Wave Interactions at Surface of Discontinuity

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In an uniform mean flow, linear waves of vorticity, sound, and heat propagate independently. They only couple at boundaries. If the mean flow is not uniform, waves will be diffracted by the mean flow gradient, different waves may interact with each other, and new wave modes may be generated.

To simply the problem, in this chapter we will discuss a special case of the nonuniform mean flow: surface of discontinuity (interface). Media on each side of the surface are uniform, in which analytical wave solutions apply. Across the surface one or more flow parameters are discontinuous. At the surface partial differential equations break down. Integral equations have to be used to establish the relations between the solutions on both sides. The final solutions are obtained by matching solutions at the surface according to these relations.

The interface itself is assumed to be stable in this chapter.

Surface of Discontinuity

Assume the surface of discontinuity is steady on the $y-z$ plane at $x=0$. (If the interface moves at a constant velocity, the coordinate system can be established moving with the same speed so that the interface is steady in this coordinate.) Across the interface derivatives and partial differential equations don't apply. However, mass, momentum, and energy of the flow must be conserved. Therefore one should use the integral form of the Euler equations. Suppose there is no mass injection, no external force or heat, and no viscous effects, then integral conservative equations for mass, momentum, and energy are (cht23.doc):

$$\iiint_V \frac{\partial \rho}{\partial t} dV + \oiint_S \rho u_j n_j dS = 0, \tag{1}$$

$$\iiint_V \frac{\partial (\rho u_i)}{\partial t} dV + \oiint_S (\rho u_i u_j + p\delta_{ij}) n_j dS = 0, \tag{2}$$

$$\iiint_V \frac{\partial}{\partial t}\left[\rho(E+\frac{1}{2}u_i u_i)\right] dV + \oiint_S \left[\rho(E+\frac{p}{\rho}+\frac{1}{2}u_i u_i)u_j - \kappa\frac{\partial T}{\partial x_j}\right] n_j dS = 0. \tag{3}$$

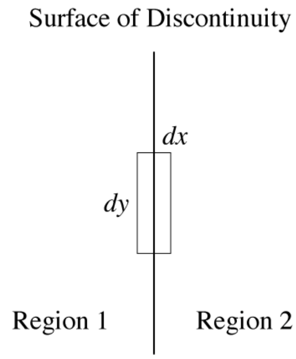

Fig.1, Steady surface of discontinuity.

Choose a fluid element with volume $dxdydz$ as in Fig.1. From continuity equation (1) we have:

$$
\begin{aligned}
&(\frac{1}{2}dxdydz)\left(\frac{\partial\rho}{\partial t}\bigg|_{-dx/2,y,z}+\frac{\partial\rho}{\partial t}\bigg|_{dx/2,y,z}\right)+[\rho u_x]dydz \\
&+\left(\rho u_y\big|_{-dx/2,y+dy/2,z}-\rho u_y\big|_{-dx/2,y-dy/2,z}+\rho u_y\big|_{dx/2,y+dy/2,z}-\rho u_y\big|_{dx/2,y-dy/2,z}\right)(\frac{1}{2}dxdz) \quad . \qquad (4)\\
&+\left(\rho u_z\big|_{-dx/2,y,z+dz/2}-\rho u_z\big|_{-dx/2,y,z-dz/2}+\rho u_z\big|_{dx/2,y,z+dz/2}-\rho u_z\big|_{dx/2,y,z-dz/2}\right)(\frac{1}{2}dxdy)=0
\end{aligned}
$$

Square bracket represents the difference of a quantity at the two sides of the interface, $[\rho u_x]=\rho u_{x2}-\rho u_{x1}$. Similar equations can be established for momentum and energy equations (2) and (3).

As $dx \rightarrow 0$ we have

$$\left[\rho u_x\right]=0, \qquad (5)$$

$$\left[\rho u_x^2+p\right]=0,\ \left[\rho u_x u_y\right]=0\ \left[\rho u_x u_z\right]=0, \qquad (6)$$

$$\left[\rho u_x\left(E+\frac{p}{\rho}+\frac{1}{2}u_iu_i\right)-\kappa\frac{\partial T}{\partial x}\right]=0. \qquad (7)$$

Quantities in square brackets must be continuous across the interface. These are the matching conditions (boundary conditions BCs) for the solutions at the two sides of the interface.

Tangential discontinuity

There are two types of discontinuities depending on whether there is flow across the interface.

If there is no flow across the discontinuity surface, BCs (5)~(7) becomes:

$$u_x = 0, \tag{8}$$
$$\left[p\right] = 0, \tag{9}$$
$$\left[\kappa \frac{\partial T}{\partial x}\right] = 0. \tag{10}$$

Across the interface pressure and heat must be continuous. Tangential velocities and density can be discontinuous:

$$[u_y] \neq 0,\ [u_z] \neq 0,\ [\rho] \neq 0. \tag{11}$$

This is called *tangential discontinuity*.

Normal Discontinuity and Shock Wave

The second type of discontinuity is when mass flux across the interface is not zero. Then:

$$u_x \neq 0, \tag{12}$$
$$\left[\rho u_x\right] = 0, \tag{13}$$
$$\left[\rho u_x^2 + p\right] = 0, \tag{14}$$
$$\left[u_y\right] = 0,\ \left[u_z\right] = 0, \tag{15}$$
$$\left[\rho u_x \left(E + \frac{p}{\rho} + \frac{1}{2} u_i u_i\right) - \kappa \frac{\partial T}{\partial x}\right] = 0. \tag{16}$$

For a perfect gas,

$$E = c_v T = \frac{p}{\rho(\gamma - 1)},\ \gamma = c_p / c_v.$$

If the perfect gas is adiabatic ($\kappa = 0$), with Eq.(13), energy BC (16) is simplified to:

$$\left[\frac{\gamma p}{\rho(\gamma - 1)} + \frac{1}{2} u_x^2\right] = 0. \tag{17}$$

Equations in (15) show that tangential velocities must be continuous at the interface. Pressure, density, and normal velocity can be discontinuous. This is called *normal discontinuity*.

If any three of the six variables: u_{x1}, p_{01}, ρ_{01}, u_{x2}, p_{02}, ρ_{02}, are known, the other three can be explicitly expressed by solving Eqs.(13), (14), and (17). One example of normal discontinuity is when u_{x1} is supersonic. It is called shock wave, or shock, also called compression discontinuity. Since tangential velocities are continuous, one can always establish the coordinate in which the shock wave front is stationary and the flows on both sides are perpendicular to the shock surface. This shock is called normal shock.

For the normal shock, solving equations (13), (14), and (17), one can obtain the *Rankine-Hugoniot* relations in terms of $M_1 = u_{x1}/a_1$ (Landau&Lifshitz1959, p.331):

$$\frac{\rho_2}{\rho_1} = \frac{u_{x1}}{u_{x2}} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}, \tag{18}$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}, \tag{19}$$

$$\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - (\gamma-1)][(\gamma-1)M_1^2+2]}{(\gamma+1)^2 M_1}, \tag{20}$$

$$M_2^2 = \frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)}. \tag{21}$$

Any one of the two sets of equations, (13)~(16)&(17), or (18)~(21), can be used in applications. Sometimes it is more convenient to combine the two sets of equations in analysis.

If in the coordinate system tangential velocities are not zero, the above equations still hold except that the velocity in the equations is the normal velocity to the shock. Total velocities on both sides aren't perpendicular to the shock wave; therefore it is called oblique shock. An oblique shock can always be transformed into a normal shock by establishing a coordinate system moving with the tangential velocity along the surface.

Unsteady Surface of Discontinuity

Now we will discuss unsteady surface of discontinuity. Suppose the surface of discontinuity without disturbance is at $y = 0$ as in Fig.2. (It shouldn't matter whether the surface is horizontal or vertical. We use a horizontal surface here just to show how to develop the formulas in a coordinate system different from Fig.1.) After disturbed, the unsteady interface is at position $y = \zeta(x,t)$. A local coordinate system $\xi - \eta$ is established at the interface with ξ tangential to the interface and η normal to the interface.

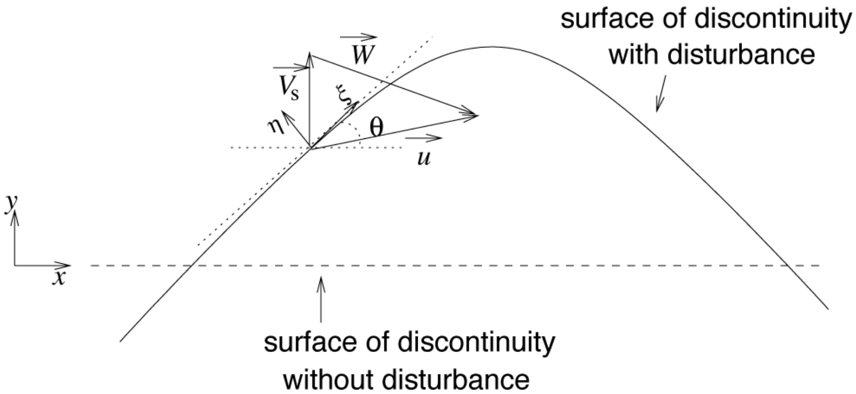

Fig.2, Unsteady surface of discontinuity.

A scalar is invariant in different coordinate systems. A vector, however, changes its component coordinates in different systems. In the $x-y$ coordinate system, flow velocity is $\vec{u}$, and velocity of the interface is

$$\vec{V}_s = V_s \vec{j}\ ,\ V_s = \frac{\partial \zeta}{\partial t}\ ,\ \vec{j}\ \text{: unit vector in } y. \tag{22}$$

In the $\xi - \eta$ system,

$$u_\xi = u_x \cos\theta + u_y \sin\theta\ ,\ u_\eta = -u_x \sin\theta + u_y \cos\theta\ , \tag{23}$$
$$V_{s\xi} = V_s \sin\theta\ ,\ V_{s\eta} = V_s \cos\theta\ ,$$

where $\theta = \mathrm{tg}^{-1} \dfrac{\partial \zeta}{\partial x}$.

Flow velocity in the $\xi - \eta$ system is

$$\vec{W} = \vec{u} - \vec{V}_s,$$
$$W_\eta = u_\eta - V_{s\eta} = -u_x \sin\theta + \left(u_y - \frac{\partial \zeta}{\partial t}\right)\cos\theta\ , \tag{24}$$
$$W_\xi = u_\xi - V_{s\xi} = u_x \cos\theta + \left(u_y - \frac{\partial \zeta}{\partial t}\right)\sin\theta\ , \tag{25}$$

Derivative in the normal direction of the interface is:

$$\frac{\partial}{\partial \eta} = -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}. \tag{26}$$

If the disturbance is small, equations (23)~(26) can be approximated at $y = 0$:

$$\theta \approx \sin\theta \approx \frac{\partial \eta}{\partial x},$$

$$u_\xi \approx u_x + u_y \frac{\partial \eta}{\partial x},\ u_\eta \approx -u_x \frac{\partial \eta}{\partial x} + u_y, \tag{27}$$

$$V_{s\xi} \approx V_s \frac{\partial \eta}{\partial x},\ V_{s\eta} \approx V_s, \tag{28}$$

$$W_\eta \approx -u_x \frac{\partial \eta}{\partial x} + u_y - \frac{\partial \eta}{\partial t}, \tag{29}$$

$$W_\xi \approx u_x + \left(u_y - \frac{\partial \eta}{\partial t}\right)\frac{\partial \eta}{\partial x}, \tag{30}$$

$$\frac{\partial}{\partial \eta} \approx -\frac{\partial \eta}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \tag{31}$$

These are the relations between the two coordinate systems we will need.

The $\xi - \eta$ system is not an inertial system. The conservation laws in this moving volume must be established in the $x - y$ system (cht23.doc):

$$\iiint_{V(t)} \frac{d\rho}{dt} dV + \oint\!\!\oint_{S(t)} \rho(u_j - V_{sj}) n_j d\vec{S} = 0, \tag{32}$$

$$\iiint_{V(t)} \frac{d(\rho u_i)}{dt} dV + \oint\!\!\oint_{S(t)} \left(\rho u_i(u_j - V_{sj}) + p\delta_{ij}\right) n_j dS = 0, \tag{33}$$

$$\iiint_{V(t)} \frac{d}{dt}\left[\rho(E + \frac{1}{2}u_i u_i)\right] dV + \oint\!\!\oint_{S(t)} \left[\rho(E + \frac{1}{2}u_i u_i)(u_j - V_{sj}) + p u_j - \kappa \frac{\partial T}{\partial x_j}\right] n_j dS = 0. \tag{34}$$

A similar fluid element as in Fig.1 can be established at the unsteady interface in Fig.2. Therefore across the interface at $y = \eta(x,t)$:

$$\left[\rho W_\eta\right] = 0, \tag{35}$$

$$\left[\rho u_i W_\eta + p\delta_{i\eta}\right] = 0,\ i.e.,\ \left[\rho W_i W_\eta + p\delta_{i\eta}\right] = 0,\ i = \xi, \eta, \tag{36}$$

$$\left[\rho(E + \frac{1}{2}u_i u_i) W_\eta + p u_\eta - \kappa \frac{\partial T}{\partial \eta}\right] = 0. \tag{37}$$

For *tangential discontinuity* at $y = \eta(x,t)$,

$$W_\eta = 0,\ \text{or}\ \left[u_\eta\right] = 0, \tag{38}$$

$$[p] = 0, \tag{39}$$

$$\left[\kappa \frac{\partial T}{\partial n}\right] = 0. \tag{40}$$

Expand p at $y = 0$,

$$p_1\big|_{y=0} + \left.\frac{\partial p_1}{\partial y}\right|_{y=0} \eta + \cdots = p_2\big|_{y=0} + \left.\frac{\partial p_2}{\partial y}\right|_{y=0} \eta + \cdots.$$

If the disturbance is small then

$$p_1 \approx p_2 \text{ at } y = 0.$$

Further using (27) ~ (31), equations (38)~(40) can be rewritten in terms of variables in the $x-y$ system at $y = 0$:

$$\frac{\partial \eta}{\partial t} + u_x \frac{\partial \eta}{\partial x} - u_y = 0, \tag{41}$$

$$[p] = 0, \tag{42}$$

$$\frac{\partial \eta}{\partial x}\left[\kappa \frac{\partial T}{\partial x}\right] = \left[\kappa \frac{\partial T}{\partial y}\right]. \tag{43}$$

It is noted that all the variables are total variables which are the sum of mean variables and perturbations. These three equations are the kinematic and dynamic boundary conditions at the mean surface of tangential discontinuity. Across the mean interface at $y = 0$, pressure must be continuous [Eq.(42)]. Although velocity normal to the unsteady interface (u_n) is continuous [Eq.(38)], u_y is not necessarily continuous. Instead Eq.(41) implies the continuity of displacement:

$$\eta_1(x,t) = \eta_2(x,t) = \eta(x,t). \tag{44}$$

To prove it, we define function $f(\vec{x},t)$:

$$f(\vec{x},t) = y - \eta(x,t).$$

$f(\vec{x},t) = 0$ at the interface, $f(\vec{x},t) > 0$ in medium 1, and $f(\vec{x},t) < 0$ in medium 2. At the interface $y = \eta(x,t)$,

$$\frac{Df(\vec{x},t)}{Dt} = 0, \text{ or,}$$

$$
\begin{aligned}
\frac{Df(\vec{x},t)}{Dt} &= \lim_{\Delta t\to 0}\frac{f(\vec{x}+\vec{u}\Delta t, t+\Delta t)-f(\vec{x},t)}{\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{f(\vec{x},t+\Delta t)-f(\vec{x},t)}{\Delta t}+\lim_{\Delta t\to 0}\frac{f(\vec{x}+\vec{u}\Delta t, t+\Delta t)-f(\vec{x},t+\Delta t)}{\Delta t}\\
&= \frac{\partial f}{\partial t}+\vec{u}\cdot\nabla f = 0
\end{aligned}
$$

Since $\frac{\partial f}{\partial t} = -\frac{\partial \xi}{\partial t}$, $\frac{\partial f}{\partial x} = -\frac{\partial \xi}{\partial x}$, $\frac{\partial f}{\partial y} = 1$, Eq.(41) is recovered.

Eqs.(41)~(43) are the boundary conditions for immiscible flows such as air and water. Sometimes this model is extended to other situations. For a jet, the two sides of the interface separating the jet and its environment may have the same gas. However the gas is often assumed immiscible at the interface, so that tangential discontinuity across the interface is applicable. Another example is plug flow over an impedance wall. The impedance wall is often considered a surface of tangential discontinuity with continuous displacement.

For *normal discontinuity*, Eqs.(35)~(37) are,

$$W_n \neq 0\,, \tag{45}$$

$$[\rho W_n] = 0\,, \tag{46}$$

$$[\rho W_n^2 + p] = 0\,, \tag{47}$$

$$[W_\tau] = 0\,, \tag{48}$$

$$\left[\rho(E+\frac{1}{2}u_i u_i)W_n + p u_n - \kappa\frac{\partial T}{\partial n}\right] = 0\,. \tag{49}$$

For small disturbance, at $y=0$:

$$\left[\rho\left(\frac{\partial \xi}{\partial t} + u_x\frac{\partial \xi}{\partial x} - u_y\right)\right] = 0\,, \tag{50}$$

$$\left[\rho\left(\frac{\partial \xi}{\partial t} + u_x\frac{\partial \xi}{\partial x} - u_y\right)^2 + p\right] = 0\,, \tag{51}$$

$$\left[u_x\right] + \left[u_y\right]\frac{\partial \xi}{\partial x} = 0\,, \tag{52}$$

$$\left[\left(\frac{\gamma p}{\rho(\gamma-1)} + \frac{1}{2}u_i u_i\right)\rho\left(u_x\frac{\partial \xi}{\partial x} + u_y - \frac{\partial \xi}{\partial t}\right) + p\frac{\partial \xi}{\partial t} - \kappa\left(-\frac{\partial \xi}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\right)\right] = 0\,. \tag{53}$$

These four equations are the matching conditions for normal discontinuity with small disturbance.

As the first example, we discuss the sound wave reflection/refraction at an interface of two immiscible media such as air and water as in Fig.3. The two fluids are assumed ideal, adiabatic, and quiescent ($\vec{u}_{01} = 0$, $\vec{u}_{02} = 0$). The interface is steady before sound is introduced from medium 1. Since there is no flow across the interface, the interface has tangential discontinuity. When there is no sound, pressure is continuous across the interface [BC (9)]:

$$p_{01} = p_{02}. \tag{54}$$

Density is discontinuous, and so is sound speed:

$$\rho_{01} \neq \rho_{02},\ a_{01} \neq a_{02}. \tag{55}$$

Impedance of a uniform medium to sound is $\rho_0 a_0$. Eq.(55) means there is impedance mismatch at the interface, which inevitably gives rise to sound reflection/refraction.

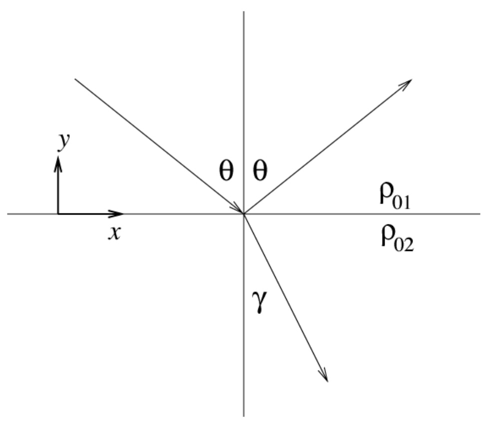

Fig.3, Sound transmission at surface of density discontinuity.

The interface itself is assumed stable. If there is gravity, the lighter fluid must be on the top of the heavier fluid, otherwise the interface will have Rayleigh-Taylor Instability.

Sound wave, introduced into the first medium, propagates towards the interface at an angle θ as in Fig.3. At the interface part of the sound wave is reflected back and part of it transmits into the second medium. The interface is perturbed with displacement $y = \eta(x,t)$.

When disturbance is small, tangential discontinuity requires continuous pressure and displacement across the interface at $y = 0$ [Eqs.(42)&(44)]. Normal velocity is generally not continuous. However, when there is no mean flow, normal velocity is continuous according to Eq.(41). Therefore the dynamic and kinematic boundary conditions of disturbance (the prime is omitted) at $y = 0$ are:

$$p_1 = p_2, \tag{56}$$
$$v_1 = v_2. \tag{57}$$

There is no vortical motion in the ideal, quiescent medium according to cht1.doc and cht7.doc. General acoustic solution in an ideal, quiescent medium with density ρ_0 and sound velocity a_0 has been given by Eqs.(22)~(24) in cht1.doc. The solutions are:

$$\begin{pmatrix} p \\ \vec{u} \end{pmatrix} = \text{Real}\left\{ \begin{pmatrix} \hat{p} \\ \hat{\vec{u}} \end{pmatrix} e^{-i\omega t} \right\}, \tag{58}$$
$$\hat{p} = A e^{i(\alpha x \pm \beta y)}, \tag{59}$$
$$\hat{u} = \frac{\alpha}{\rho_0 \omega} \hat{p}, \tag{60}$$
$$\hat{v} = \frac{1}{i \rho_0 \omega} \frac{\partial \hat{p}}{\partial y}. \tag{61}$$

The equation of computing β from α is in cht1.doc. If α is real and $-\omega / a_0 \le \alpha \le \omega / a_0$, the wave is a pure propagating plane sound wave. If α is real but $\alpha > \omega / a_0$ or $\alpha < -\omega / a_0$, the wave only propagates in x direction with phase speed ω / α, with decaying amplitude in y direction. For all other α, the wave is decayed plane wave propagating in an oblique direction. Except for the propagating waves, effect of disturbances is only local to the interface.

Incident wave

Assume the incident wave is a propagating wave with incident wave angle θ to the normal direction (Fig.3):

$$\alpha = k_1 \sin\theta, \; \beta_1 = k_1 \cos\theta, \; k_1 = \omega / a_{01}, \tag{62}$$

$$\hat{p}_i = e^{ik_1 (x \sin\theta - y \cos\theta)}, \; \hat{u}_i = \frac{1}{\rho_{01} a_{01}} \hat{p}_i \sin\theta, \; \hat{v}_i = -\frac{1}{\rho_{01} a_{01}} \hat{p}_i \cos\theta. \tag{63}$$

To ensure propagating wave in +x direction, α must be real and $0 < \alpha \le \omega / a_{01}$, then β_1 is also real, and the reflected wave is a propagating wave.

Reflected wave

From Eqs.(56)&(57), the reflected wave and transmitted wave must have the same α as the incident wave. Thus the reflected wave solutions are:

$$\hat{p}_r = R \, e^{ik_1 (x \sin\theta + y \cos\theta)}, \; \hat{u}_r = \frac{1}{\rho_{01} a_{01}} \hat{p}_r \sin\theta, \; \hat{v}_r = \frac{1}{\rho_{01} a_{01}} \hat{p}_r \cos\theta. \tag{64}$$

Refracted wave

The transmitted (refracted) wave solutions are:

$$\hat{p}_t = T\, e^{i(\alpha x - \beta_2 y)},\ \hat{u}_t = \frac{\alpha}{\rho_{02}\omega}\hat{p}_t,\ \hat{v} = -\frac{\beta_2}{\rho_{02}\omega}\hat{p}_t. \tag{65}$$

Suppose the incident wave is from the medium with smaller density, *i.e.*, $\rho_{01} < \rho_{02}$. Since $a_0 = \sqrt{\gamma p_0/\rho_0}$, we have $a_{01} > a_{02}$. For the propagating incident wave, α is real and $\alpha \le \omega/a_{01}$. Therefore $\alpha < \omega/a_{02}$, β_2 is real and thus the refracted wave is a propagating wave in the direction with angle θ_t as in Fig.3. In this case the refraction wave solutions are:

$$\alpha = k_1 \sin\theta = k_2 \sin\theta_t,\ k_2 = \omega/a_{02},\ \beta_2 = k_2 \cos\theta_t.$$

$$\hat{p}_t = T\, e^{ik_2(x\sin\theta_t - y\cos\theta_t)},\ \hat{u}_t = \frac{1}{\rho_{02}a_{02}}\hat{p}_t \sin\theta_t,\ \hat{v}_t = -\frac{1}{\rho_{02}a_{02}}\hat{p}_t \cos\theta_t. \tag{66}$$

The refracted wave angle is determined by:

$$\sin\theta_t = a_{02}/a_{01}\sin\theta. \tag{67}$$

This is the *Snell's law*. When the incident wave is in the lighter fluid, the refracted wave is bent towards the normal direction.

On the other hand, if the incident wave is in the heavier fluid, then $a_{02} > a_{01}$. Eq.(67) shows the refracted wave is bent away from the normal direction. It is possible that $\omega/a_{02} < \alpha \le \omega/a_{01}$, *i.e.* $a_{02}/a_{01}\sin\theta > 1$, for which no angle θ_t can be determined from (67) and $\beta_2 = i\sqrt{\alpha^2 - k_2^2}$ is a pure imaginary complex number. In this case, the refracted wave only propagates in *x* direction (along the interface) with phase speed ω/α, but the amplitude decays in -*y* direction. (Check out Fig. 12 in paper Tam&Ju, Computation of the Aliasing and the Interface Transmission Benchmark Problems by the Dispersion-Relation-Preserving Scheme, 4th CAA workshop NASA/CP-2004-212954). There is a critical angle,

$$\sin\theta_c = a_{01}/a_{02}. \tag{68}$$

When the incident wave angle is bigger than this angle, the incident wave will be totally reflected.

Reflection coefficient

Coefficients R and T in equations (64) and (65) can be determined from boundary conditions (56)&(57):

$$1 + R = T,$$

$$1 - R = \frac{\rho_{01} a_{01}}{\rho_{02}\omega}\frac{\beta_2}{\cos\theta} T.$$

Therefore,

$$T = \frac{2}{1 + \frac{\rho_{01} a_{01}}{\rho_{02}\omega}\frac{\beta_2}{\cos\theta}}, \tag{69}$$

$$R = T - 1. \tag{70}$$

When $a_{02} > a_{01}$ and the incident angle is larger than critical angle θ_c, $\beta_2 = i\sqrt{\alpha^2 - k_2^2}$ is purely imaginary, and $|R| = 1$, that means the incident wave is totally reflected.

Baroclinic vorticity production

The two uniform media are vorticity free. However, as we know in cht5.doc, in a region with nonuniform density, the mass center and the geometric center of the a fluid element doesn't coincide. When sound waves propagate into this region, vorticity will be generated. This is called Baroclinic Vorticity Production. According to cht5.doc, the indicator of vorticity generation is circulation instead of vorticity, since vorticity of a material fluid element may change while the circulation does not. To quantify the circulation at the surface of discontinuity, one may define the circulation density γ as in Fig.4:

$$\gamma = \frac{\text{Circulation around ABCD}}{\text{Length AB}}$$

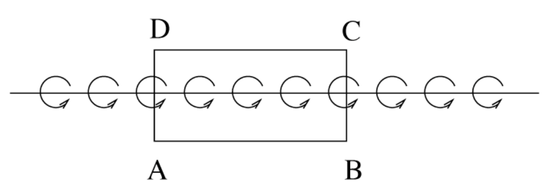

Fig.4, Circulation at the interface.

Than:

$$\hat{\omega} = \hat{u}_t - \hat{u}_r - \hat{u}_i = Te^{i\alpha x}\frac{\sin\theta}{a_{01}}\left(\frac{1}{\rho_{02}} - \frac{1}{\rho_{01}}\right). \tag{71}$$

Vorticity is generated because of the misaligned acoustic pressure and the mean density gradients.

Physical explanation

Suppose the wave front of the incident sound hits the interface at point p_1 at time t_1. It generates one cylindrical wave in medium 1 and one cylindrical wave in medium 2. At t_2 the incident wave front hits p_2 and generates two other cylindrical waves, one on each side of the interface. Fig.5 shows the wave pattern at time t_3 when the incident wave just hits p_3. At this time waves generated at previous times (t_1 and t_2) have propagated. The envelope of these waves in medium 1 forms the reflected wave front; the envelope of the waves in medium 2 forms the transmitted wave front. Sound speeds on both sides are different, so are angles of the reflected and transmitted waves.

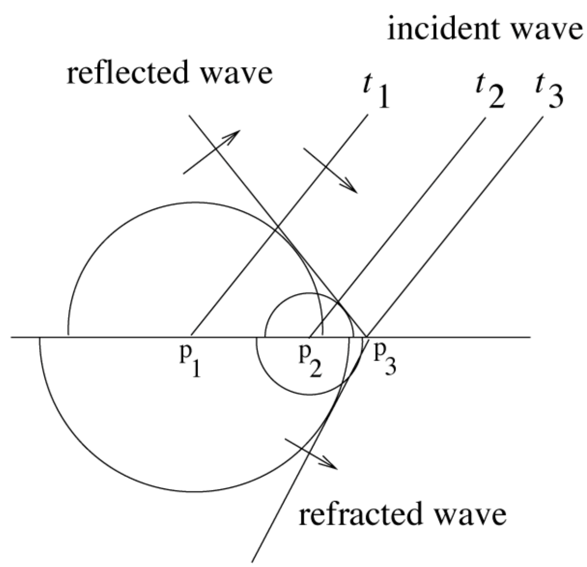

Fig.5, Sound wave transmission at surface of density discontinuity at time t_3.

Wave Interaction with Shock Wave

One-dimensional analysis of wave interaction with shock wave was given by Powell. In one dimension there is no vorticity wave and there is no sound wave propagating upstream in the supersonic medium. If the incident wave before the shock is a sound wave, two waves exist after the shock: the transmitted, amplified sound wave and the convective entropy wave. If the incident wave is an entropy wave, the downstream waves are transmitted entropy wave and intense sound wave.

In two or three dimensions, when a vortex hits the shock wave, there exist all three modes after the shock: vorticity wave, sound wave, and entropy wave. This is believed to be the mechanism of broadband shock associated noise in a supersonic jet. Vortex/shock interaction depends on whether the flow after the shock is subsonic or supersonic.

Vortex/shock interaction can be investigated analytically by fitting the shock with Rankine-Hugoniot relation. Moore1954 analyzed the interaction of a normal shock with an oblique plane sound wave or vortex wave. Ribner discussed plane vortex wave/shock interaction (Ribner1954) and cylindrical vortex with axis aligned with the shock (Ribner1985). Erlebacher, Hussaini&Shu discussed cylindrical vortex with axis normal to the shock (*i.e.*, longitudinal vortex). Numerical investigation was done by many researchers, such as Meadows, Kumar&Hussaini1991, Erlebacher, Hussaini&Shu, etc.

Suppose the undisturbed shock is at $x = 0$. The flow comes in from region 1 to region 2. The BCs at $x = 0$ are from (13)~(15)&(17):

$$[\rho_0 u_0] = 0, \qquad (72)$$

$$[\rho_0 u_0^2 + p_0] = 0, \qquad (73)$$

$$v_{01} = v_{02} = 0, \qquad (74)$$

$$\left[\frac{\gamma p_0}{\rho_0(\gamma - 1)} + \frac{1}{2}u_0^2\right] = 0. \qquad (75)$$

Or one can use Rankine-Hugoniot equations (18)~(21) directly.

With small disturbance, the shock wave surface is disturbed to position $x = \xi(y,t)$. The BCs satisfied at $x = 0$ Eqs.(50)~(53) are linearized to be:

$$\left[\rho_0\left(u' - \frac{\partial \xi}{\partial t}\right) + \rho' u_0\right] = 0, \qquad (71)$$

$$\left[2\rho_0 u_0\left(u' - \frac{\partial \xi}{\partial t}\right) + \rho' u_0^2 + p'\right] = 0, \qquad (72)$$

$$[v'] + [u_0]\frac{\partial \xi}{\partial y} = 0, \qquad (73)$$

$$\left[\left(\frac{\gamma p_0}{\gamma - 1} + \frac{3}{2}\rho_0 u_0^2\right)\left(u' - \frac{\partial \xi}{\partial t}\right) + u_0\left(\frac{\gamma p'}{\gamma - 1} + \frac{1}{2}u_0^2 \rho'\right)\right] = 0. \qquad (74)$$

Upstream sound wave/shock interaction

Suppose the incident wave from region 1 is a plane sound wave. According to Eqs.(61)~(64) in cht1.doc, a plane sound wave can be represented by:

$$p_1 = Ae^{i(\alpha_{1a}x+\beta y-\omega t)}, \tag{75}$$

$$u_1 = \frac{A\alpha_{1a}}{\rho_{01}(\omega-\alpha_{1a}u_{01})}e^{i(\alpha_{1a}x+\beta y-\omega t)}, \tag{76}$$

$$v_1 = \frac{A\beta}{\rho_{01}(\omega-\alpha_{1a}u_{01})}e^{i(\alpha_{1a}x+\beta y-\omega t)}, \tag{77}$$

$$s_1 = 0,\ p_1 = a_{01}^2\rho_1. \tag{78}$$

Physical variables in time domain are real parts of these variables in Eqs.(75)~(78). *A* and ω are amplitude (strength) and annular frequency of the sound wave. β is the wave number in *y* direction. Wave number in *x* direction is computed by Eq.(66) of cht1.doc:

$$\alpha_{1a} = \frac{M_{01}\omega/a_{01} + \sqrt{M_{01}^2-1}\sqrt{\beta^2+\omega^2/(u_{01}^2-a_{01}^2)}}{M_{01}^2-1}. \tag{79}$$

Check Fig.10 of cht1.doc for the branch cut.

No sound wave is reflected at the shock since the flow in region 1 is supersonic. The sound wave is refracted in region 2 behind the shock. Vorticity and entropy waves are generated at the shock. All the waves after the shock have the same frequency and wave number in *y* direction as the incident sound wave.

For the sound wave in region 2:

$$p_{2a} = Be^{i(\alpha_{2a}x+\beta y-\omega t)}, \tag{80}$$

$$u_{2a} = \frac{B\alpha_{2a}}{\rho_{02}(\omega-\alpha_{2a}u_{02})}e^{i(\alpha_{2a}x+\beta y-\omega t)}, \tag{81}$$

$$v_{2a} = \frac{B\beta}{\rho_{02}(\omega-\alpha_{2a}u_{02})}e^{i(\alpha_{2a}x+\beta y-\omega t)}, \tag{82}$$

$$s_{2a} = 0,\ p_{2a} = a_{02}^2\rho_{2a}. \tag{83}$$

Assume flow after the shock is subsonic, then,

$$\alpha_{2a} = \frac{-M_{02}\omega/a_{02} + i\sqrt{1-M_{02}^2}\sqrt{\beta^2-\omega^2/(a_{02}^2-u_{02}^2)}}{1-M_{02}^2}. \tag{84}$$

Solution of vorticity wave in region 2 is expressed by Eqs.(72)~(75) in Cht1.doc. They are:

$$u_{2v} = Ce^{i(\alpha_{2c}x+\beta y-\omega t)}, \tag{85}$$

$$v_{2v} = -C\frac{\alpha_{2c}}{\beta}e^{i(\alpha_{2c}x+\beta y-\omega t)}, \tag{86}$$

$$p_{2v} = 0,\ s_{2v} = 0,\ \rho_{2v} = 0. \tag{87}$$

where $\alpha_{2c} = \omega / u_{02}$.

Entropy wave in region 2 is represented by Eqs.(76)~(79) in cht1.doc:

$$p_{2e} = 0,\ u_{2e} = 0,\ v_{2e} = 0, \tag{88}$$

$$s_{2e} = -D\frac{c_p}{\rho_0} e^{i(\alpha_{2c}x+\beta y-\omega t)},\ \rho_{2e} = De^{i(\alpha_{2c}x+\beta y-\omega t)}. \tag{89}$$

The total waves in region 2 are sum of these three waves:

$$p_2 = p_{2a} = Be^{i(\alpha_{2a}x+\beta y-\omega t)}, \tag{90}$$

$$u_2 = u_{2a} + u_{2v} = \left[\frac{B\alpha_{2a}}{\rho_{02}(\omega - \alpha_{2a}u_{02})}e^{i\alpha_{2a}x} + Ce^{i\alpha_{2c}x}\right]e^{i(\beta y-\omega t)}, \tag{91}$$

$$v_2 = v_{2a} + v_{2v} = \left[\frac{B\beta}{\rho_{02}(\omega - \alpha_{2a}u_{02})}e^{i\alpha_{2a}x} - C\frac{\alpha_{2c}}{\beta}e^{i\alpha_{2c}x}\right]e^{i(\beta y-\omega t)}, \tag{92}$$

$$\rho_2 = \rho_{2a} + \rho_{2e} = \left[\frac{B}{a_{02}^2}e^{i\alpha_{2a}x} + De^{i\alpha_{2c}x}\right]e^{i(\beta y-\omega t)}, \tag{93}$$

$$s_2 = s_{2e} = -D\frac{c_p}{\rho_{02}}e^{i(\alpha_{2c}x+\beta y-\omega t)}. \tag{94}$$

Shock wave displacement $\xi = \hat{\xi}e^{i(\beta y-\omega t)}$. Substitute incident wave Eqs.(75)~(78) and waves after the shock Eqs.(90)~(93) into the four matching conditions, Eqs.(71)~(74). From the four equations, *B, C, D*, and $\hat{\xi}$ can be solved in terms of *A*.

Upstream plane vorticity wave/shock interaction

Suppose a plane vorticity wave is propagating towards the shock:

$$u_1 = Ae^{i(\alpha_{1c}x+\beta y-\omega t)}, \tag{95}$$

$$v_1 = -A\frac{\alpha_{1c}}{\beta}e^{i(\alpha_{1c}x+\beta y-\omega t)}, \tag{96}$$

$$p_1 = 0,\ s_1 = 0,\ \rho_1 = 0. \tag{97}$$

where $\alpha_{1c} = \omega / u_{01}$.

The three waves on the downstream of the shock wave can be expressed by equations (80)~(94). Substitute incident wave Eqs.(95)~(97) and waves after the shock Eqs.(90)~(93) into the four matching conditions, Eqs.(71)~(74), from which *B, C, D*, and $\hat{\xi}$ can be solved in terms of *A*.

Upstream plane entropy wave/shock interaction

If the upstream incident wave is a plane entropy wave, then,

$$p_1 = 0,\ u_1 = 0,\ v_1 = 0, \tag{98}$$

$$s_1 = -A\frac{c_p}{\rho_{01}} e^{i(\alpha_{1c}x+\beta y-\omega t)},\ \rho_1 = A e^{i(\alpha_{1c}x+\beta y-\omega t)}. \tag{99}$$

In the same way as before, *B, C, D*, and $\hat{\xi}$ can be solved by matching waves on both side of the shock.

Downstream plane sound wave/shock interaction

Now let's discuss the case where incident wave is from downstream of the shock. Since no wave can propagate upstream against the supersonic flow in region 1, one has:

$$p_1 = 0,\ u_1 = 0,\ v_1 = 0,\ s_1 = 0,\ \rho_1 = 0. \tag{100}$$

The only possible incident wave from the downstream is sound. It is represented by equations:

$$p^-_{2a} = A e^{i(\alpha^-_{2a}x+\beta y-\omega t)}, \tag{101}$$

$$u^-_{2a} = \frac{A\alpha^-_{2a}}{\rho_{02}(\omega-\alpha^-_{2a}u_{02})} e^{i(\alpha^-_{2a}x+\beta y-\omega t)}, \tag{102}$$

$$v^-_{2a} = \frac{A\beta}{\rho_{02}(\omega-\alpha^-_{2a}u_{02})} e^{i(\alpha^-_{2a}x+\beta y-\omega t)}, \tag{103}$$

$$s^-_{2a} = 0,\ p^-_{2a} = a_{02}^2\rho^-_{2a}, \tag{104}$$

where,

$$\alpha^-_{2a} = \frac{-M_{02}\omega/a_{02} - i\sqrt{1-M_{02}^2}\sqrt{\beta^2-\omega^2/(a_{02}^2-u_{02}^2)}}{1-M_{02}^2}. \tag{105}$$

The reflected waves are represented by Eqs.(80)~(89). The total waves in region 2 are:

$$p_2 = \left(Ae^{i\alpha^-_{2a}x} + Be^{i\alpha_{2a}x}\right)e^{i(\beta y-\omega t)}, \tag{106}$$

$$u_2 = \left[\frac{A\alpha^-_{2a}}{\rho_{02}(\omega-\alpha^-_{2a}u_{02})}e^{i\alpha^-_{2a}x} + \frac{B\alpha_{2a}}{\rho_{02}(\omega-\alpha_{2a}u_{02})}e^{i\alpha_{2a}x} + Ce^{i\alpha_{2c}x}\right]e^{i(\beta y-\omega t)}, \tag{107}$$

$$v_2 = \left[\frac{A\beta}{\rho_{02}(\omega-\alpha^-_{2a}u_{02})}e^{i\alpha^-_{2a}x} + \frac{B\beta}{\rho_{02}(\omega-\alpha_{2a}u_{02})}e^{i\alpha_{2a}x} - C\frac{\alpha}{\beta}e^{i\alpha_{2c}x}\right]e^{i(\beta y-\omega t)}, \tag{108}$$

$$\phi_2 = \left[\frac{A}{a_{02}^2} e^{-i\alpha_{2a}x} + \frac{B}{a_{02}^2} e^{i\alpha_{2a}x} + De^{i\alpha_{2c}x}\right] e^{i(\beta y - \omega t)}, \tag{109}$$

$$s_2 = s_{2e} = -D\frac{c_p}{\rho_{02}} e^{i(\alpha_{2c}x + \beta y - \omega t)}. \tag{110}$$

Substitute Eqs.(100) and (106)~(110) into the four matching conditions, Eqs.(71)~(74). *B*, *C*, *D*, and $\hat{\alpha}$ can then be solved in terms of *A*.