from the case $(2,2,2,3)$ we have

$$\alpha_{p_1,\ldots,p_k} > 1 - \chi(2,2,2,4) = \frac{5}{4} = 1.25 > \alpha_{2,3,7}.$$

The remaining case $(2,2,2,3)$, for $k = 4$, can be checked by computer:

$$\alpha_{2,2,2,3} = 1.72208\cdots > \alpha_{2,3,7},$$

as can the minimal cases for $k = 3$, which finishes the proof:

$$\begin{aligned} \alpha_{3,3,4} &= 1.40127\cdots > \alpha_{2,3,7} \\ \alpha_{2,4,5} &= 1.28064\cdots > \alpha_{2,3,7}. \end{aligned}$$

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