Waves in Uniform Flow on Half Plane

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The normal mode method is employed to identify normal modes of linear Euler equations in uniform mean flow. The physical domain is a two-dimensional half open space. Three normal modes: acoustic wave, vorticity wave, and entropy wave, are explored and discussed.

Mean Flow Parallel to the Interface

Linear Euler Equations and the Normal Mode Method

The Euler equations for an ideal adiabatic gas are:

$$\frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla)\rho + \rho \nabla \cdot \vec{u} = 0, \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho}, \tag{2}$$

$$\frac{\partial s}{\partial t} + (\vec{u} \cdot \nabla)s = 0, \tag{3}$$

$$\frac{\rho}{c_p}\delta s = \frac{1}{a^2}\delta p - \delta \rho. \tag{4}$$

Consider the uniform mean velocity u_0 in x direction. Quantities with prime represent perturbation, and those with subscript 0 represent ambient values: $s = s_0 + s'$, $p = p_0 + p'$, $\vec{u} = \vec{u}_0 + \vec{u}'$, etc. The perturbation is small compared with the ambient quantities, so that the linearized equations apply:

$$\frac{1}{\rho_0 a_0^2}\frac{D_0 p'}{Dt} + \nabla \cdot \vec{u}' = 0, \tag{5}$$

$$\frac{D_0 \vec{u}'}{Dt} = -\frac{1}{\rho_0}\nabla p', \tag{6}$$

$$\frac{D_0 s'}{Dt} = 0, \tag{7}$$

$$\frac{\rho_0}{c_p} s' = \frac{1}{a_0^2} p' - \rho'. \tag{8}$$

where $\frac{D_0}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}$.

Consider the mean flow parallel to the interface $y = 0$. The half space is above x-axis. The physical domain is:

$$x \in (-\infty, \infty),\ y \in [0, \infty).$$

We will use the normal mode method to solve the problem. The normal mode method can identify all the normal modes of a system. The normal modes are building blocks for the solutions of the Euler equations with an initial field. Instabilities and its characteristics can also be identified with this method.

The mean flow is homogeneous and the physical domain extends infinitely in x direction. One can use exponential functions as the trial solutions on x and t and assume the next form of solution:

$$\begin{pmatrix} p'(x,y,t) \\ \vec{u}'(x,y,t) \\ s'(x,y,t) \end{pmatrix} = \begin{pmatrix} \hat{p}(\alpha,y,\omega) \\ \hat{\vec{u}}(\alpha,y,\omega) \\ \hat{s}(\alpha,y,\omega) \end{pmatrix} e^{i(\alpha x - \omega t)}. \tag{9}$$

Substituting (9) into linear Euler equations (5)~(8), one obtains:

$$(\omega - \alpha u_0)\hat{p} - \rho_0 a_0^2 \alpha \hat{u} + i\rho_0 a_0^2 \frac{\partial \hat{v}}{\partial y} = 0, \tag{10}$$

$$(\omega - \alpha u_0)\hat{u} = \frac{1}{\rho_0} \alpha \hat{p}, \tag{11}$$

$$(\omega - \alpha u_0)\hat{v} = \frac{1}{i\rho_0} \frac{d\hat{p}}{dy}, \tag{12}$$

$$(\omega - \alpha u_0)\hat{s} = 0, \tag{13}$$

$$\frac{\rho_0}{c_p}\hat{s} = \frac{1}{a_0^2}\hat{p} - \hat{\rho}. \tag{14}$$

Acoustic Wave

Now one needs to solve $\hat{p}$, $\hat{\vec{u}}$, and $\hat{s}$ from (10)~(14). First assume $\omega - \alpha u_0 \neq 0$, then one has:

$$\frac{\partial^2 \hat{p}}{\partial y^2} + \lambda^2 \hat{p} = 0, \tag{15}$$

$$\hat{u} = \frac{\alpha}{\rho_0(\omega - \alpha u_0)}\hat{p}, \tag{16}$$

$$\hat{v} = \frac{1}{i\rho_0(\omega - \alpha u_0)}\frac{d\hat{p}}{dy}, \tag{17}$$

$$\hat{s} = 0. \tag{18}$$

where

$$\beta^2 = \left(\frac{\omega}{a_0} - \alpha M_0\right)^2 - \alpha^2, \; M_0 = u_0 / a_0. \tag{19}$$

We will choose appropriate branch cuts of $\left[(\omega / a_0 - \alpha M_0)^2 - \alpha^2\right]^{1/2}$ in α-plane to ensure that β always has nonnegative imaginary part:

$$0 \le \arg(\beta) < \pi.$$

Then the general solution to Eq.(15) is:

$$\hat{p} = A(\alpha,\omega)e^{i\beta y} + B(\alpha,\omega)e^{-i\beta y}. \tag{20}$$

A and B are to be determined by boundary conditions at $y = 0$ and $y = \infty$.

From Eqs.(16) and (17), vorticity on the half space is zero:

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \hat{\Omega}e^{i(\alpha x - \omega t)}, \; \hat{\Omega} = i\alpha\hat{v} - \frac{d\hat{u}}{dy} = 0. \tag{21}$$

There is no entropy either in this solution, Eq.(18). Therefore this set of solutions represents only acoustics waves. With subscript *a*, the acoustic solutions can be written as:

$$\hat{p}_a = A(\alpha,\omega)e^{i\beta y} + B(\alpha,\omega)e^{-i\beta y}, \tag{22}$$

$$\hat{u}_a = \frac{\alpha}{\rho_0(\omega - \alpha u_0)}\left[A(\alpha,\omega)e^{i\beta y} + B(\alpha,\omega)e^{-i\beta y}\right], \tag{23}$$

$$\hat{v}_a = \frac{\beta}{\rho_0(\omega - \alpha u_0)}\left[A(\alpha,\omega)e^{i\beta y} - B(\alpha,\omega)e^{-i\beta y}\right], \tag{24}$$

$$\hat{s}_a = 0, \; \hat{\rho}_a = \hat{p}_a / a_0^2. \tag{25}$$

In this set of equations, A and B are determined by boundary conditions at $y = 0$ and $y = \infty$. β is computed from Eq.(19). Eq.(19) is the relation between frequency ω and wave numbers α and β. It is the dispersion relation for sound waves. In general, ω and α can be any complex numbers. If they are Fourier Transform variables, which means Eqs.(10)~(14) are Fourier Transforms of Eqs.(5)~(8), then ω and α are real numbers. To illustrate how to compute β, we will assume α and ω are real. From (19), depending on M_0, there are three situations for choosing the branch cuts in α-plane to determine β.

If $M_0 < 1$, ω is real and positive, choose the branch cuts as in Fig.1:

$$\beta = \sqrt{1 - M_0^2}\sqrt{\gamma_+\gamma_-}\, e^{i(\theta_+ + \theta_- + \pi)/2}, \quad \frac{\theta_+ + \theta_-}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{26}$$

With these branch cuts, the imaginary part of β is non-negative.

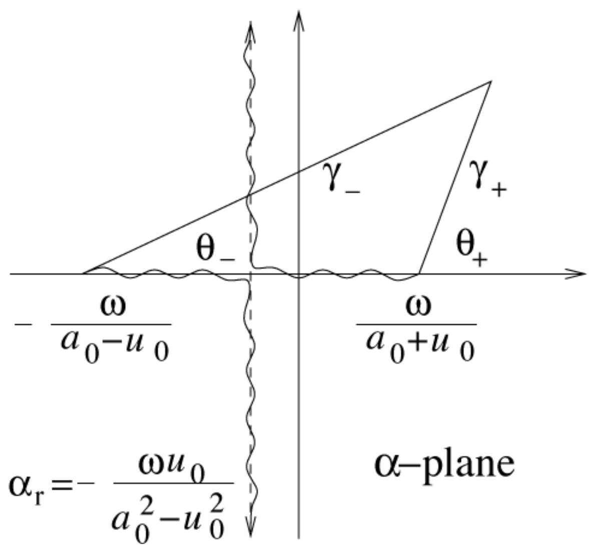


Fig.1, Branch cuts in α-plane when $M_0 < 1$, ω is real and positive.

When α is real,

$$\beta = \begin{cases} \sqrt{1 - M_0^2}\sqrt{\gamma_+\gamma_-}, & \text{if } -\dfrac{\omega}{a_0 - u_0} \le \alpha \le \dfrac{\omega}{a_0 + u_0} \\ i\sqrt{1 - M_0^2}\sqrt{\gamma_+\gamma_-}, & \text{if } \alpha < -\dfrac{\omega}{a_0 - u_0} \text{ or } \alpha > \dfrac{\omega}{a_0 + u_0} \end{cases}. \tag{27}$$

For incompressible flow, $a_0 \to \infty$,

$$\begin{cases} \beta = i\alpha, & \text{real}(\alpha) \ge 0; \\ \beta = -i\alpha, & \text{real}(\alpha) < 0. \end{cases} \tag{28}$$

From Eq.(22), the time domain sound pressure is:

$$p_a(x,y,t) = A(\alpha,\omega)e^{i(\alpha x + \beta y - \omega t)} + B(\alpha,\omega)e^{i(\alpha x - \beta y - \omega t)}. \tag{29}$$

When α is real and $-\omega/(a_0-u_0) \le \alpha \le \omega/(a_0+u_0)$, β is real and Eq.(29) represents two propagating plane waves. Plane wave $A(\alpha,\omega)e^{i(\alpha x+\beta y-\omega t)}$ propagates to the far field $y=\infty$; plane wave $B(\alpha,\omega)e^{i(\alpha x-\beta y-\omega t)}$ propagates towards the boundary at $y=0$. The propagation angle of plane wave $A(\alpha,\omega)e^{i(\alpha x+\beta y-\omega t)}$ as α changes from 0 to $\omega/(a_0+u_0)$ is shown in Fig.2. The wave propagates normal to the interface to the far field when $\alpha=0$, Fig.3(a). As α increases, the wave slants toward the flow direction, Fig.3(b). Eventually the wave propagates along the interface when $\alpha=\omega/(a_0+u_0)$, Fig.3(c). The wave amplitude is constant along the phase perpendicular to the propagation direction. Phase speed of the wave is $\omega/\sqrt{\alpha^2+\beta^2} = a_0/|1-\alpha u_0/\omega|$, shown in Fig.4. It increases from a_0 to maximum $a_0|1+M_0|$ as α changes from 0 to $\omega/(a_0+u_0)$. In this α range, the phase speed of the wave in x direction , $\omega/\alpha \ge a_0+u_0$, is supersonic.

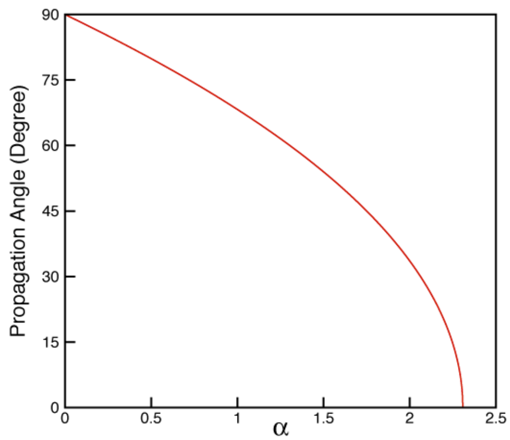
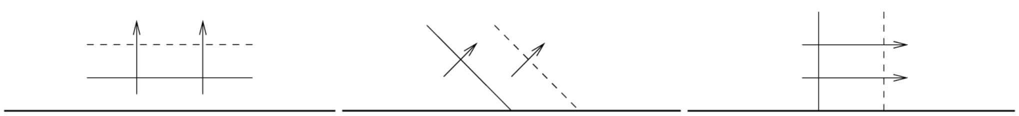


Fig.2, Plane wave propagation angle vs. α. ($\omega/a_0=3$, $M_0=0.3$)



(a) $\alpha=0$ (b) $0<\alpha<\omega/(a_0+u_0)$ (c) $\alpha=\omega/(a_0+u_0)$

Fig.3, Plane waves for $0 \le \alpha \le \omega/(a_0+u_0)$.

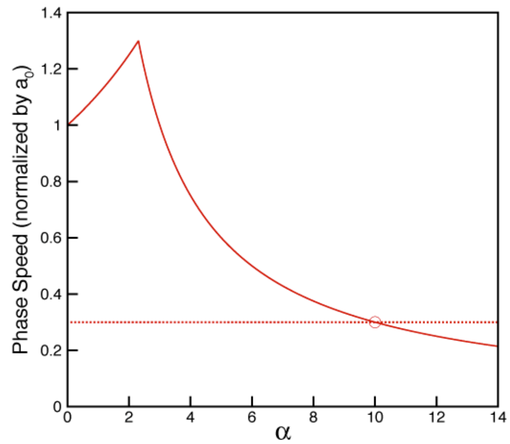


Fig.4, Phase speed vs. $|\alpha|$. ($\omega / a_0 = 3,\ M_0 = 0.3$)

When $\alpha > \omega/(a_0 + u_0)$ and $\alpha \neq \omega/u_0$, or $\alpha < -\omega/(a_0 - u_0)$, β is purely imaginary. Fig.5 shows the variation of β as $\alpha > \omega/(a_0 + u_0)$. The wave pattern is shown in Fig.6. The wave only propagates in x direction and the amplitude decays in y direction. The effect of disturbances is local to the interface. In this $|\alpha|$ range, the wave doesn't propagate to the far field. Phase speed of the wave is subsonic: $\omega/\alpha < a_0 + u_0$, which is not related to sound speed a_0. Fig.4 shows that the phase speed decreases as $|\alpha|$ increases from $\omega/(a_0 + u_0)$. This is the only wave pattern for incompressible flow. (Check out sound wave reflection/transmission at density interface, the vortex/shock wave interaction, where this kind of wave is excited.)

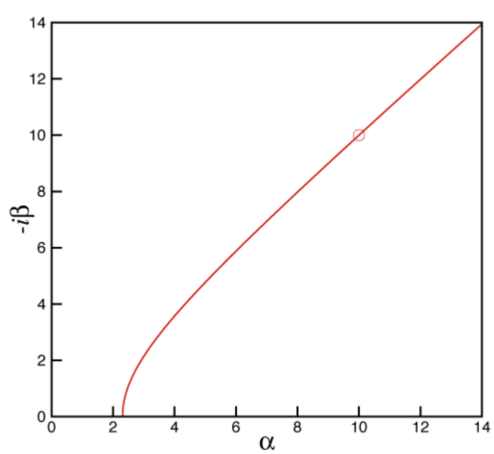


Fig.5, Imaginary β for $\alpha > \omega/(a_0 + u_0)$ and $\alpha \neq \omega/u_0$.

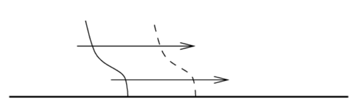


Fig.6, waves for $\alpha > \omega/(a_0 + u_0)$ and $\alpha \neq \omega/u_0$.

In the above analysis, $\alpha \neq \omega / u_0$ is assumed. When $\alpha = \omega / u_0$, the phase speed is equal to the mean flow velocity. This point is indicated by a circle In Figs.(4)&(5). Mathematically this is a singular point for acoustic equations (15)~(17). As $\alpha \to \omega / u_0$, acoustic velocities approach infinity. In physics this corresponds to the resonance phenomena. In this case the exponential functions we used as the trial solutions are not suitable. We have to use other trial functions, or appeal to Laplace Transform method. However even the mathematical solution is possible, there is no physically acceptable acoustic solution. The physical model (invisid and linear) breaks down. The sound wave can not be damped by viscosity; it grows infinitively due to linearity. One way to obtain a physically acceptable acoustic solution is to use a better model including viscosity and nonlinearity, such as the full N.S. equations. Another way is to appeal to analytical continuation. α can be treated as a complex number with positive imaginary part, and the integration path in the Inverse Fourier Transform should be detoured accordingly to avoid the singular point. This is equivalent to introduce artificial damping into the system, or apply Laplace Transform on *t*. In Laplace transform method ω is the Laplace transform variable, whose integral contour is above all singularities. The inverse Laplace transform has no singularity problem along this integral contour.

For a complex α, the wave is a decayed plane wave propagating in an oblige direction.

If $M_0 = 1$, choose the branch cut is as in Fig.7:

$$\beta = i\sqrt{\frac{2\omega}{a_0}}\sqrt{\alpha - \frac{\omega}{2a_0}} \ . \tag{30}$$

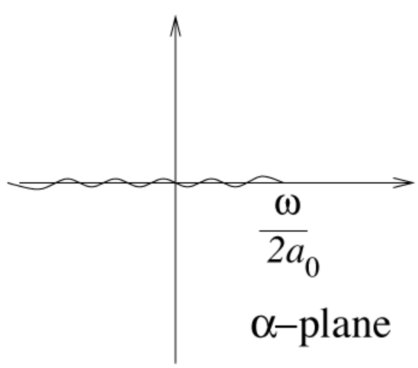


Fig.7, Branch cut in α-plane when $M_0 = 1$.

If $M_0 > 1$, choose the branch cuts as in the Fig.8:

$$\beta = \sqrt{M_0^2 - 1}\sqrt{r_+ r_-}\, e^{i(\theta_+ + \theta_-)/2}, \ \frac{\theta_+ + \theta_-}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{31}$$

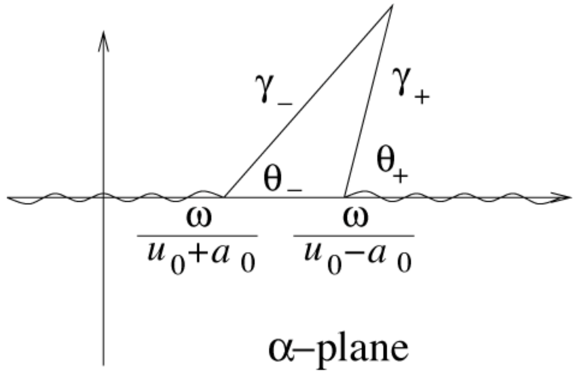


Fig.8, Branch cuts in α-plane when $M_0 > 1$.

An important property of acoustic wave can be derived from Eqs.(10), (11) and (12):

$$\frac{1}{2}\rho_0(\omega - \alpha u_0)^*\left(\hat{u}_a\hat{u}_a^* + \hat{v}_a\hat{v}_a^*\right) = \frac{\omega - \alpha u_0}{2\rho_0 a_0^2}\hat{p}_a\hat{p}_a^*.$$

Since ω and α are real,

$$\frac{1}{2}\rho_0\left(\left|\hat{u}_a\right|^2 + \left|\hat{v}_a\right|^2\right) = \frac{1}{2\rho_0 a_0^2}\left|\hat{p}_a\right|^2. \tag{32}$$

$\frac{1}{2}\rho_0\left(\left|\hat{u}_a\right|^2 + \left|\hat{v}_a\right|^2\right)$ is the time averaged acoustic kinetic energy, and $\frac{1}{2\rho_0 a_0^2}\left|\hat{p}_a\right|^2$ the time averaged acoustic potential energy. Eq.(32) means that the time averages of acoustic kinetic energy and potential energy are equal.

Vorticity Wave

Now let's assume:

$$\omega - \alpha u_0 = 0. \tag{33}$$

If $\alpha = 0$ or $u_0 = 0$, then $\omega = 0$, the solution will be time-invariant, which is of no interest in the discussion. Therefore we can further assume $\alpha \neq 0$ and $u_0 \neq 0$. The next equations can be derived from (10)~(14):

$$\hat{p} = 0, \tag{34}$$

$$\alpha\hat{u} = i\frac{d\hat{v}}{dy}, \tag{35}$$

$$\frac{\rho_0}{c_p}\hat{s} = -\hat{\rho}. \tag{36}$$

From Eq.(34), there is no pressure fluctuations in this solution. From (35) and (36), velocities $\hat{u}$ and $\hat{v}$, and thermodynamic variable $\hat{s}$ can be determined separately.

First we discuss the velocity solution. One has this set of solutions:

$$\hat{p}_v = 0, \tag{37}$$
$$\hat{v}_v = f(\alpha, y, \omega), \tag{38}$$
$$\hat{u}_v = \frac{i}{\alpha}\frac{df}{dy}, \tag{39}$$
$$\hat{s}_v = 0. \tag{40}$$

$f(\alpha, y, \omega)$ is function of y. It can be set at any line $x = x_b$. Usually it is determined at upstream boundary.

The vorticity is

$$\hat{\omega} = i\alpha f - \frac{i}{\alpha}\frac{d^2 f(y)}{dy^2}, \tag{41}$$

and the dilation

$$\Delta = \nabla \cdot \vec{u} = \hat{\Delta} e^{i(\alpha x - \omega t)},\ \hat{\Delta} = i\alpha\hat{u} + {d\hat{v}}/{dy} = 0. \tag{42}$$

This solution is dilation free. It represents a vorticity wave, which is denoted by subscript v. Eq.(33) is the dispersion relation for vorticity waves.

One can study one Fourier component of the vorticity wave:

$$\hat{v}_v = f(\alpha, y, \omega) = e^{i\beta y}, \tag{43}$$
$$\hat{u}_v = -\frac{\beta}{\alpha} e^{i\beta y}. \tag{44}$$

Since $\left(-\frac{\beta}{\alpha}, 1\right) \cdot (\alpha, \beta) = 0$, the velocity direction is perpendicular to the wave propagation direction. Therefore a vorticity wave is a transverse wave.

Entropy Wave

The other set of solution to Eqs.(34)~(36), is:

$$\hat{p}_e = 0, \tag{45}$$
$$\hat{u}_e = 0, \tag{46}$$
$$\hat{v}_e = 0, \tag{47}$$
$$\hat{s}_e = g(\alpha, y, \omega) \tag{48}$$

Similar to $f(\alpha,y,\omega)$, $g(\alpha,y,\omega)$ can be determined at the upstream boundary. In this set of solutions only heat is involved. It is the entropy wave. An entropy wave only consists entropy fluctuation, or density fluctuation $\tilde{\rho}_e = -\rho_0 g(\alpha,y,\omega)/c_p$. Eq.(33) is also the dispersion relation for entropy waves.

Solutions in Time Domain

In the uniform mean flow, the three normal modes: acoustic, vorticity and entropy waves, have been identified. Once four boundary conditions are set: $f(\alpha,y,\omega)$ and $g(\alpha,y,\omega)$ at the upstream boundary, $A(\alpha,\omega)$ and $B(\alpha,\omega)$ at $y=0$ and $y=\infty$, the three waves for each pair of α,ω are uniquely determined from Eqs.(22)~(25), Eqs. (37)~(40), and Eqs. (45)~(48). Then the total solution for this α and ω is:

$$\hat{p} = \hat{p}_a,\ \hat{\vec{u}} = \hat{\vec{u}}_a + \hat{\vec{u}}_v,\ \hat{s} = \hat{s}_e. \tag{49}$$

In all the boundary conditions if there is no complex α with positive imaginary part for any real ω, there are no instabilities. Then the normal mode analysis is equivalent to the Fourier Transform method, and the time domain solutions can be obtained by the Inverse Fourier Transform. For example, the time domain pressure is:

$$p'(x,y,t) = \frac{1}{(2\pi)^2}\int_{\omega}\int_{\alpha}\hat{p}(\alpha,y,\omega)e^{i(\alpha x-\omega t)}d\alpha d\omega.$$

Mean Flow Perpendicular to the Interface

In this section we will consider the mean flow that is perpendicular to the interface. Mean flow u_0 is in x direction. The half open space is:

$$x \in [0,\infty),\ y \in (-\infty,\infty).$$

The physical domain extends infinitely in *y* direction. The normal mode method will be applied on *y* and *t*. Consider the next form of solutions:

$$\begin{Bmatrix} p'(x,y,t) \\ \vec{u}'(x,y,t) \\ s'(x,y,t) \end{Bmatrix} = \begin{Bmatrix} \hat{p}(x,\beta,\omega) \\ \hat{\vec{u}}(x,\beta,\omega) \\ \hat{s}(x,\beta,\omega) \end{Bmatrix} e^{i(\beta y-\omega t)}. \tag{50}$$

β and ω are real numbers. Substituting (50) into linear Euler equations (5)~(8), one has:

$$(-i\omega + u_0\frac{\partial}{\partial x})\hat{p} + \rho_0 a_0^2(\frac{\partial \hat{u}}{\partial x} + i\beta\hat{v}) = 0, \tag{51}$$

$$(-i\omega + u_0 \frac{\partial}{\partial x})\hat{u} = -\frac{1}{\rho_0}\frac{\partial \hat{p}}{\partial x}, \tag{52}$$

$$(-i\omega + u_0 \frac{\partial}{\partial x})\hat{v} = -\frac{ik}{\rho_0}\hat{p}, \tag{53}$$

$$(-i\omega + u_0 \frac{\partial}{\partial x})\hat{s} = 0, \tag{54}$$

$$\frac{\rho_0}{c_p}\hat{s} = \frac{1}{a_0^2}\hat{p} - \tilde{\rho}. \tag{55}$$

Acoustic Wave

To eliminate $\hat{u}$ and $\hat{v}$ from (51), one can apply $(-i\omega + u_0 \frac{\partial}{\partial x})$ on (51) and plug (52) and (53) into the equation to obtain:

$$(u_0^2 - a_0^2)\frac{\partial^2 \hat{p}}{\partial x^2} - 2i\omega u_0 \frac{\partial \hat{p}}{\partial x} - (\omega^2 - a_0^2 k^2)\hat{p} = 0\,. \tag{56}$$

This is a homogeneous linear second order ordinary differential equation when $M_0 \neq 1$. There is a trivial solution $\hat{p} = 0$. But first we will discuss the nontrivial solution.

If $M_0 < 1$, the nontrivial solution of (56) is:

$$\hat{p} = A(k,\omega)e^{i\kappa_+ x} + B(k,\omega)e^{i\kappa_- x}, \tag{57}$$

$$\kappa_\pm = \frac{-M_0\omega / a_0 \pm i\sqrt{1-M_0^2}\sqrt{k^2 - \omega^2/(a_0^2 - u_0^2)}}{1-M_0^2}. \tag{58}$$

$A(k,\omega)$ and $B(k,\omega)$ are to be determined by boundary conditions at $x = 0$ and $x = \infty$.

The branch cuts of $\sqrt{k^2 - \omega^2/(a_0^2 - u_0^2)}$ in ω-plane are shown in the Fig.9:

$$\sqrt{k^2 - \omega^2/(a_0^2 - u_0^2)} = \sqrt{r_+ r_-}\, e^{i(\theta_+ + \theta_-)/2},\ \frac{\theta_+ + \theta_-}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{59}$$

κ_+ always has positive imaginary part, and κ_- always has negative imaginary part.

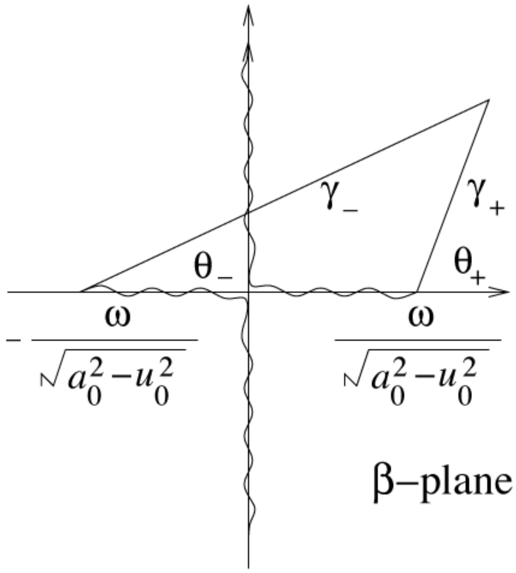


Fig.9, Branch cuts in β-plane when $M_0 < 1$.

Since β is real:

$$\sqrt{\beta^2 - \omega^2/(a_0^2 - u_0^2)} = \begin{cases} \sqrt{\gamma_+\gamma_-}, & \text{if } |\beta| \geq \dfrac{\omega}{a_0 + u_0}; \\ -i\sqrt{\gamma_+\gamma_-}, & \text{if } -\dfrac{\omega}{\sqrt{a_0^2 - u_0^2}} < \beta < \dfrac{\omega}{\sqrt{a_0^2 - u_0^2}}. \end{cases} \tag{60}$$

When $-\omega/\sqrt{a_0^2 - u_0^2} < \beta < \omega/\sqrt{a_0^2 - u_0^2}$, $\lambda_\pm$ are real and the waves are pure propagating plane waves. When $|\beta| \geq \omega/\sqrt{a_0^2 - u_0^2}$, the wave only propagates in y direction with phase speed ω/β, and the amplitude decays in x direction; disturbances are only local near the interface. For β with imaginary part, the wave is decayed plane waves propagating in an oblige direction.

Acoustic wave is an isentropic process. Therefore the set of acoustic solutions is:

$$\hat{p}_a = A(\beta,\omega)e^{i\lambda_+ x} + B(\beta,\omega)e^{i\lambda_- x}, \tag{61}$$

$$\hat{u}_a = \frac{\lambda_+ A(\beta,\omega)}{\rho_0(\omega - \lambda_+ u_0)}e^{i\lambda_+ x} + \frac{\lambda_- B(\beta,\omega)}{\rho_0(\omega - \lambda_- u_0)}e^{i\lambda_- x}, \tag{62}$$

$$\hat{v}_a = \frac{\beta}{\rho_0}\left[\frac{A(\beta,\omega)}{\omega - \lambda_+ u_0}e^{i\lambda_+ x} + \frac{B(\beta,\omega)}{\omega - \lambda_- u_0}e^{i\lambda_- x}\right], \tag{63}$$

$$\hat{s}_a = 0, \ \hat{p}_a = a_0^2 \hat{\rho}_a. \tag{64}$$

There is no entropy in the solution from Eq.(64). One can show that the vorticity in the physical domain is zero:

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \hat{\Omega} e^{i(\beta y - \omega t)},\ \hat{\Omega} = \frac{\partial \hat{v}}{\partial x} - i\beta \hat{u} = 0\,. \tag{65}$$

So this set of solutions only represent acoustics waves (denoted by subscript *a*), excluding any vortical and heat motions in the physical domain.

If $M_0 > 1$,

$$\alpha_{\pm} = \frac{M_0\omega / a_0 \pm \sqrt{M_0^2 - 1}\sqrt{\beta^2 + \omega^2/(u_0^2 - a_0^2)}}{M_0^2 - 1}. \tag{66}$$

The branch cut of $\sqrt{\beta^2 + \omega^2/(u_0^2 - a_0^2)} = \sqrt{\gamma_+\gamma_-}\, e^{i(\theta_+ + \theta_-)/2}$ in β-plane is shown in Fig.10, so that α_+ always has positive imaginary part, and α_- always has negative imaginary part.

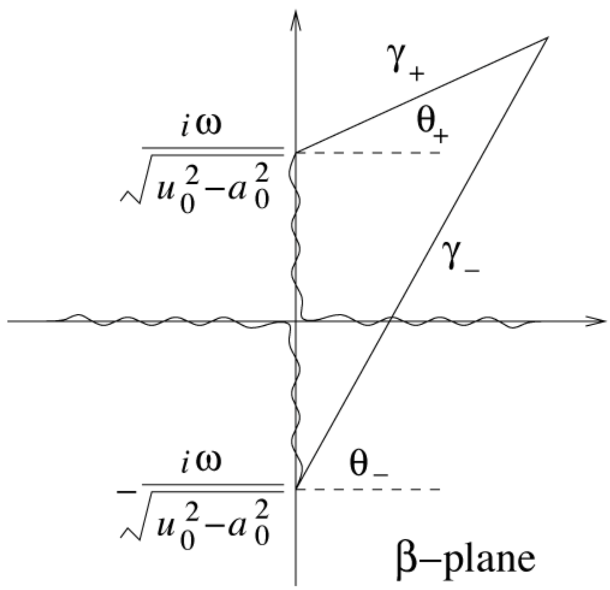


Fig.10, Branch cuts in β-plane when $M_0 > 1$.

If $M_0 = 1$, Eq.(56) degenerates to a first order PDE with the nontrivial solution:

$$\hat{p} = A(\beta,\omega)e^{i\alpha x},\ \alpha = \frac{\omega}{2a_0} - \beta^2 \frac{a_0}{2\omega}. \tag{67}$$

Vorticity wave

Now we return to the trivial solution of (56):

$$\hat{p} = 0\,. \tag{68}$$

Plugging (68) into (51)~(53), we have:

$$\frac{\partial \hat{u}}{\partial x} + i\square\hat{v} = 0, \tag{69}$$

$$(\square i\square + u_0 \frac{\partial}{\partial x})\hat{u} = 0\,, \tag{70}$$

$$(\square i\square + u_0 \frac{\partial}{\partial x})\hat{v} = 0\,. \tag{71}$$

Eq.(69) means it is dilation free. The solutions of the vortical wave are:

$$\hat{u}_v = C(\square,\square)e^{i(\square/u_0)x}, \tag{72}$$

$$\hat{v}_v = \square\frac{\square}{\square u_0}C(\square,\square)e^{i(\square/u_0)x}, \tag{73}$$

$$\hat{p}_v = 0, \tag{74}$$

$$\hat{s}_v = 0. \tag{75}$$

$C(\square,\square)$ should be determined by upstream boundary condition.

Entropy Wave

The other set of solutions is:

$$\hat{p}_e = 0, \tag{76}$$

$$\hat{u}_e = 0, \tag{77}$$

$$\hat{v}_e = 0, \tag{78}$$

$$\hat{s}_e = De^{i(\square/u_0)x},\ \tilde{\square}_e = \square\square_0\hat{s}_e/c_p. \tag{79}$$

Once the four boundary conditions: three at upstream boundary and one at downstream boundary, are set, unique solutions for acoustic, vorticity, and entropy waves will be determined.