Sound Propagation in Ducts

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In this section we discuss sound propagation in ducts. This is a typical case of acoustic waves in a confined environment. In the wall bounded directions, acoustic waves bounce back and forth to form standing wave patterns. In the axial direction acoustic waves propagate freely. We will show how to solve acoustic equations for these boundary conditions. The theory developed is very useful in applications such as the prediction and control of aircraft engine noise.

Sound Propagation in Rectangular Duct

Assume a uniform subsonic mean flow in a rectangular duct with width *d* and height *h*. The mean flow density is $\bar{\rho}$, pressure $\bar{p}$, and velocity $\bar{u}$ in +*x*-axis direction.

The scales used are: length *L*, velocity: sound speed a_0, time L/a_0, density: ambient density ρ_0, pressure: $\rho_0 a_0^2$, impedance: $\rho_0 a_0$. The linear Euler equations for the fluctuation quantities are:

$$\frac{\partial u_x}{\partial t} + \bar{u}\frac{\partial u_x}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x}, \tag{0-1}$$

$$\frac{\partial u_y}{\partial t} + \bar{u}\frac{\partial u_y}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial y}, \tag{0-2}$$

$$\frac{\partial u_z}{\partial t} + \bar{u}\frac{\partial u_z}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z}, \tag{0-3}$$

$$\frac{\partial p}{\partial t} + \bar{u}\frac{\partial p}{\partial x} + \gamma\bar{p}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0. \tag{0-4}$$

Variables without bars are perturbation quantities. Isentropic process is assumed; therefore density can be calculated from pressure: $\rho = p/\bar{a}^2$, $\bar{a} = \sqrt{\gamma\bar{p}/\bar{\rho}}$.

We use the *trial solution* and the *method of separation of variables* to solve the equations. In the *y*- and *z*- directions there are wall boundaries. In the *x*- direction the duct extends to infinity. Assume the next form of solutions:

$$\begin{bmatrix} u_x \\ u_y \\ u_z \\ p \end{bmatrix} = \text{Real}\left\{ \begin{bmatrix} \hat{u}_x(y,z) \\ \hat{u}_y(y,z) \\ \hat{u}_z(y,z) \\ \hat{p}(y,z) \end{bmatrix} e^{i(k_x x - \omega t)} \right\}. \tag{1}$$

Substituting (1) into the linear Euler equations, one obtains the following acoustic equations:

$$\frac{d^2\hat{p}}{dy^2} + \frac{d^2\hat{p}}{dz^2} + \beta^2\hat{p} = 0,\ \beta^2 = \left(\overline{\omega}/\overline{a}\right)^2 - k_x^2, \tag{2}$$

$$\hat{u}_x = \frac{k_x}{\overline{\rho}\overline{\omega}}\hat{p},\ \hat{u}_y = \frac{1}{i\overline{\rho}\overline{\omega}}\frac{d\hat{p}}{dy},\ \hat{u}_z = \frac{1}{i\overline{\rho}\overline{\omega}}\frac{d\hat{p}}{dz}, \tag{3}$$

when $\overline{\omega} = \omega - \overline{u}k_x = (\omega/k_x - \overline{u})k_x \neq 0$. The linear Euler equations with the isentropic process have two normal modes: acoustic mode ($\overline{\omega} \neq 0$, phase speed not equal to flow velocity), and vortical mode ($\overline{\omega} = 0$, phase speed equal to flow velocity). We are only interested in the acoustic modes; therefore $\overline{\omega} \neq 0$ and the acoustic equations in Eqs.(2&3) are chosen according to cht1.doc.

Rigid Wall Duct with Mean Flow

If the duct walls are rigid, the boundary conditions are

$$\hat{u}_y = 0, \text{ at } y = 0 \text{ and } y = h; \tag{4}$$

$$\hat{u}_z = 0, \text{ at } z = 0 \text{ and } z = d. \tag{5}$$

In the flow direction the wave amplitude must remain finite as $x \to \pm\infty$.

We try the separation of variable:

$$\hat{p}(y,z) = R(y)T(z).$$

Substituting it into (2) leads to:

$$\frac{1}{R}\frac{d^2R}{dy^2} + \frac{1}{T}\frac{d^2T}{dz^2} + \beta^2 = 0.$$

The first term on the left depends on *y* only. The second term depends on *z* only. β is constant with respect to *y* and *z*. Therefore the following equations must be true:

$$\frac{1}{R}\frac{d^2R}{dy^2} = k_y^2,\ \frac{1}{T}\frac{d^2T}{dz^2} = k_z^2,\ k_y^2 + k_z^2 = \beta^2. \tag{5-1}$$

k_y and k_z are constants. One can use the trial solution method to solve the two ODEs in (5-1). The general solution is:

$$\hat{p}(y,z) = (Ae^{ik_y y} + Be^{-ik_y y})(Ce^{ik_z z} + De^{-ik_z z}) . \tag{6}$$

To satisfy boundary conditions (4) and (5), we must have

$$A = B\,,\ k_y = m\pi / h\,; \tag{7}$$

$$C = D\,,\ k_z = n\pi / d\ . \tag{8}$$

m and n can be any integers. Usually we use non-negative integers: $m = 0,1,\cdots$, $n = 0,1,\cdots$. Due to the wall confinements, wave numbers k_y and k_z can no longer be any real numbers. They must be multiples of π / h and π / d respectively. For this reason, k_y and k_z are called eigenvalues. With (7) and (8), solution (6) is:

$$\hat{p}_{mn}(y,z) = A_{mn} \cos(m\pi y / h) \cos(n\pi z / d) . \tag{9}$$

It represents the standing waves in both the *y* and *z* directions. (Compared with the plane wave solutions discussed in cht1.doc.) Amplitude $A_{mn} = 4AC$ is to be determined by acoustic sources in the duct and the boundary conditions in the *x* direction. $\cos(m\pi y / h)$ and $\cos(n\pi z / d)$ are called the eigenfunctions. Each pair of (*m*,*n*) represents one acoustic mode with the solution (9) to Eq.(2). It is called a *normal mode* of the acoustic field.

According to Eq.(5-1), (7), and (8), eigenvalue β_{mn} is determined by the duct geometry:

$$\beta_{mn} = \pi \sqrt{(m/h)^2 + (n/d)^2}\ . \tag{10}$$

Eq.(2) implies only β_{mn}^2 matters; therefore only the positive square root is chosen. β_{mn} is *real* and *positive* for the rigid wall duct. For each eigenvalue β_{mn}, there are two solutions of k_{xmn} from Eq.(2):

$$k_{xmn}^{\pm} = \frac{-M\omega / \bar{a} \pm \sqrt{\omega^2 / \bar{a}^2 - (1-M^2)\beta_{mn}^2}}{1-M^2}\,,\ M = \bar{u} / \bar{a}\,. \tag{11}$$

They represent waves propagating in opposite directions in *x*. The direction of a wave is determined by group velocity. If $\beta_{mn} \le (\omega / \bar{a}) / \sqrt{1-M^2}$, $k_{xmn}^{\pm}$ is real and the mode is cut-on. Taking the derivative with respect to $k_{xmn}^{\pm}$ in Eq.(11), we obtain the group velocity

$$v_{gxmn}^{\pm} = \frac{\partial \omega}{\partial k_{xmn}^{\pm}} = \frac{1 - M^2}{-M/\bar{a} \pm \dfrac{\omega/\bar{a}^2}{\sqrt{\omega^2/\bar{a}^2 - (1-M^2)\beta_{mn}^2}}}.$$

Since $v_{gxmn}^{-} < 0$, the wave with k_{xmn}^{-} propagates in the -*x* direction. Noting

$$\frac{\omega/\bar{a}^2}{\sqrt{\omega^2/\bar{a}^2 - (1-M^2)\beta_{mn}^2}} \geq \frac{1}{\bar{a}},$$
$$v_{gxmn}^{+} > 0,$$

the wave with k_{xmn}^{+} propagates in the +*x* direction.

If $\beta_{mn} > (\omega/\bar{a})/\sqrt{1-M^2}$, we have

$$\sqrt{\omega^2/\bar{a}^2 - (1-M^2)\beta^2} = +i\sqrt{1-M^2}\ (\beta-\beta_-)^{1/2}\ (\beta-\beta_+)^{1/2},$$
$$k_{xmn}^{\pm} = \frac{-M\omega/\bar{a} \pm i\sqrt{1-M^2}(\beta-\beta_-)^{1/2}(\beta-\beta_+)^{1/2}}{1-M^2}, \tag{11-0}$$
$$\beta_{\pm} = \pm(\omega/\bar{a})/\sqrt{1-M^2}.$$

i, not –*i*, is chosen for the square root so $k_{xmn}^{+} = k_r + ik_i$ has positive imaginary part and $k_{xmn}^{-} = k_r - ik_i$ has negative imaginary part. The imaginary part of $k_{xmn}^{\pm}$ provides exponential attenuation factor $e^{\mp k_i x}$ in $e^{i(k_x^{\pm}x - \omega t)} = e^{\mp k_i x} e^{i(k_r x - \omega t)}$ in Eq.(1), with which the wave amplitude remains finite as $x \to \pm\infty$. $k_i > 0$ represents an evanescent or cut-off mode.

The mode becomes cut-on if the frequency increases so that:

$$\omega > \omega_{mn} = \bar{a}\beta_{mn}\sqrt{1-M^2}. \tag{11-1}$$

ω_{mn} is the cut-off frequency of mode (*m*,*n*). The cutoff ratio is defined as (Hubbard1995, p.159):

$$\sigma_{mn} = \frac{\omega}{\omega_{mn}} = \frac{\omega}{\bar{a}\beta_{mn}\sqrt{1-M^2}}. \tag{11-2}$$

$\sigma_{mn} \geq 1$ represents propagating, or cut-on, modes. The number of cut-on modes is limited at any frequency. A mean flow makes a mode easier to cut on. A plane wave is always cut-on: $m = 0$, $n = 0$, $\beta_{00} = 0$, $\sigma_{00} \to \infty$.

It is noted that the eigenvalues and eigenfunctions in the y or z direction are solely determined by the boundary conditions. They are independent of wave propagating directions. This is not true when there is mean flow and the walls are not rigid, as will be discussed later.

Substituting (9) into (1) and summing up all possible modes, we obtain the general solution of acoustic pressure:

$$p(x,y,z,t) = \mathrm{Real}\{\tilde{p}(x,y,z)e^{-i\omega t}\},$$
$$\tilde{p}(x,y,z) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \cos(j\pi y/h)\cos(l\pi z/d)(A_{jl}^{+}e^{ik_{xjl}^{+}x} + A_{jl}^{-}e^{ik_{xjl}^{-}x}). \tag{12}$$

Acoustic velocities can be calculated from Eq.(3). This solution represents a general sound field in a rectangular rigid-wall duct. Any specific pressure field can be expanded to the sum of normal modes of the rectangular duct. The amplitude of each mode $A_{mn}^{\pm}$ is determined by the sound sources and the boundary conditions in the x direction. Mode (m,n) is present in the sound field, or excited by the source, if $|A_{mn}^{\pm}| \neq 0$.

Given the total pressure field $\tilde{p}(x,y,z)$, how to calculate amplitude $A_{mn}^{\pm}$ of each individual mode? It is noted that eigenfunctions $\cos(m\pi y/h)$ are orthogonal:

$$\int_0^h \cos(m\pi y/h)\cos(j\pi y/h)dy = \begin{cases} h, & m = j = 0; \\ h/2, & m = j \neq 0; \\ 0, & m \neq j. \end{cases} \tag{12-1}$$

The same is true for $\cos(n\pi z/d)$. One can calculate the jth component of pressure using the orthogonality. Applying (12-1) on (12) and integrate the equation on the duct section at x, we have:

$$A_{mn}^{+}e^{ik_{xmn}^{+}x} + A_{mn}^{-}e^{ik_{xmn}^{-}x} = \frac{1}{\Lambda_{mn}}\int_0^d\int_0^h \tilde{p}(x,y,z)\cos(m\pi y/h)\cos(n\pi z/d)dydz, \tag{12-2}$$
$$\Lambda_{mn} = \begin{cases} hd, & m = n = 0; \\ hd/2, & (m = 0 \text{ and } n \neq 0) \text{ or } (m \neq 0 \text{ and } n = 0); \\ hd/4, & m \neq 0, n \neq 0. \end{cases}$$

Modes (m,n) are obtained by filtering out all other modes through the integration. Depending on locations of sound sources and duct terminations, one can further separate amplitudes A_{mn}^{+} and A_{mn}^{-}. For example, if sound sources are at the left side of the section and at the right hand side the duct extends to infinity (no reflection from this end), then

$$
\begin{aligned}
&A_{mn}^{-} = 0, \\
&A_{mn}^{+} = e^{-ik_{xmn}^{+}x} \frac{1}{\Lambda_{mn}} \int_0^d \int_0^h \tilde{p}(x,y,z)\cos(m\pi y/h)\cos(n\pi z/d)\,dydz.
\end{aligned} \tag{12-3}
$$

If waves propagate towards this section from both sides, the integration (12-2) at two axial locations x_1 and x_2 are needed and A_{mn}^{+} and A_{mn}^{-} can be solved from the linear equations:

$$
\begin{aligned}
A_{mn}^{+}e^{ik_{xmn}^{+}x_1} + A_{mn}^{-}e^{ik_{xmn}^{-}x_1} &= \frac{1}{\Lambda_{mn}} \int_0^d \int_0^h \tilde{p}(x_1,y,z)\cos(m\pi y/h)\cos(n\pi z/d)\,dydz, \\
A_{mn}^{+}e^{ik_{xmn}^{+}x_2} + A_{mn}^{-}e^{ik_{xmn}^{-}x_2} &= \frac{1}{\Lambda_{mn}} \int_0^d \int_0^h \tilde{p}(x_2,y,z)\cos(m\pi y/h)\cos(n\pi z/d)\,dydz.
\end{aligned}
$$

For a rigid-wall duct, β_{mn} is real so k_x can be calculated by (11) for cut-on modes and (11-0) for cut-off modes. In many practical problems such as in lined ducts, β_{mn} is complex. Then branch cuts must be inserted for $(\beta-\beta_-)^{1/2}$ and $(\beta-\beta_+)^{1/2}$ in (11-0) to make them single-valued. The branch cuts must be chosen so that the imaginary part of $(\beta-\beta_-)^{1/2}\,(\beta-\beta_+)^{1/2}$ is not negative so the magnitude of the wave is finite as $x \to \pm\infty$.

We choose the branch cuts as shown in Fig.1, with which the phase range of $(\beta-\beta_-)^{1/2}\,(\beta-\beta_+)^{1/2}$ is $[-\pi/2, \pi/2]$. The phase range for k_{xmn}^{+} in (11-0) is $[0,\pi]$ so the amplitude of the right-propagating wave with factor $e^{ik_{xmn}^{+}x_1}$ remains finite as $x \to +\infty$. The phase range for k_{xmn}^{-} is $[-\pi, 0]$ and amplitude of the left-propagating wave remains finite as $x \to -\infty$.

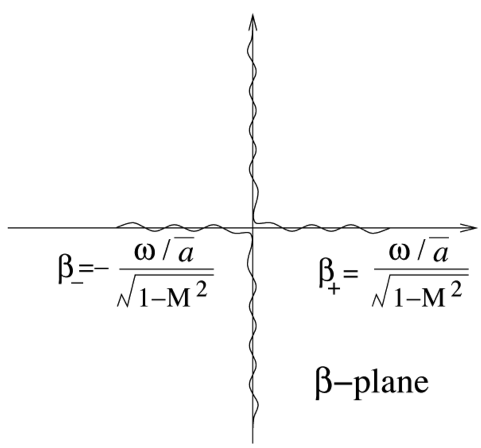


Fig.1, Branch cuts for $(\beta-\beta_-)^{1/2}$ and $(\beta-\beta_+)^{1/2}$ on the β-plane.

Sometimes we need to calculate eigenvalue β from wave number k_x:

$$\beta = \sqrt{(\overline{\omega} / \overline{a})^2 - k_x^2} = -i(k_x - k_d)^{1/2}(k_x - k_u)^{1/2}(1 - M^2)^{1/2}, \tag{13-1}$$
$$k_u = -(\omega/\overline{a})/(1-M),\ k_d = (\omega/\overline{a})/(1+M).$$

Branch cuts for $(k_x - k_d)^{1/2}$ and $(k_x - k_u)^{1/2}$ are chosen as in Fig.2. The phase range of $(k_x - k_d)^{1/2}(k_x - k_u)^{1/2}$ is $[0, \pi]$. The phase range of β is $[-\pi/2, \pi/2]$ so the real part of β is always positive.

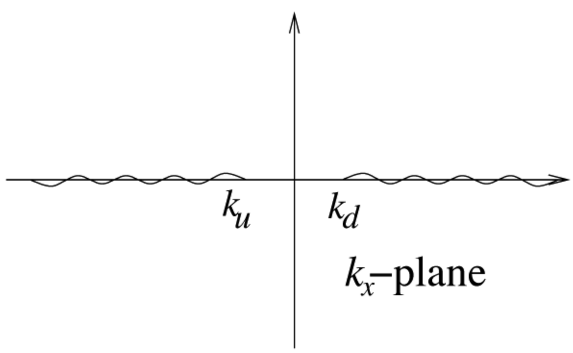


Fig.2, Branch cuts for $(k_x - k_d)^{1/2}$ and $(k_x - k_u)^{1/2}$ on the k_x-plane.

Soft Wall Duct without Mean Flow

To simplify the problem, we assume no mean flow in the duct and the same impedance in the opposite walls. The boundary conditions are:

$$\frac{\partial \hat{p}}{\partial y} = -i\omega\xi_y \hat{p} \text{ at } y = 0,\ \frac{\partial \hat{p}}{\partial y} = i\omega\xi_y \hat{p} \text{ at } y = h; \tag{13}$$
$$\frac{\partial \hat{p}}{\partial z} = -i\omega\xi_z \hat{p} \text{ at } z = 0,\ \frac{\partial \hat{p}}{\partial z} = i\omega\xi_z \hat{p} \text{ at } z = d. \tag{14}$$

ξ_y and ξ_z are admittance of the soft walls.

We can still use the trial solution Eq.(6). To facilitate the derivation, we rewrite Eq.(6) as:

$$\hat{p}(y,z) = A'C'\cos(k_y y + \phi_y)\cos(k_z z + \phi_z) \tag{15}$$

by defining $A = A'e^{i\phi_y}$, $B = A'e^{-i\phi_y}$, $C = C'e^{i\phi_z}$, and $D = C'e^{-i\phi_z}$.

To satisfy boundary conditions (13) in the *y* direction, we must have

$$k_y \mathrm{tg}(k_y h + \phi_y) = -i\omega\xi_y; \tag{16}$$
$$k_y \mathrm{tg}\phi_y = i\omega\xi_y. \tag{17}$$

From which we have,

$$\left[\left(\frac{\mathrm{i}\omega\xi_y}{k_y}\right)^2 - 1\right]\mathrm{tg}(k_y h) = 2\frac{\mathrm{i}\omega\xi_y}{k_y}\,. \tag{18}$$

Eigenvalue k_y can be solved from this equation using the grid search/Newton iteration method.

Another way to solve equations (16) and (17) is to note:

$$\mathrm{tg}(k_y h + \phi_y) = -\mathrm{tg}(\phi_y)\,. \tag{19}$$

One possible solution is $k_y h + \phi_y = -\phi_y$, then:

$$\begin{aligned}\phi_y &= -\frac{1}{2}k_y h,\\ k_y \mathrm{tg}(\frac{1}{2}k_y h) &= -i\omega\xi_y,\end{aligned} \tag{20}$$

which corresponds to the even symmetric solution in Morse&Ingard1968, p.504.

The other possible solution to Eq.(19) is $k_y h + \phi_y = -\phi_y + \pi/2$, *i.e.*,

$$\begin{aligned}\phi_y &= -\frac{1}{2}k_y h + \frac{\pi}{2},\\ k_y \mathrm{ctg}(\frac{1}{2}k_y h) &= i\omega\xi_y.\end{aligned} \tag{21}$$

which corresponds to the odd symmetric solution in Morse&Ingard1968, p.504.

Similarly k_z can be solved. Eigenvalue k_y or k_z is the same for the wave propagating in +*x* or -*x* direction. However, eigenfunctions $\cos(k_y y + \phi_y)$ or $\cos(k_z z + \phi_z)$ are no longer orthogonal as in the hard wall case.

If there is a mean flow in the soft wall duct, eigenvalues k_y and k_z are no longer independent of the wave propagation direction in *x*. They are different for the different propagation directions. This will be further discussed in the following section about the soft wall circular duct.

Sound Propagation in Circular Duct

Scales used here are: length: duct diameter D, velocity: sound speed a_0, time: D/a_0, density: density ρ_0, pressure: $\rho_0 a_0^2$, impedance: $\rho_0 a_0$.

The mean flow: $\bar{\rho}$ and $\bar{p}$ are constant; mean velocity $\bar{u}$ is in +x-axis direction and constant.

The Euler equations for the fluctuation quantities are:

$$\frac{\partial u_x}{\partial t} + \bar{u}\frac{\partial u_x}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x}, \tag{21-1}$$

$$\frac{\partial u_r}{\partial t} + \bar{u}\frac{\partial u_r}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial r}, \tag{21-2}$$

$$\frac{\partial u_\phi}{\partial t} + \bar{u}\frac{\partial u_\phi}{\partial x} = -\frac{1}{\bar{\rho}}\frac{\partial p}{r\partial \phi}, \tag{21-3}$$

$$\frac{\partial p}{\partial t} + \bar{u}\frac{\partial p}{\partial x} + \gamma\bar{p}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\phi}{r\partial \phi}\right) = 0. \tag{21-4}$$

(Density can be calculated directly from pressure: $\rho = p/\bar{a}^2$, $\bar{a} = \sqrt{\gamma\bar{p}/\bar{\rho}}$.)

In the radial direction there is a wall boundary and in the circumferential direction there is a periodic boundary. In the x direction the duct extends to infinity. We assume the form of solutions:

$$\begin{bmatrix} u_x \\ u_r \\ u_\phi \\ p \end{bmatrix} = \mathrm{Real}\left\{\begin{bmatrix} \hat{u}_x(r,\phi) \\ \hat{u}_r(r,\phi) \\ \hat{u}_\phi(r,\phi) \\ \hat{p}(r,\phi) \end{bmatrix} e^{i(k_x x - \omega t)}\right\}. \tag{22}$$

Substituting it into the Euler equations, we have:

$$r^2\frac{d^2\hat{p}}{dr^2} + r\frac{d\hat{p}}{dr} + \frac{d^2\hat{p}}{d\phi^2} + r^2\beta^2\hat{p} = 0, \tag{23}$$

$$\hat{u}_x = \frac{k_x}{\bar{\rho}\bar{\varpi}}\hat{p},\ \hat{u}_r = \frac{1}{i\bar{\rho}\bar{\varpi}}\frac{d\hat{p}}{dr},\ \hat{u}_\phi = \frac{1}{i\bar{\rho}\bar{\varpi}}\frac{d\hat{p}}{rd\phi}. \tag{24}$$

where $\bar{\varpi} = \omega - \bar{u}k_x$. $\beta^2 = (\bar{\varpi}/\bar{a})^2 - k_x^2$ is the dispersion relationship for circular/annular ducts.

To solve $\hat{p}(r,\phi)$ from Eq.(23), assume the separation of variables:

$$\hat{p}(r,\phi) = R(r)\Phi(\phi)\,. \tag{25}$$

Substitute it into Eq.(23),

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + r^2\beta^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}.$$

The left hand side of the equation depends on r only. The right hand side is only a function of ϕ. Then they must be equal to the same constant:

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \nu^2, \tag{26}$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + r^2\beta^2 = \nu^2. \tag{27}$$

The general solution to Eq.(26) is:

$$\Phi(\phi) = Ae^{-i\nu\phi} + Be^{i\nu\phi}. \tag{28}$$

A, B and ν are to be determined by two boundary conditions in the ϕ direction.

Solutions to Eq.(27) are Bessel functions. It is either Bessel function of the first kind $J_\nu(r)$ representing standing waves in the radial direction in a circular duct, or the second kind $Y_\nu(r)$ for standing waves in annular ducts, or Hankel functions $H_\nu^{(1)}(r)$ and $H_\nu^{(2)}(r)$ (the third kind of Bessel functions) representing propagating waves (outward/inward, no ducts). The general solution in the duct is:

$$R(r) = CJ_\nu(\beta r) + DY_\nu(\beta r). \tag{29}$$

$J_\nu(\beta r)$, $Y_\nu(\beta r)$ are respectively the νth order first and second kinds of Bessel functions. They are two independent solutions to Bessel equation (27). C, D and β are determined by two boundary conditions in the radial direction.

For a *circular duct*, the periodic boundary condition applies in ϕ direction. The general solution to Eq.(26) is:

$$\Phi(\phi) = Be^{im\phi}, \tag{30}$$

where $\nu = m$ is an integer. In the radial direction, the boundary condition at $r = 0$ is that $\hat{p}$ must be finite. $Y_m(\beta r)$ is eliminated since it is infinitive at $r = 0$. Therefore the general solution of $R(r)$ is

$$R(r) = CJ_m(\beta r). \tag{31}$$

The general solution of $\hat{p}(r,\phi)$ for a circular duct is:

$$\hat{p}(r,\phi) = AJ_m(\beta r)e^{im\phi}. \tag{32}$$

β is to be determined by the boundary condition at $r = R_o$. ($R_o = 1/2$ is the radius of the duct.) All other variables can be computed from $\hat{p}(r,\phi)$ by Eq.(24).

As $|m|$ increases, $J_m(r)$ is small for small r and large for large r. Sound energy concentrates near the cylindrical duct wall for high $|m|$ mode.

Rigid Wall Circular Duct

For a rigid wall duct, the boundary condition at $r = R_o$ is

$$\hat{u}_r = 0. \tag{33}$$

To satisfy (33), one has the eigenvalue:

$$\beta_{mn} = j'_{mn} / R_0, \tag{34}$$

where j'_{mn} is the nth root of $J'_m(x) = 0$. (See Appendix for how to find j'_{mn}.) Since $J_{-m}(x) = (-1)^m J_m(x)$, eigenvalues are the same for $\pm m$: $\beta_{mn} = \beta_{-mn}$. And eigenfunctions satisfy: $J_{-m}(\beta_{-mn} r) = (-1)^m J_m(\beta_{mn} r)$.

The time domain solution of p is:

$$p = \text{Real}\left\{ \sum_{m=-\infty}^{\infty} e^{i(m\phi - \omega t)} \sum_{n=0}^{\infty} (A_{mn}^{+} e^{ik_{xmn}^{+} x} + A_{mn}^{-} e^{ik_{xmn}^{-} x}) J_m(r\beta_{mn}) \right\}. \tag{35}$$

β_{mn} and $J_m(r\beta_{mn})$ are the same for both directions in x. $J_m(r\beta_{mn})$ is orthogonal. (This is not true when there is mean flow and the duct wall is not rigid.) Wave number $k_{xmn}^{\pm}$ can be computed from Eq.(11) and the cut-on frequency from Eq.(11-1). For larger azimuthal mode (larger $|m|$), j'_{mn} is larger, so is the cut-on frequency. Therefore there is less number of cut-on radial modes for larger $|m|$.

Soft Wall Circular Duct

Assume a plug flow in the duct. The tangential discontinuity at the disturbed surface leads to the continuity of displacement at the wall (cht21.doc) and the Myer's impedance boundary condition (cht24.doc):

$$\left(1-\frac{\bar{u}k_x}{\omega}\right)\hat{p}=Z\hat{u}_r \text{ at } r=R_o. \tag{35-1}$$

Plugging $\hat{p}$ in Eq.(32) and $\hat{u}_r$ in Eq.(24) into the Myer's boundary condition, we have:

$$i\bar{\rho}(\omega-\bar{u}k_x)^2 J_m(\beta_m R_o)=Z\omega\beta J'_m(\beta_m R_o). \tag{35-2}$$

It is not trivial to solve eigenvalue β_m from this equation. The grid search/Newton's iteration method can be used. With mean flow, β_m is coupled with k_x. Therefore, it is no longer independent of the wave propagation direction as in the no flow soft wall case or as in the solid wall with mean flow case. β_m is different for waves propagating in +*x* and –*x* direction for soft wall duct with mean flow.

Sound Propagation in Annular Duct

An *annular duct* has an inner circular wall with radius R_i and an outer circular wall with radius R_o. The equations are (21-1)~(21-4), with scales: length: outer duct diameter $D=2R_o$, velocity: sound speed a_0, time: D/a_0, density: density ρ_0, pressure: $\rho_0 a_0^2$, impedance: $\rho_0 a_0$. And the solution form is (22).

The analysis of the periodic boundary condition and the general solution (30) for the circular duct still apply in the annular duct. In the radial direction, $r=0$ is not in the physical domain as in the circular duct. Therefore $Y_m(\beta_m r)$ in Eq.(29) should be kept in the general solution:

$$R(r)=CJ_m(\beta r)+DY_m(\beta r). \tag{36}$$

C, *D* and β are to be determined by two boundary conditions at $r=R_i$ and $r=R_o$. With $Y_m(\beta r)$, sound energy doesn't concentrate near the outer duct wall for high mode *m* as in a cylindrical duct.

Rigid Wall Annular Duct

For a rigid wall annular duct, the boundary conditions are

$$\hat{u}_r=0, \text{ at } r=R_i \text{ and } r=R_o. \tag{37}$$

First of all, one should check if $\beta=0$ is the eigenvalue. Since $Y_m(0r)\to-\infty$, *D* must be zero and $R(r)=CJ_m(0r)$. Since $J_m(0r)=1$ for $m=0$, $J_m(0r)=0$ for $m\neq 0$, the only nontrivial solution to Eq.(36) is:

$$R(r) = CJ_0(\beta_{00} r) = C,\ \beta_{00} = 0. \tag{38}$$

This represents a plane wave. The plane wave in a circular or annular duct only propagates in the axial direction.

For all the cases other than $m = 0$ and $n = 0$, we have $\beta \neq 0$. From boundary condition (37), the dispersion relation is obtained from (36):

$$J'_m(\beta R_i) Y'_m(\beta R_o) = J'_m(\beta R_o) Y'_m(\beta R_i). \tag{39}$$

The prime denotes the derivative with respect to the whole argument. It is not trivial to solve eigenvalue β from Eq.(39). When the gap of annular duct $R_i - R_o$ is much smaller than R_o, we can obtain an approximate solution of β. We begin with the Bessel equation (27). By replacing r by the medium radius $R_c = (R_i + R_0)/2$, the Bessel equation becomes a second order ODE with constant coefficients:

$$R_c^2 \frac{d^2 R}{dr^2} + R_c \frac{dR}{dr} + (R_c^2 \beta^2 - m^2) R = 0\,. \tag{39-1}$$

Its general solution is:

$$R(r) = Ce^{\lambda_+ r} + De^{\lambda_- r},\ \lambda_\pm = \frac{-1 \pm \sqrt{1 - 4(R_c^2 \beta^2 - m^2)}}{2R_c}\,.$$

λ_+ and λ_- are both real, or a conjugate pair.

To satisfy boundary conditions (37),

$$e^{-(\lambda_+ - \lambda_-)(R_o - R_i)} = e^{-\frac{R_o - R_i}{R_c}\sqrt{1 - 4(R_c^2\beta^2 - m^2)}} = 1,$$

i.e.,

$$\frac{R_o - R_i}{R_c}\sqrt{4(R_c^2 \beta^2 - m^2) - 1} = 2\pi n,\ n\text{=0,1,2,....}$$

Therefore the eigenvalue for an annular duct with small gap is

$$\beta_{mn} = \sqrt{\frac{\pi^2 n^2}{(R_o - R_i)^2} + \frac{1}{R_c^2}(m^2 + \frac{1}{4})}\ ,\ n\text{=0, 1, 2, ...} \tag{40}$$

According to Eq.(11-2), for a cut-on mode in a narrow annular duct, *n* must be zero. Therefore

$$\frac{\beta_{mn}R_c}{m} = \sqrt{1+\frac{R_c^2}{4m^2}} > 1.$$

For large *m*,

$$\beta_{mn}R_c / m \to 1.$$

As $R_c \to \infty$, does the narrow annular approaches a rectangular duct? Compare Eq.(40) with Eq.(10). The radial wavenumber in the annular duct $\pi n/(R_o - R_i)$ approaches the height mode, $R_o - R_i \sim h$. The spinning mode wavenumber m/R_c approaches the depth mode. However, there is an extra wavenumber $1/(2R_c)$ in the annular duct. It comes from term $R_c \dfrac{dR}{dr}$ in Eq.(39-1), which does not exist in the rectangular duct Eq.(5-1). If *m* is kept constant as $R_c \to \infty$, then $\beta_{mn} \to \sqrt{\dfrac{\pi^2 n^2}{(R_o - R_i)^2}}$. It is equivalent to regular duct with uniform mode shape in the depth direction. On the other hand, if m/R_c remains constant as $R_c \to \infty$, then $m \to \infty$ and $\beta_{mn} \to \sqrt{\dfrac{\pi^2 n^2}{(R_o - R_i)^2} + \dfrac{m^2}{R_c^2}}$. The annular duct approaches a rectangular duct. For the mode shape, as $R_c \to \infty$, $|\beta_{mn} r| \to \infty$, Eq.(36)

$$J_m(\beta_{mn} r) \to \sqrt{\frac{2}{\pi \beta_{mn} r}} \cos\left(\beta_{mn} r - \frac{1}{2} m\pi - \frac{1}{4}\pi\right),$$

$$Y_m(\beta_{mn} r) \to \sqrt{\frac{2}{\pi \beta_{mn} r}} \sin\left(\beta_{mn} r - \frac{1}{2} m\pi - \frac{1}{4}\pi\right).$$

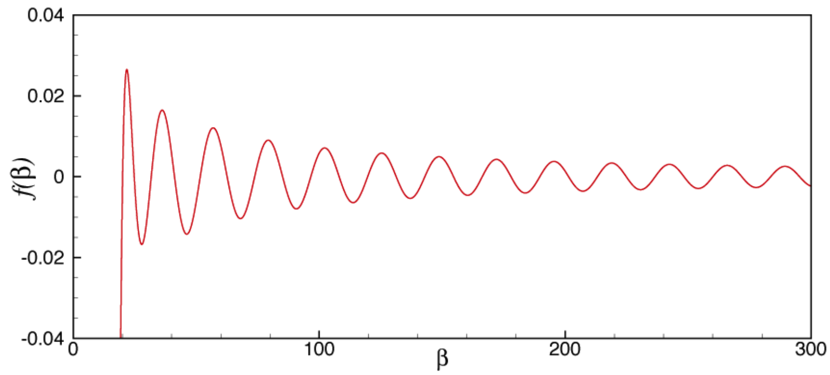
Compare it with Eq.(12), it approaches rectangular duct height mode shape.

Another asymptotic formula for the eigenvalues based on the WKB method was developed by Envia. (Envia1998, Eq.9)

One may solve Eq.(39) numerically using the Newton iteration method. Define function:

$$f_m(\beta) = J'_m(\beta R_i) Y'_m(\beta R_o) - J'_m(\beta R_o) Y'_m(\beta R_i).$$

For initial eigenvalue $\beta^{(0)}$, the improved eigenvalue is $\beta^{(1)} = \beta^{(0)} + \Delta\beta$, $\Delta\beta = -f_m(\beta^{(0)}) / f_m'(\beta^{(0)})$. This process is repeated until $\Delta\beta$ is small. A successful Newton method depends on a good initial eigenvalue $\beta^{(0)}$. The next figure shows a typical $f_m(\beta)$. Separate β-axis into intervals, ..., $\beta^{(n)}, \beta^{(n+1)}$,.... If $f_m(\beta^{(n)}) f_m(\beta^{(n+1)}) \le 0$, $\beta^{(n)}$ is a good choice of $\beta^{(0)}$. A FORTRAN90 code, modenumber_annularduct.f, using this method to compute eigenvalues in a solid wall annular duct is attached.



m=18, $R_o = 1$, $R_i = 11/15$.

One can show that $f_{-m}(\beta) = f_m(\beta)$. Therefore eigenvalues for *m* are also the eigenvalues for –*m*: $\beta_{mn} = \beta_{(-m)n}$. Eigenfunction for (-*m*,*n*) is the same as for (*m*,*n*).

A general solution of $\hat{p}(r,\phi)$ is then:

$$\hat{p}_{00}(r,\phi) = A_{00}\text{, } m=n=0;$$

$$\hat{p}_{mn}(r,\phi) = A_{mn}\left[J_m(\beta_{mn}r) - \frac{J'_m(\beta_{mn}R_i)}{Y'_m(\beta_{mn}R_i)}Y_m(\beta_{mn}r)\right]e^{im\phi}\text{, other.} \qquad (40\text{-}1)$$

Velocities are computed by Eq.(24). Wave number $k^{\pm}_{xmn}$ can be computed from Eq.(11) & (11-0).

The time domain solution of *p* is:

$$p = \text{Real}(A^+_{00}e^{ik^+_{x00}x} + A^-_{00}e^{ik^-_{x00}x})e^{-i\omega t}$$
$$+\text{Real}\sum_{m=-\infty}^{\infty} e^{i(m\phi-\omega t)} \sum_{\substack{n=0 \text{ if } m\neq 0 \\ n=1 \text{ if } m=0}}^{\infty} (A^+_{mn}e^{ik^+_{xmn}x} + A^-_{mn}e^{ik^-_{xmn}x})\left[J_m(\beta_{mn}r) - \frac{J'_m(\beta_{mn}R_i)}{Y'_m(\beta_{mn}R_i)}Y_m(\beta_{mn}r)\right]. \quad (41)$$

Orthogonality:

According to Abramowitz&Stegun1964, Eq.(11.4.2), p.485,

$$\Psi_{mn}(r,\phi) = \Re_m(\beta_{mn}r)e^{im\phi}$$

$$\Re_m(\beta_{mn}r) = \begin{cases} 1, m=n=0 \\ J_m(\beta_{mn}r) - \dfrac{J'_m(\beta_{mn}R_i)}{Y'_m(\beta_{mn}R_i)}Y_m(\beta_{mn}r), other \end{cases}$$

$$\int_{R_i}^{R_o}\int_0^{2\pi}\Psi_{mn}(r,\phi)\Psi_{jl}^{*}(r,\phi)rd\phi dr=\begin{cases}\Lambda_{mn},\ j=-m,l=n\\ 0,\ j\neq -m,\text{or},(j=-m,n\neq l)\end{cases}$$

$$\Lambda_{mn}=\begin{cases}\pi\left(R_o^2-R_i^2\right),m=n=0\\ \pi\left(R_o^2-\dfrac{m^2}{\beta_{mn}^2}\right)\Re_m^2(\beta_{mn}R_o)-\pi\left(R_i^2-\dfrac{m^2}{\beta_{mn}^2}\right)\Re_m^2(\beta_{mn}R_i),other\end{cases}$$

Cut-on ratio

For annular ducts the cut-on ratio in Eq.(11-2) can be rewritten as

$$\begin{aligned}\sigma_{mn}&=\frac{\omega}{\omega_{mn}}=\frac{\omega}{\bar{a}\beta_{mn}\sqrt{1-M^2}}\\ &=\frac{\omega R_o/\bar{a}m}{(\beta_{mn}R_o/m)\sqrt{1-M^2}}\\ &=\frac{M_m}{M_m^*\sqrt{1-M^2}}\end{aligned},$$

$$M_m=\frac{\omega R_c}{\bar{a}m},\ M_m^*=\frac{\beta_{mn}R_c}{m}.$$

M_m is the phase speed in the circumferential direction. M_m^* is called the cut-off Mach number. It is roughly the ratio of radial wavenumber and the circumferential wavenumber. For narrow ducts, $M_m^*>1$. This is true for any ducts. For narrow ducts and large *m,* $M_m^*\to 1$. For arbitrary annular ducts, M_m^* depends on the hub/tip ratio and m:

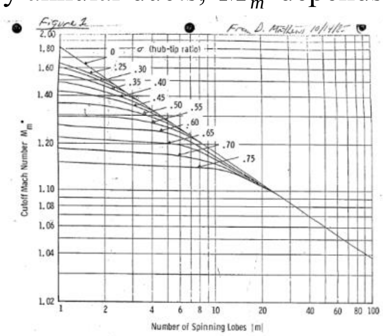


Figure 1. Cutoff Mach Number for Spinning Modes.

If for large *m* in narrow ducts without axial mean flow, the phase speed of the mode in the circumferential direction must be supersonic to be cut-on.

Sound Propagation C-Duct

In the outlet of jet engines, the annular duct is often separated by installations, as illustrated by Fig.3. The duct has a C-shape. It is here called *C-duct*. The C-duct has an inner duct with radius R_i and an outer duct with radius R_o. The separation plates are at ϕ_s and ϕ_e.

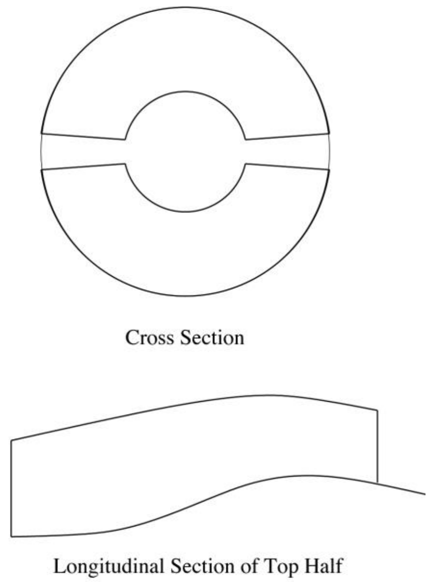


Fig.3, C-duct.

The first major difference of C-duct from the annular duct is the periodic boundary condition is no longer applicable in the ϕ direction. Because of the presence of the bifurcations, spinning duct modes are not possible. Reflections at the bifurcations lead to the formation of a standing wave pattern in the cross-section of the duct. Similar to the argument for Eq.(9) in rectangular duct, the general solution of $\Phi(\phi)$ is:

$$\Phi(\phi) = B\cos[\nu_m(\phi - \phi_s)], \tag{42}$$

where

$$\nu_m = m\pi/(\phi_e - \phi_s), \; m = 0,1,2,\cdots. \tag{43}$$

m is any nonnegative integer. ν_m may not be an integer as in the circular or annular duct unless when $\phi_e - \phi_s = \pi$. (When $\phi_e - \phi_s = \pi$, the solution in a C duct is the same as that in an annular duct.)

In the radial direction the general solution is:

$$R(r) = CJ_{\nu_m}(\beta r) + DY_{\nu_m}(\beta r). \tag{44}$$

The second major difference of the C-duct from annular duct is the Bessel functions here may have non-integer orders.

Rigid Wall C-Duct

For the rigid wall C-duct, the boundary conditions in radial direction are

$$\hat{u}_r = 0, \text{ at } r = R_i \text{ and } r = R_o. \tag{45}$$

The similar argument as for Eq.(38) leads to the plane wave solution:

$$R(r) = J_0(\beta_{00} r), \ \beta_{00} = 0. \tag{46}$$

If $\beta \neq 0$, the next dispersion relation can be derived from (44) and (45):

$$J'_{\nu_m}(\beta R_i) Y'_{\nu_m}(\beta R_o) = J'_{\nu_m}(\beta R_o) Y'_{\nu_m}(\beta R_i). \tag{47}$$

Eigenvalue β is to be determined by this equation. The difficulty involved here is the Bessel functions with non-integer order.

From the similar procedures for Eq.(40), when the gap of annular duct, $(R_i - R_o)/R_o$, is small, the eigenvalue is approximated by:

$$\beta_{mn} = \sqrt{\frac{\pi^2 n^2}{(R_o - R_i)^2} + \frac{1}{R_c^2}(\nu_m^2 + \frac{1}{4})}, \ n=0,1,2,..... \tag{48}$$

If the gap of the annular duct is not small, we can use (48) as an initial value in a Newton iteration method. The final results need to be reorganized.

The time domain solution of p is:

$$p = \text{Real}\left\{ \begin{array}{l} (A_{00}^{+} e^{ik_{x00}^{+} x} + A_{00}^{-} e^{ik_{x00}^{-} x}) e^{-i\omega t} + \\ \displaystyle\sum_{m=0}^{\infty} \sum_{\substack{n=0 \text{ if } m\neq 0 \\ n=1 \text{ if } m=0}}^{\infty} \cos[\nu_m(\phi - \phi_s)] \left[J_{\nu_m}(\beta_{mn} r) - \frac{J'_{\nu_m}(\beta_{mn} R_o)}{Y'_{\nu_m}(\beta_{mn} R_o)} Y_{\nu_m}(\beta_{mn} r) \right] (A_{mn}^{+} e^{ik_{xmn}^{+} x} + A_{mn}^{-} e^{ik_{xmn}^{-} x}) e^{-i\omega t} \end{array} \right\}. \tag{49}$$

With solution of $\hat{p}$, all other variables can be computed from Eq.(24).

Sound Propagation in Circular Duct with Nonuniform Mean Flow (Boundary Layer)

Scales used will be: length: duct radius R, velocity: sound speed a_0, density: ambient density ρ_0, pressure: $\rho_0 a_0^2$, impedance: $\rho_0 a_0$.

The mean flow: $\rho_0 = 1$ and $p_0 = 1/\gamma$ ($\gamma = 1.401$) are constant; Shear mean velocity $u_0(r)$ is in +x-axis direction.

Assume the next form of solutions:

$$\begin{bmatrix} u_x \\ u_r \\ u_\phi \\ p \end{bmatrix} = \mathrm{Real}\left\{ \begin{bmatrix} \hat{u}_x(r) \\ \hat{u}_r(r) \\ \hat{u}_\phi(r) \\ \hat{p}(r) \end{bmatrix} e^{i(\alpha x + m\phi - \omega t)} \right\}. \tag{50}$$

(Density can be computed directly from pressure: $\rho = p$.)

From Euler equations, one can obtain the next equations for shear flow:

$$\frac{d\hat{u}_r}{dr} - i\left[\varpi - \frac{1}{\varpi}\left(\alpha^2 + \frac{m^2}{r^2}\right)\right]\hat{p} + \left(\frac{\alpha}{\varpi}\frac{du_0}{dr} + \frac{1}{r}\right)\hat{u}_r = 0, \tag{51}$$

$$\frac{d\hat{p}}{dr} - i\varpi\hat{u}_r = 0, \tag{52}$$

where $\varpi = \omega - u_0\alpha$. For a wall with impedance Z, the boundary condition is

$$\hat{p} = Z\hat{u}_r, \text{ at } r = 1. \tag{53}$$

Equations (51) and (52) and boundary condition (53) form an eigenvalue problem. Only some special values of α can satisfy these equations. These α are *eigenvalues*, and the respective functions $\hat{p}$ and $\hat{u}_r$ are *eigenfunctions*.

An important property of eigenvalues of Eqs.(51)&(52) is given here. Suppose α, $\hat{p}$, $\hat{u}_r$ are a set of eigenvalue and eigenfunctions of this problem, then α^*, $\hat{p}^*$, $\hat{u}_r^*$ form a set of eigenvalue and eigenfunctions for Eqs.(51)&(52) satisfying the next B.C.:

$$\hat{p}^* = Z^*\hat{u}_r^*, \text{ at } r = 1. \tag{54}$$

Here superscript * represents conjugate. Under some circumstances B.C.(53) is exactly the same as B.C.(54), such as for hard wall, $\hat{u}_r = \hat{u}_r^* = 0$, or the impedance with only resistance, $Z = Z^*$. Then α^*, $\hat{p}^*$, $\hat{u}_r^*$ are also the set of eigenvalue and eigenfunctions for the original problem [Eqs.(51)&(52) with B.C.(53)].

Newton iteration method can be used to solve the eigenvalues and eigenfunctions. For an initial guess of the eigenvalue α_0, integrate Eqs.(51)&(52) numerically from the outside of the boundary layer to $r=1$ to get $\hat{u}_r$ and $\hat{p}$ at $r=1$. $\hat{u}_r$ and $\hat{p}$ are functions of α_0: $\hat{u}_r(\alpha_0)$, $\hat{p}(\alpha_0)$, which may not satisfy B.C. (53) at $r=1$. Similarly we can have another solution $\hat{u}_r(\alpha_1)$ and $\hat{p}(\alpha_1)$ for initial eigenvalue α_1 near α_0. Suppose the exact eigenvalue is $\alpha_0+\Delta\alpha$, *i.e.*,

$$\hat{p}(\alpha_0+\Delta\alpha)=Z\hat{u}_r(\alpha_0+\Delta\alpha).$$

To the first order of Taylor expansion, we have:

$$\Delta\alpha=\frac{\hat{p}(\alpha_0)-Z\hat{u}_r(\alpha_0)}{Z\dfrac{d\hat{u}_r(\alpha_0)}{d\alpha}-\dfrac{d\hat{p}(\alpha_0)}{d\alpha}}.$$

If we use $\dfrac{d\hat{u}_r(\alpha_0)}{d\alpha}\approx\dfrac{\hat{u}_r(\alpha_1)-\hat{u}_r(\alpha_0)}{\alpha_1-\alpha_0}$, $\dfrac{d\hat{p}(\alpha_0)}{d\alpha}\approx\dfrac{\hat{p}(\alpha_1)-\hat{p}(\alpha_0)}{\alpha_1-\alpha_0}$, we can get $\Delta\alpha$ to have a better approximation to the accurate eigenvalue. The process can be repeated until certain accuracy is achieved. This is the Newton's iteration method.

Choosing appropriate initial value α_0 is critical for a successful Newton's iteration method. Eigenvalues of plug flow without boundary layer, Eq.(34) and Eq.(11), are good choice of the initial eigenvalues. In most of the time these choices work well. But in some cases Newton's method fail when using these initial eigenvalues.

Another way for choosing initial eigenvalues is the *grid search*. Compute function:

$$f(\alpha)=\hat{p}(\alpha)/Z-\hat{u}_r(\alpha) \tag{55}$$

for a matrix in α-plane, draw the contours of real(*f*)=0 and imag(*f*)=0 in the α-plane using a commercial software. The intersection points in the α-plane should be the solution of the eigenvalues. Read in these data and use them as the initial eigenvalues in the Newton iteration.

As an example, let's find eigenvalues for mean flow with turbulent boundary layer in a circular rigid wall duct with radius: 62.33". The velocity profile of the turbulent boundary layer is shown in Fig.4. Mach number: 0.452, boundary layer thickness: 1",

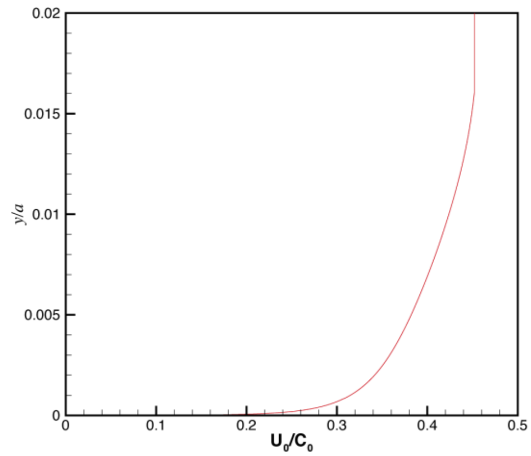


Fig.4, Velocity profile of turbulent boundary layer, Mach number: 0.452, boundary layer thickness: 1".

The grid search method is used to get the initial guess of eigenvalues. Since the wall is rigid, both α and α^* are the eigenvalues according to the property discussed above. Therefore in Fig.5 only the up half of the α-plane is needed. Read in the data of the intersection points of solid and dashed lines as the initial values in the Newton's method.

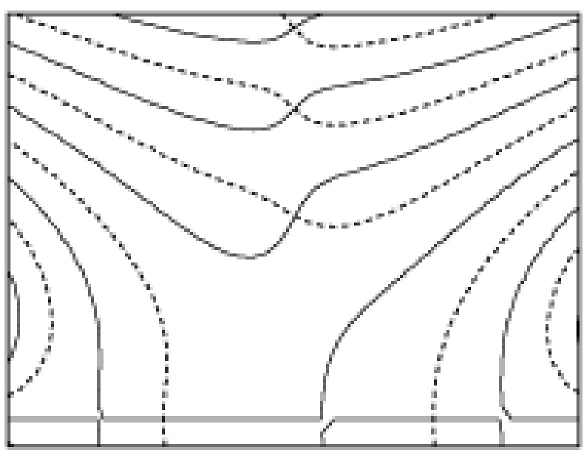


Fig.5, Contours of real(f)=0 (solid lines) and imag(f)=0 (dashed lines) in the α-plane

The next figures show the results. Fig.6 shows the eigenvalues when frequency is just above the cut-on frequency. Fig.7 is for frequency is well above the cut-on frequency. Fig.8 is for frequency well below the cut-on frequency. In most of the cases, the effect of shear flow is to move real parts of plug flow eigenvalues to right while the imaginary parts keep nearly the same. This change is very small. Using plug flow eigenvalues as initial values in Newton method works well. The only exception is when the frequency is just above the cut-on frequency (Fig6). There are two cut-on modes in the plug flow. The boundary layer makes these propagating modes into cut-off modes. This explains why a cut-on mode wave was damped in experiments. In this case the plug flow eigenvalues as the initial value fails.

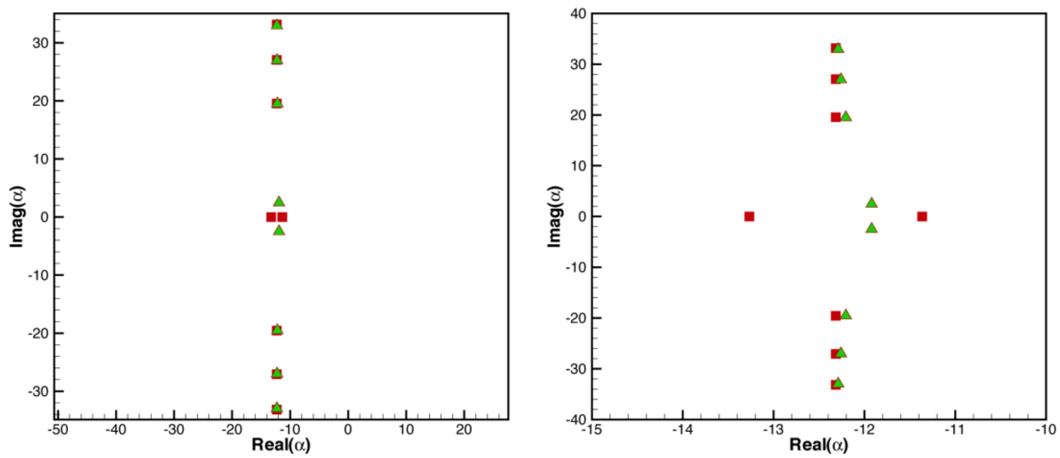


Fig.6, Eigenvalues at 741Hz for plug flow (square) and shear flow with turbulent boundary layer (triangle). Mach number: 0.452, boundary layer thickness: 1”, duct radius: 62.33”, azimuthal mode number m=22, cut-on frequency for the 1st radial mode of the plug flow: 740.6Hz. [Real(α) in the right figure is rescaled to show the difference.]

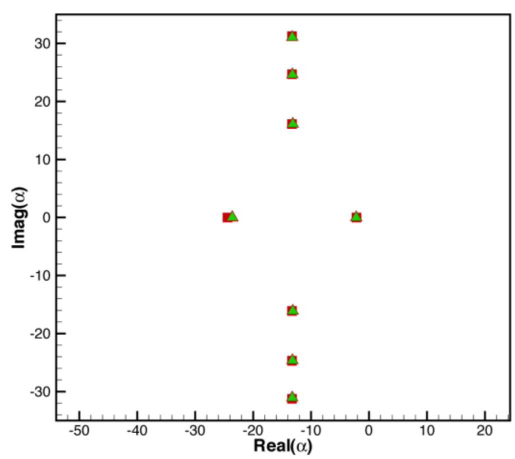


Fig.7, Eigenvalues at 800Hz for plug flow (square) and shear flow with turbulent boundary layer (triangle). Mach number: 0.452, boundary layer thickness: 1”, duct radius: 62.33”, azimuthal mode number m=22, cut-on frequency for the 1st radial mode of the plug flow: 740.6Hz.

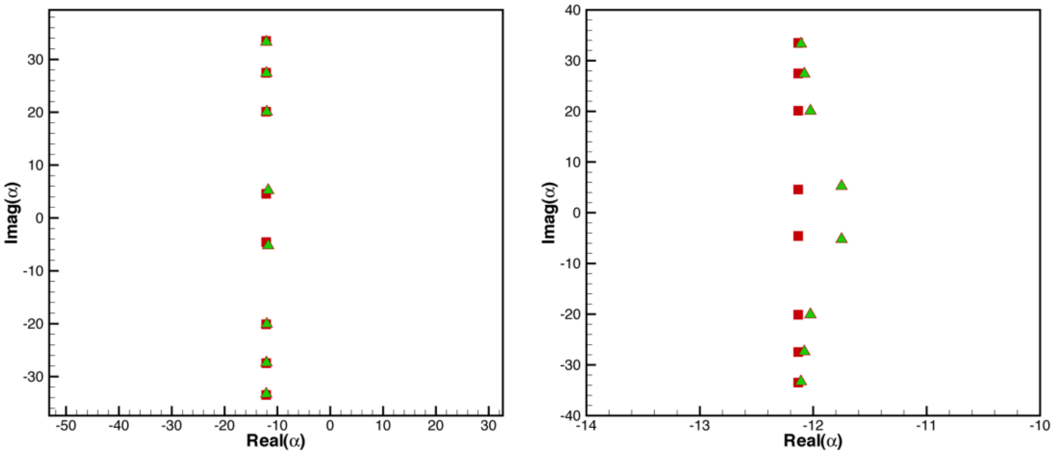


Fig.8, Eigenvalues at 730Hz for plug flow (square) and shear flow with turbulent boundary layer (triangle). Mach number: 0.452, boundary layer thickness: 1”, duct radius: 62.33”, azimuthal mode number m=22, cut-on frequency for the 1st radial mode of the plug flow: 740.6Hz. [Real(α) in the right figure is rescaled to show the difference.]

A numerical integration method can also be used.