Low Mach Number Flow Noise: Flow as the Quadrupole Source

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In a uniform flow, linear waves of vorticity, sound and heat are independent. They interact with each other only at boundaries. When the flow is not uniform, or disturbances in the flow are not small, different waves will interact by *scattering* and/or *nonlinearity*: one type of wave can grow, be generated or dampened by other types of waves.

It is difficult to analyze wave interactions in a nonlinear nonuniform flow. In this chapter we will discuss a simple case of *nonlinear interaction*: the sound generation by a compact vortical flow with low Mach number M, shown by Fig.1. The flow moves at typical speed u_0 which is much smaller than sound speed a. During one cycle of the typical oscillation with period T, the ratio of distance l traveled by the flow to sound wavelength λ is $l/\lambda = u_0 T/(aT) = M$. Because Mach number M is small, the two lengths are disparate: $l << \lambda$. On flow length scale l, the flow in its low order M expansion is vortical and incompressible. On sound wavelength scale λ, the flow is irrotational in its low order M expansion. That means the short and very long lengths are only weakly coupled in the flow and sound regions. The physical explanation of this weak coupling is, during one cycle of the flow parameter, the sound wave oscillates many cycles so its net effect is almost cancelled; on the other hand, the slowly varying flow is almost constant over one cycle of the sound oscillation. (Crighton, et.al.1992, p.209) Therefore, one may only need to solve an incompressible flow in the flow region and solve the sound wave equation in the sound wave region. The two solutions are then coupled in an overlap region. This coupling is fulfilled only mathematically. There is no physical solution in the overlap region. In the final solution the incompressible flow acts as a quadrupole source to the far sound field, which is the same as in the Lighthill's Analogy Theory.

The mathematic tool for this analysis is the perturbation method. We first assume the flow is incompressible ($M = 0$). An incompressible flow is localized. Then a small compressibility is admitted. Admission of compressibility has two effects. First, the flow energy can propagate away in the form of compressible waves (sound) from the local flow region. Second, there is time lag between the flow and its generated sound. Mach number M is the expansion parameter. It is the ratio of two lengths. The disparity of the two length scales implies that the low Mach number flow sound is a multiple scale problem, and the perturbation will be singular instead of regular (Dyke1975, p.80: a perturbation solution is uniformly valid in the space and time coordinates unless the perturbation quantity is the ratio of two lengths or two times).

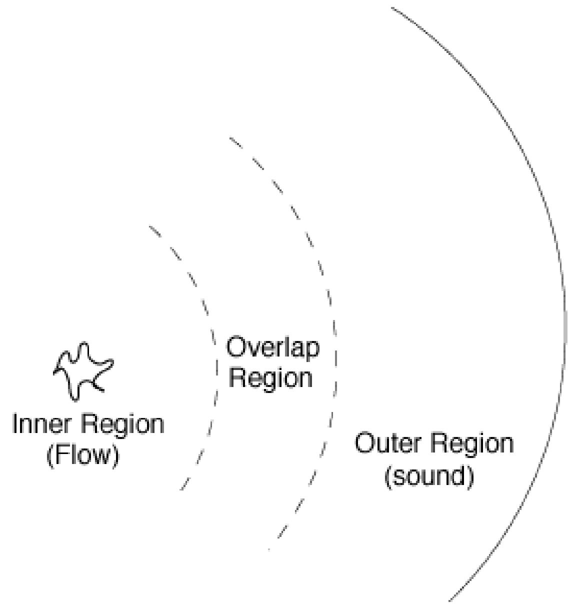

Fig.1, Sound generation by low Mach number compact flow.

There are two different singular perturbation methods. Landau and Lifshitz first used the *Matched Asymptotic Expansion* (MAE) to compute sound from a small pulsating and oscillating body. Crow1970, Obermeier1985 extended this MAE method to low Mach number flow sound problem. Different length scales are used in the near field flow region and the far field sound region. Regular expansions are developed based on their local length scales in different regions. The two expansions are then matched in the overlap region to form a composite solution.

The other method is *Multiple Scale Method.* One may choose one time scale and two length scales (l and λ) in the analysis as in Fortenbach&Munz2003. Since for the same distance, the time spent by the sound wave is much shorter than that by the flow, one may also use one length scale and two time scales in the Multiple Scale method as in Müller1999.

MAE method similar to Crow1970 will be employed here. The following are the equations for inviscid, non-heat-conducting flow used in this chapter:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0, \tag{1}$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i}, \tag{2}$$

$$\frac{p}{p} = \left(\frac{\rho}{\rho}\right)^{\gamma}. \tag{3}$$

Inner Expansion

Success of MAE depends on choosing right scales in the region analyzed. Different choice of scales gives different results. In the flow region, the flow is fully characterized by typical eddy speed u_0 (velocity scale), typical eddy size *l* (length scale), and time scale l/u_0. From continuity equation (1), density scale has no significance. Generally we use density in the far field ρ_∞ as the density scale. From momentum equation (2), the relative scale of pressure to that of density is critical for the analysis. However, the appropriate pressure scale can't be found from the momentum equation. It can be found from isentropic equation (3). It is straightforward to choose pressure in the far field p_∞ as the scale. Just for convenience, we usually choose $\gamma p_\infty = \rho_\infty a_\infty^2$ (sound speed $a_\infty = \sqrt{\gamma p_\infty / \rho_\infty}$) as the pressure scale. Then the dimensionless equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0, \tag{4}$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = -\frac{1}{M^2}\frac{\partial p}{\partial x_i},\ M = u_0/a_\infty, \tag{5}$$

$$p = \rho^\gamma/\gamma. \tag{6}$$

All variables in (4)~(6) and hereafter are dimensionless quantities unless specified.

Let's try the regular expansion series:

$$\rho = \rho_0 + M\rho_1 + M^2\rho_2 + \cdots, \tag{7}$$

$$p = p_0 + Mp_1 + M^2 p_2 + \cdots, \tag{8}$$

$$u_i = u_{0i} + Mu_{1i} + M^2 u_{2i} + \cdots. \tag{9}$$

Substituting them into (4)~(6) and equating the terms of like powers of *M*, we have:

$O(1)$ *equations*:

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial(\rho_0 u_{0j})}{\partial x_j} = 0,\ \frac{\partial p_0}{\partial x_i} = 0,\ p_0 = \rho_0^\gamma/\gamma. \tag{10}$$

The general solution of p_0 is $p_0(t)$, which should be determined by matching it to the outer expansion in the overlap region. From physics intuition, it should be the atmospheric pressure in the far field $p_\infty = 1/\gamma$. Therefore,

$$p_0 = 1/\gamma,\ \rho_0 = 1, \text{ and } \nabla\cdot\vec{u}_0 = 0. \tag{11}$$

$O(M)$ *equations*:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial(\rho_0 u_{1j} + \rho_1 u_{0j})}{\partial x_j} = 0, \ \frac{\partial p_1}{\partial x_i} = 0, \ p_1 = \rho_1. \tag{12}$$

The same argument as in $O(1)$ equations indicates:

$$p_1 = 0, \ \rho_1 = 0, \ \nabla \cdot \vec{u}_1 = 0. \tag{13}$$

(11) and (13) have similar properties. There is no advantage to separate these two orders of equations (Crow1970, Müller1999, Eldredge2002). Therefore, $O(1)$ and $O(M)$ order variables can be combined. The proper expansion series should be:

$$\rho = \rho_0 + M^2 \rho_2 + \cdots, \tag{14}$$
$$p = p_0 + M^2 p_2 + \cdots, \tag{15}$$
$$u_i = u_{0i} + M^2 u_{2i} + \cdots. \tag{16}$$

Now p_0, ρ_0, and $\vec{u}_0$ are variables of $O(1)$ and $O(M)$ orders, instead of variables only of $O(1)$ order as in (7)~(8).

Substituting (14)~(16) into (4)~(6), we have:

$O(1)+O(M)$ *equations*:

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial(\rho_0 u_{0j})}{\partial x_j} = 0, \ \frac{\partial p_0}{\partial x_i} = 0, \ p_0 = \rho_0^{\gamma}/\gamma. \tag{17}$$

The same argument as for (11) brings to:

$$p_0 = 1/\gamma, \ \rho_0 = 1, \ \nabla \cdot \vec{u}_0 = 0. \tag{18}$$

In this order the flow is solenoidal (dilation free).

$O(M^2)$ *equations*:

$$\frac{\partial \rho_2}{\partial t} + u_{0j}\frac{\partial \rho_2}{\partial x_j} + \rho_0 \frac{\partial u_{2j}}{\partial x_j} = 0, \tag{19}$$
$$\frac{\partial u_{0i}}{\partial t} + u_{0j}\frac{\partial u_{0i}}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p_2}{\partial x_i}, \tag{20}$$
$$p_2 = \rho_0^{\gamma-1}\rho_2 = \rho_2. \tag{21}$$

From (18) we know in the limit of low compressibility, pressure converges to a constant thermodynamic background pressure. When small compressibility is admitted, from equations (18) and (20), we can form the following complete system of equations:

$$\nabla \cdot \vec{u}_0 = 0, \tag{22}$$

$$\frac{\partial u_{0i}}{\partial t} + u_{0j}\frac{\partial u_{0i}}{\partial x_j} = -\frac{\partial p_2}{\rho_0 \partial x_i}. \tag{23}$$

These are the well known incompressible flow equations. They are actually the low Mach number flow equations. Velocity field is solenoidal. We will use $\vec{v}_0$ instead of $\vec{u}_0$ in the equations. Taking divergence on both sides of (23), we have the Poisson equation about pressure:

$$\nabla^2 p_2 = -\rho_0 \frac{\partial^2 (v_{0i} v_{0j})}{\partial x_i \partial x_j}. \tag{24}$$

Its solution in three dimensions if $\vec{v}_0$ is known is:

$$p_2(\vec{x},t) = \frac{\rho_0}{4\pi} \iiint_V \frac{1}{R} \frac{\partial^2 (v_{0i} v_{0j})}{\partial y_i \partial y_j} d\vec{y}, \tag{25}$$

where $R = |\vec{x} - \vec{y}|$, $\partial / \partial y_i$ with R fixed. Any harmonic solution J (solution of $\nabla^2 J = 0$) can be added to $p_2(\vec{x},t)$ to form another solution. But J is analytical and will grow algebraically in the far field (Crighton, et.al.1992). Thus J should not be included in the solution.

From (18) and (25), to second order $O(M^2)$, the formal solution of pressure in the flow region is:

$$p(\vec{x},t) = \frac{1}{\gamma} + M^2 \frac{\rho_0}{4\pi} \iiint_V \frac{1}{R} \frac{\partial^2 (v_{0i} v_{0j})}{\partial y_i \partial y_j} d\vec{y} + O(M^3). \tag{26}$$

Solutions (25) and (26) are only formal since $\vec{v}_0$ has to be solved together with p_2 from incompressible equations (22) and (23). p_2 is called *hydrodynamic pressure*. Hydrodynamic pressure appears in incompressible equation (23) to establish the divergence-free velocity $\vec{v}_0$. Associated with p_2 is *hydrodynamic density* ρ_2 [Eq.(21)] and second order velocity $\vec{u}_2$ [Eq.(19)]. $\vec{u}_2$ is not necessarily dilation free. That means p_2 may compress the fluid element, although in this low order [$O(M^2)$] there is no sound, Eq.(24).

An acoustic/viscous splitting numerical scheme can be established based on this asymptotic analysis. One may use the next asymptotic series:

$$\rho = \rho_0 + M^2 \rho_2 + \rho', \tag{27}$$

$$p = p_0 + M^2 p_2 + p', \tag{28}$$

$$u_i = u_{0i} + M^2 u_{2i} + u'_i. \tag{29}$$

u_{0i}, p_2, ρ_2 and u_{2i} have small length scale l. They are solved from incompressible equations (22)&(23) and equations (21)&(19) on a fine grid system. Substitute expansions (27)~(29) into the N.S. equations and subtract the incompressible equations from them to obtain a set of equations for ρ', p', and u'_i. These equations are *acoustic equations*. Since only the long length scale, sound wavelength, exists in the acoustic equations, they can be solved on a coarse grid system. By splitting the incompressible flow and the sound waves, the singularity due to length scale disparity is removed. A similar acoustic/viscous splitting numerical scheme was proposed by Hardin&Pope1994, in which u_{2i} is not solved in the incompressible flow.

Outer Expansion

In the sound region, the appropriate length scale is sound wavelength λ. Use λ as the new length scale to define the dimensionless coordinate:

$$X_i = x_i l/\lambda = M x_i . \tag{30}$$

Eq.(30) is a stretching transformation of the coordinate. The sound is generated by the fluid flow, therefore the inner and outer regions have the same time scale: $\lambda/a_\infty = l/u_0$. ρ_∞ is the density scale and $p_\infty = \rho_\infty a_\infty^2$ the pressure scale. We may still use u_0 as the velocity scale. However we may have a better choice. From the $O(M^2)$ momentum equation of the inner expansion, $M^2 p_2 \sim \frac{1}{2} M^2 \rho_0 u_0^2$ (on the inner expansion scales), then the pressure fluctuation in dimensional form is $p' \sim \frac{1}{2}\rho_0 u_0^2$. As we know for sound waves, $p' \sim \rho_0 a_0 u_a$ (u_a is the acoustic velocity). Which means the acoustic velocity is in the order of $u_a \sim u_0 M$, which seems to be a better choice for velocity scale in the outer region: $\vec{U} = \vec{u} u_0/(u_0 M) = \vec{u}/M$. Then the dimensionless equations in the outer region are:

$$\frac{\partial \rho}{\partial t} + M^2 \frac{\partial(\rho U_j)}{\partial X_j} = 0 , \tag{31}$$

$$M^2 \frac{\partial(\rho U_i)}{\partial t} + M^4 \frac{\partial(\rho U_i U_j)}{\partial X_j} = -\frac{\partial p}{\partial X_i} , \tag{32}$$

$$p = \rho^\gamma/\gamma . \tag{33}$$

The next regular expansion series are used:

$$\rho = \rho_0 + M^2 \rho_2 + M^3 \rho_3 + \cdots , \tag{34}$$

$$p = P_0 + M^2 P_2 + M^3 P_3 + \cdots , \tag{35}$$

$$U_i = U_{0i} + M^2 U_{2i} + M^3 U_{3i} + \cdots . \tag{36}$$

We have:

$O(1)+O(M)$ *equations*:

$$\frac{\partial \rho_0}{\partial t}=0,\ \frac{\partial P_0}{\partial X_i}=0,\ P_0=\rho_0^{\gamma}/\gamma. \qquad (37)$$

The $O(1)$ solutions are:

$$P_0=1/\gamma,\ \rho_0=1. \qquad (38)$$

$O(M^2)$ *equations*:

$$\frac{\partial \rho_2}{\partial t}+\frac{\partial(\rho_0 U_{0j})}{\partial X_j}=0,\ \frac{\partial(\rho_0 U_{0i})}{\partial t}=-\frac{\partial P_2}{\partial X_i},\ P_2=\rho_2. \qquad (39)$$

From the momentum equation in (39), one has:

$$\frac{\partial(\nabla\times\vec{U}_0)}{\partial t}=0.$$

In the far field there is no vorticity initially, therefore the velocity field is vorticity free in this order:

$$\nabla\times\vec{U}_0=0. \qquad (40)$$

The wave equation is obtained from equations in (39):

$$\frac{\partial^2 P_2}{\partial t^2}-\nabla_X^2 P_2=0. \qquad (41)$$

$O(M^3)$ *equations*:

$$\frac{\partial \rho_3}{\partial t}=0,\ \frac{\partial P_3}{\partial X_i}=0,\ P_3=\rho_3. \qquad (42)$$

The solutions are:

$$P_3=\rho_3=0. \qquad (43)$$

$O(M^4)$ *equations*:

$$\frac{\partial \rho_4}{\partial t}+\frac{\partial(\rho_0 U_{2j}+\rho_2 U_{0j})}{\partial X_j}=0, \qquad (44)$$

$$\frac{\partial(\rho_0 U_{2i} + \rho_2 U_{0i})}{\partial t} + \frac{\partial(\rho_0 U_{0i} U_{0j})}{\partial X_j} = -\frac{\partial P_4}{\partial X_i}, \tag{45}$$

$$P_4 = \rho_4 + (\gamma - 1)\rho_2/2. \tag{46}$$

In this order vorticity $\nabla \times \vec{U}_2$ is not necessarily zero.

$O(M^5)$ *equations*:

$$\frac{\partial \rho_5}{\partial t} + \frac{\partial(\rho_0 U_{3j})}{\partial X_j} = 0, \tag{47}$$

$$\frac{\partial(\rho_0 U_{3i})}{\partial t} = -\frac{\partial P_5}{\partial X_i}, \tag{48}$$

$$P_5 = \rho_5 + (\gamma - 1)\rho_0^{\gamma-2}\rho_2\rho_3 = \rho_5. \tag{49}$$

From (48), $\nabla \times \vec{U}_3 = 0$.

The wave equation is:

$$\frac{\partial^2 P_5}{\partial t^2} - \nabla_X^2 P_5 = 0. \tag{50}$$

Here we will only discuss the solution to $O(M^5)$ wave equation (50) because it is the term which will match the inner expansion. The general solution to this wave equation is composed of monopoles, dipoles, quadrupoles, and even higher order components. But only the quadrupole is needed for matching to the inner solution:

$$P_5 = \frac{\partial^2}{\partial X_i \partial X_j}\left[\frac{1}{R} A_{ij}(t - R)\right], \text{ where } R = \left|\vec{X}\right|. \tag{51}$$

Therefore, pressure in the sound field is:

$$p_{out}(\vec{X}, t) = \frac{1}{\gamma} + M^5 \frac{\partial^2}{\partial X_i \partial X_j}\left[\frac{1}{R} A_{ij}(t - R)\right] + O(M^6). \tag{52}$$

Matching

Both the inner and outer expansions are the approximations to the same function, but just in different regions. They have to match with each other in the overlap region. There are two matching methods: Intermediate Matching Principle and Asymptotic Matching Principle (Crighton, et.al. 1992, Van Dyke1975, Holmes1995). We will use Intermediate Matching Principle here. In this method, the two expansions match in an overlap region. As $M \to 0$, the overlap region is the far field for the inner region, but a near field for the outer expansion. To describe this region mathematically, a new coordinate system is introduced: $\xi_i = Mx_i/\varepsilon$. $\varepsilon(M)$ is chosen so that as $M \to 0$ when keeping ξ_i fixed,

$$x_i = \frac{\varepsilon}{M}\xi_i \to \infty \quad , \; X_i = \varepsilon\xi_i \to 0. \tag{53}$$

One may choose $\varepsilon = M^{\alpha}$, $0 < \alpha < 1$. In the new coordinate system,

$$\frac{1}{r} = \frac{1}{|\vec{x} - \vec{y}|} = \frac{M}{\varepsilon}\frac{1}{\left|\vec{\xi} - \frac{M}{\varepsilon}\vec{y}\right|} = \frac{M}{\varepsilon}\frac{1}{\rho} - \frac{M^2}{\varepsilon^2} y_j \frac{\partial}{\partial \xi_j}\frac{1}{\rho} + \cdots, \tag{54}$$

$$\rho = |\vec{\xi}|.$$

Inner solution (26) can be written in the new coordinate system as:

$$\begin{aligned} p_{in} &= 1/\gamma + M^2 \frac{\rho_0}{4\pi}\iiint_V \frac{\partial^2 (v_{0i}v_{0j}/r)}{\partial y_i \partial y_j} d\vec{y} + O(M^3) \\ &= 1/\gamma + M^2 \frac{\rho_0}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\iiint_V \frac{v_{0i}v_{0j}}{r} d\vec{y} + O(M^3) \\ &= 1/\gamma + \frac{M^5}{\varepsilon^3}\frac{\rho_0}{4\pi}\left[\iiint_V v_{0i}v_{0j} d\vec{y}\right]\frac{\partial^2}{\partial \xi_i \partial \xi_j}\frac{1}{\rho} + O(\frac{M^6}{\varepsilon^4}) + O(M^3) \end{aligned} \tag{55}$$

(Note that the scale of $\vec{y}$ is still l.)

Outer solution (52) is rewritten as:

$$\begin{aligned} p_{out} &= 1/\gamma + \frac{M^5}{\varepsilon^3}\frac{\partial^2}{\partial \xi_i \partial \xi_j}\left[\frac{1}{\rho} A_{ij}(t - \varepsilon\rho)\right] + O(M^6) \\ &= 1/\gamma + \frac{M^5}{\varepsilon^3} A_{ij}(t)\frac{\partial^2}{\partial \xi_i \partial \xi_j}\frac{1}{\rho} + O(\frac{M^5}{\varepsilon^2}) + O(M^6) \end{aligned} \tag{56}$$

Matching the terms with M^5/ε^3 in (55) and (56), we have:

$$A_{ij}(t) = \frac{\rho_0}{4\pi}\iiint_V v_{0i}v_{0j} d\vec{y}. \tag{57}$$

Final Solutions

Now we summarize the final solutions with scales: length l, velocity u_0, time l/u_0, density $\bar{\rho}$, and pressure $\bar{p} = \bar{\rho} a^2$.

Inner solution:

$$p = 1/\gamma + M^2 p_2 + O(M^3), \tag{58}$$
$$\rho = 1 + M^2 \rho_2 + O(M^3), \tag{59}$$
$$u_i = v_{0i} + M^2 u_{2i} + O(M^3); \tag{60}$$
$$\nabla \cdot \vec{v}_0 = 0, \tag{61}$$
$$p_2 = \frac{\rho_0}{4\pi} \iiint_V \frac{1}{r} \frac{\partial^2 (v_{0i} v_{0j})}{\partial y_i \partial y_j} d\vec{y}, \tag{62}$$
$$\rho_2 = p_2, \tag{63}$$
$$\frac{\partial u_{2j}}{\partial x_j} = -\frac{1}{\rho_0} \left(\frac{\partial \rho_2}{\partial t} + u_{0j} \frac{\partial \rho_2}{\partial x_j} \right). \tag{64}$$

Outer solution:

$$\rho = 1 + M^5 \rho_5 + O(M^6), \tag{65}$$
$$p = 1/\gamma + M^5 P_5 + O(M^6), \tag{66}$$
$$u_i = M^3 u_{3i} + O(M^5); \tag{67}$$
$$P_5 = \frac{\rho_0}{4\pi M^2} \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{R} \iiint_V v_{0i} v_{0j} d\vec{y} \right]_{t - M|\vec{x}|}, \tag{68}$$
$$\rho_5 = P_5, \tag{69}$$
$$\frac{\partial u_{3i}}{\partial t} = -\frac{\partial P_5}{\partial x_i}. \tag{70}$$

Acoustic Analogy of the Low Speed Flow Sound

From sound wave equation (50) and solution (68), the equivalent inhomogeneous wave equation in the sound field is:

$$\frac{\partial^2 P_5}{\partial t^2} - \nabla_X^2 P_5 = \frac{\partial^2 (\rho_0 v_{0i} v_{0j})}{M^3 \partial X_i \partial X_j}, \text{ or} \tag{71}$$

$$\frac{\partial^2 p}{\partial t^2} - \frac{1}{M^2} \nabla_x^2 p = \frac{\partial^2 (\rho_0 v_{0i} v_{0j})}{\partial x_i \partial x_j}. \tag{72}$$

In dimensional form, inhomogeneous wave equation (72) is (the same variable names are used):

$$\frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} - \nabla_x^2 p = \frac{\partial^2(\rho_0 v_{0i} v_{0j})}{\partial x_i \partial x_j}. \tag{73}$$

The nonlinear incompressible flow in the near field acts as a quadrupole source driving the far field acoustic field. Strength of the quadrupole source is $\rho_0 v_{0i} v_{0j}$, which is the *Lighthill stress tensor* in Lighthill's Analogy Theory. From Eq.(24),

$$\frac{1}{a^2}\frac{\partial^2 p}{\partial t^2} - \nabla_x^2 p = -\nabla_x^2 p_2. \tag{74}$$

p_2 is the hydrodynamic pressure in the near field. $\nabla_x^2 p_2$ represents the relative value of p_2 at one point compared to the average pressure around this point. $\nabla_x^2 p_2 \geq 0$ means p_2 is the local minimum, otherwise it is the local maximum. Therefore, the sound source is the pressure at one point relative to the averaged pressure of its neighbor caused by the turbulent eddies. $\nabla_x^2 p_2$ is called jetlets by Ribner.

The same result can also be obtained by weakly nonlinear analysis.