Time Domain Impedance Boundary Conditions

Hongbin Ju
Department of Mathematics
Florida State University, Tallahassee, FL.32306
www.aeroacoustics.info
Please send comments to: hju@math.fsu.edu

Impedance in Frequency Domain

We have discussed equations for sound waves in cht12.doc. For the acoustic mode in an ideal stationary medium, the linear Euler equations are:

$$-\frac{\partial p}{\partial t} = \frac{\partial u_j}{\partial x_j}, \ \frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i}, \tag{1}$$
$$\nabla \times \vec{u} = 0, \ \rho = p.$$

All the variables are perturbations over the stationary medium with density ρ_0 and sound speed a_0. The scales for the variables are: length: L, velocity: sound speed a_0, time: L/a_0, density: ρ_0, pressure p: $\rho_0 a_0^2$.

Since $\nabla \times \vec{u} = 0$, velocity potential ϕ can be introduced in a singly-connected domain:

$$\vec{u} = \nabla\phi. \tag{2}$$

Any function of time, $\phi^*(t)$, can be added to ϕ, but it will not affect the equations and analysis. With this velocity potential ϕ, $\nabla \times \vec{u} = 0$ is automatically satisfied, and equations (1) can be rewritten as:

$$-\frac{\partial p}{\partial t} = \nabla^2\phi, \ \frac{\partial \phi}{\partial t} = -p. \tag{3}$$

Which leads to:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi = 0, \text{ or } \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0. \tag{4}$$

These are called *wave equations*. They are linear second order hyperbolic equations. Any acoustic variable satisfies this acoustic wave equation.

In industry a Fourier Transform (FT) pair with time factor $e^{i\omega t}$ like the following is usually adopted:

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt\,,\ \ h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{h}(\omega) e^{i\omega t} d\omega\,.$$

By taking the FT of equations (4) and (3), we have:

$$\nabla^2 \hat{\phi} = (i\omega)^2 \hat{\phi},\ \ \hat{p} = -i\omega\hat{\phi}. \tag{5}$$

The Fourier transformation reduces the space-time initial-boundary value problem, Eqs.(1), into a boundary problem, Eq.(5). Eq.(5) is a Poisson equation. Frequency is a parameter in this equation.

Two boundary conditions are needed for the second order partial differential equation (5) to render a unique solution. There are two ways to set these boundary conditions. To facilitate the description, here we use the one dimension problem as an example. There are two boundaries in one dimension space: the two ends *A* and *B*. One may set the two boundary conditions solely at *A*. In this way you have no control at the boundary *B*. The characteristic of the boundary *B* is totally determined by the boundary conditions at *A*. The other way of setting the two boundary conditions is to set one boundary condition at each boundary end, *A* and *B*. Obviously to control the boundary characteristics both at *A* and *B*, the second method is the only way.

In almost all acoustics problems, the appropriate boundary condition for Eq.(5) is to set one boundary condition at all the boundaries. There are three types of boundary conditions. The first type is Dirichlet boundary condition in which $\hat{\phi}$ is specified at the boundary. This is equivalent to specify pressure $\hat{p} = -i\omega\hat{\phi}$ at the boundary. The second type is Neumann boundary condition. It specifies $\hat{u}_n = \partial\hat{\phi}/\partial n$, velocity normal to the boundary, on the boundary of the domain. The third type of boundary condition is the linear combination of the first two boundary conditions, which sets $a\hat{\phi} + b\partial\hat{\phi}/\partial n = 0$ on boundary. This is to set $Z = \hat{p}/\hat{u}_n$ at the boundary. *Z* is the impedance in Acoustics and Electromagnetics. The third type of boundary condition is also called impedance boundary condition:

$$\hat{p} = Z\hat{u}_n\,. \tag{6}$$

The scale of *Z* in this equation is $\rho_0 a_0$, the impedance of free air. Impedance with scale of $\rho_0 a_0$ is called specific impedance.

Impedance is used to characterize a boundary only for ideal acoustic equations (1). For viscous flow, there is vorticity in boundary layer near walls. Impedance then is used at the outside of the boundary layer. The boundary layer analysis can give the impedance at the outside of the layer. (cf. Howe1998, chapter5)

Impedance is a complex number:

$$Z = R + \mathrm{i}X\,.$$

R is the resistance, and *X* the reactance. Resistance and reactance are generally dependent on frequency, sound intensity, and adjacent mean flow. The microscopic flow at the sound absorbing material is very complicated. For engineering applications, a gross macroscopic description of all effects of the treatment panels on the incident waves is preferred. Any object such as Helmholtz resonator, cavity, etc., can be modeled by the single quantity: impedance.

Impedance can be measured experimentally. Experimental impedance data are usually on a positive, narrow banded frequency range. The majority of experiments are restricted to two classes. One is the impedance tube. It is based on the reflection of plane waves and the measurement of standing wave patterns. This method is restricted to normal reflections and only for locally reactive materials. The other class of impedance measurement is developed as the field and in-situ technique, based on the reflection of harmonic spherical waves from a point source.

Here are some examples of experimentally measured impedance.

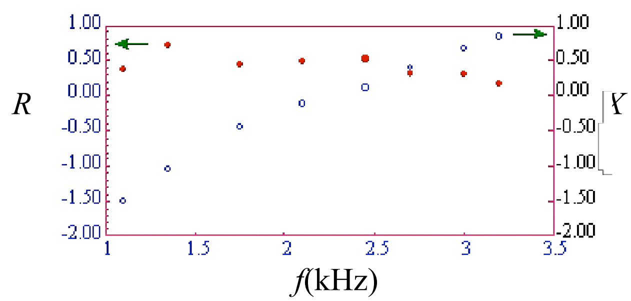

Fig.1, Measured resistance *R* (solid dots) and reactance *X* (open dots) of a 6.7%-perforated treatment panel for sound absorption at low sound intensity without flow. (Motsinger&Kraft1991, Tam&Auriault1996)

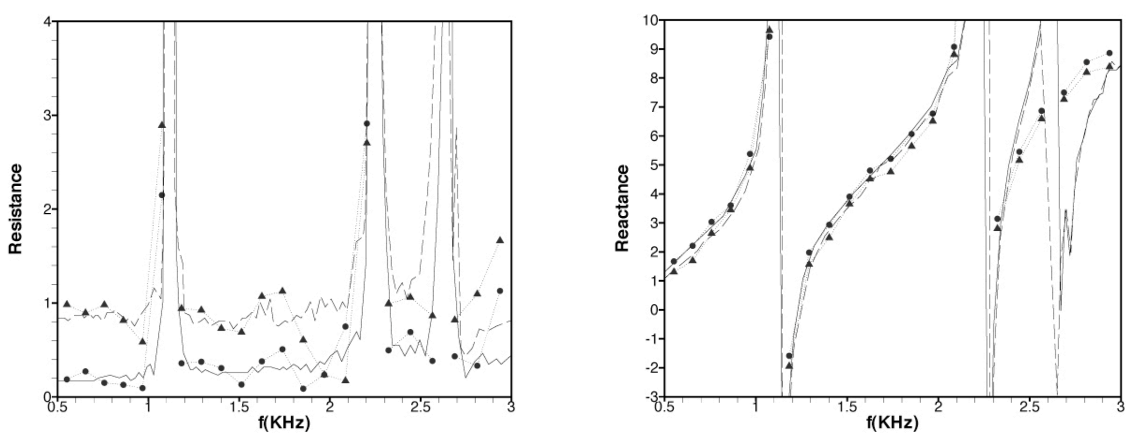

Fig.2, Comparison between calculated and measured impedance of a 90° slit with 0.05-inch width backed by 2"∏2"∏6" cavity in an impedance tube. Broadband incident sound,—— 120dB experiment, ···●··· 120dB calculation, — — —140dB experiment, ···▲··· 140dB calculation. (Tam, et.al. 2005)

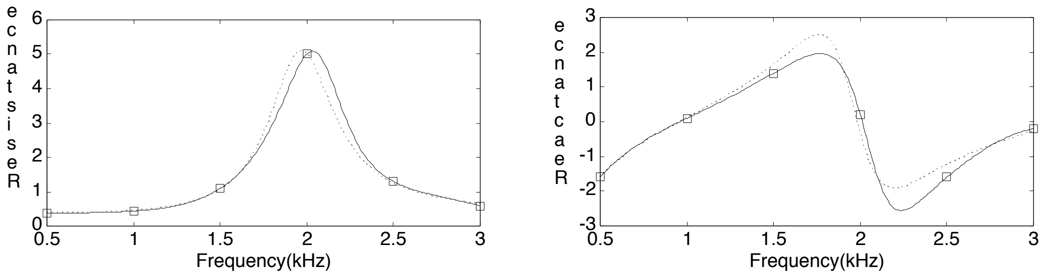

Fig.3, Impedance of constant depth ceramic tubular liner. , fitted data; □ , model; and - - -, Ozyoruk,Long,Jones1998.

Impedance can also be deducted analytically. Such as the impedance for a Helmholtz resonator:

$$Z(\omega) = R + i\left[-\cot(\omega L / a_0) + M_H \omega\right], \tag{7}$$

where resistance R is a constant, $M_H \omega$ is the mass reactance, and $-\cot(\omega L / a_0)$ is the cavity reactance (L: cavity depth). Another example is the impedance model of a pipe open end [Morse&Ingard, p.384]:

$$Z(\omega) = 1 - \frac{2J_1(\omega)}{\omega} + M(\omega)i\ ,\ M(\omega) = \frac{4}{\pi}\int_0^{\pi/2} \sin(\omega \cos x)\sin^2 x dx\ , \tag{8}$$

where $J_1(\cdot)$ is the first order Bessel function, and ω has been non-dimensionalized by sound speed C_0 and pipe diameter d.

In industry applications, impedance can also be estimated by semi-empirical formulas, such as the two-parameter impedance model for typical outdoor grounds (Li, Taherzadeh&Attenborough1997):

$$Z = 0.436(1 - i)(38 / f)^{0.5} - 0.2922 i / f\ , f \text{ in kHz.} \tag{9}$$

Time Domain Impedance Boundary

In the last section we have discussed the boundary conditions for the unique solution of the acoustic equation in time domain, Poisson equation (5). Uniqueness of the solution in time domain, Eqs.(1), (3), or (4), can be proven by energy integral method (Pierce1989 p.171). In a singly connected region, the solution is uniquely determined if the normal velocity (or velocity potential, or pressure) distribution is prescribed at surfaces enclosing the region. Setting normal velocity at the boundary is only a sufficient, not necessary, condition. Here we will show if properly modeled, impedance boundary (6) is also sufficient for the uniqueness of the time domain equations.

To facilitate the discussion, we begin with the one-dimensional equation:

$$\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}=0, \ \frac{\partial p}{\partial t}+\frac{\partial u}{\partial x}=0.$$

The equations can be uncoupled as:

$$\frac{\partial u^+}{\partial t}+\frac{\partial u^+}{\partial x}=0, \ \frac{\partial u^-}{\partial t}-\frac{\partial u^-}{\partial x}=0, \tag{10}$$

$$u^+ = u + p, \ u^- = u - p.$$

The solutions to (10) are:

$$u^+ = F(x-t), \ u^- = G(x+t). \tag{11}$$

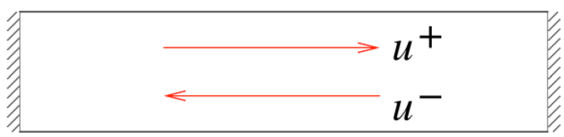

Fig.4, Propagating waves in one dimension.

u^+ represents a right propagating acoustic wave, and u^- the left propagating acoustic wave. F and G are two arbitrary functions of their arguments. They need to be determined by *two* boundary conditions. In one-dimensional problems, shown in the figure above, there are two ways to impose the two boundary conditions. The first is to set two boundary conditions at one boundary, leaving the other boundary unattained. For example, set u^+ and u^-, or u and p, at the left boundary. There must be no boundary condition imposed at the right boundary otherwise the system will be overdetermined, which means the characteristic of the right boundary can not be imposed in this method. And it is difficult to extend this kind of boundary conditions to multidimensional problems. In two- or three- dimensional spaces, waves propagate in all directions. It is difficult to figure out which part of the boundary needs the boundary condition, with the rest untouched to ensure the system well determined (neither overdetermined nor underdetermined).

The other way is to set one boundary at each boundary, such as u^- at the left boundary and u^+ at the right boundary, or u^- at the right boundary and u^+ at the left boundary. The latter makes more sense physically. This method can be used in multidimensional problems easily by just setting one boundary condition at the whole boundary.

At the right boundary $x = x_b$, u^- is the reflection wave of the incident wave u^+. u^+ is set at the left boundary $x = x_a$. It is known at the right boundary at all time. The right boundary condition must not affect the incident wave, the principal of causality, or domain of dependence. Only u^- can be specified at the boundary. It should be determined from u^+ and the boundary characteristics. This is the consequence of causality.

From Eq.(6) and Eq.(10), in frequency domain we have,

$$\hat{u}^{-} = \hat{W}\hat{u}^{+}, \ \hat{W} = \frac{1-Z}{1+Z}. \tag{12}$$

The reflection wave $\hat{u}^{-}$ is uniquely determined by the impedance boundary condition. If the impedance is properly extended to the whole ω-axis, the reflection wave should be the convolution of the incident wave u^{+} and the boundary response function $W(t)$:

$$u^{-}(t) = \int_{-\infty}^{\infty} W(\tau)u^{+}(t-\tau)d\tau = \int_{-\infty}^{0} W(\tau)u^{+}(t-\tau)d\tau + \int_{0}^{\infty} W(\tau)u^{+}(t-\tau)d\tau. \tag{13}$$

The integration from $-\infty$ to 0 means the reflection wave may depend on the future incident waves. This is obviously the violation of causality. Causality requires there is no reflection before the incident waves reach the boundary. It is a very important property of Acoustics (Pierce1989, p.172), and it is often incorporated, either explicitly or implicitly, in posing acoustic boundary value problems. The principal difference between a frequency-domain approach and a time-domain approach is the need to observe causality in time domain. The time domain boundary condition must be:

$$u^{-}(t) = \int_{0}^{\infty} W(\tau)u^{+}(t-\tau)d\tau; \ W(t)=0 \text{ when } t<0. \tag{14}$$

Causality is implied in this equation since only the integration on the positive time axis is involved. Appropriate impedance model Z or $\hat{W}$ must have the time response function satisfying $W(t)=0$ when $t<0$.

The time domain impedance boundary condition can also be established from Eq.(6):

$$p(t) = \int_{0}^{\infty} \bar{Z}(\tau)u(t-\tau)d\tau. \tag{15}$$

Here $\bar{Z}(\tau)$ is the inverse Fourier transform of $Z(\omega)$ that satisfies $\bar{Z}(\tau)=0$ for $\tau<0$. Since $u^{+}(t)$ is known at the boundary, one can calculate $u(t)$ from $u(t) + \int_{0}^{\infty} \bar{Z}(\tau)u(t-\tau)d\tau = u^{+}(t)$, and then $u^{-}(t) = u(t) - \int_{0}^{\infty} \bar{Z}(\tau)u(t-\tau)d\tau$ is uniquely determined.

Similarly one can use the next frequency domain impedance boundary and its time domain equivalent:

$$\hat{u}(\omega) = \hat{Y}(\omega)\hat{p}(\omega), \tag{16}$$

$$u(t) = \int_0^{\infty} Y(\tau) p(t - \tau) d\tau. \tag{17}$$

Here, $\hat{Y}(\omega) = 1/Z(\omega)$ is the admittance and $Y(\tau) = 0$ for $\tau < 0$.

Any of the three sets of the boundary conditions, (12)&(14), (6)&(15), or (16)&(17), can be used. We will use the reflection coefficient $\hat{W}$ and its time response $W(t)$, (12)&(14), in the next analysis. For passive boundaries, resistance $R \geq 0$, and then $|\hat{W}| \leq 1$ is assured from the bilinear transform in Eq.(12) , so that stability is assured. Other sets of IBC have no such property.

Damped Spring-Mass Oscillator: The Three Parameter Model

Damped Spring-Mass Oscillator

Fig.1 shows the measured impedance of a 6.7%-perforated treatment panel at low sound intensity without flow (Motsinger&Kraft1991). One may find a similar figure in Morse&Ingard1968, Figure 9.11. This is the typical impedance for a Helmholtz resonator (Fig.5), similar to a damped mass-spring oscillator, in the frequency range away from the resonance. In figures 2 and 3, resonance frequency is where the resistance and reactance vary drastically. Away from the resonance frequencies, the impedance shapes are pretty much like that in Fig.1.

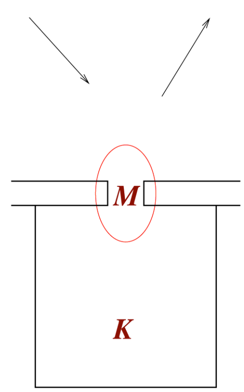

Fig.5, Helmholtz resonator.

Impedance in Fig.1 can be represented by the next three-parameter model (Tam&Auriault1996, Botteldooren):

$$Z(\omega) = R + \mathrm{i}(-K/\omega + M\omega), \tag{18}$$

R, K, and *M* are all positive real numbers. *R* is the resistance from friction, *K* is the stiffness of the spring, and *M* is the mass of the oscillator. In a Helmholtz resonator, *R* is the viscous dissipation of sound wave in the neck, *K* the stiffness of the air in the cavity, and *M* is the mass of air in and around the neck (Fig.5). Some of the energy of the oscillator of sound wave is converted to heat because of the resistance. Energy is stored as potential energy due to stiffness reactance K/ω, and as kinetic energy due to the mass reactance $M\omega$ (Speaks1992). Resistance is always in phase with velocity and is less dependent of frequency. Stiffness reactance always lags the velocity by 90^0; mass reactance always leads velocity by 90^0. Mass reactance is 180^0 out of phase with stiffness reactance. These two kinds of reactance always act in opposite direction. When one is storing energy, the other is giving it up. Resonance happens when reactance is zero, when magnitudes of stiffness reactance and mass reactance are equal, $K/\omega = M\omega$. The resonance frequency is then $f_0 = \sqrt{K/M}/2\pi$. This is called the natural frequency of the system. Near the natural frequency, the system is mainly resistance-controlled. Magnitude of stiffness reactance increases as frequency becomes lower. When frequency is much lower than the natural frequency, the stiffness reactance is large, and the system is called stiffness-controlled. Mass reactance increases as frequency goes frequency goes higher. At frequency well above the natural frequency, mass reactance is large, and the system is called mass-controlled.

We will use the three parameter model as an example to discuss the TDIBC of Eq.(14).

Cauchy's Residue Theorem

We will use *Cauchy's Residue Theorem* extensively in this chapter. It is necessary to give some details about this theorem first.

In Fourier transforms t and ω are real quantities. Mathematically they can be viewed as complex quantities, and thus the integral in FT can be evaluated from contour integral and *Cauchy's Residue Theorem*.

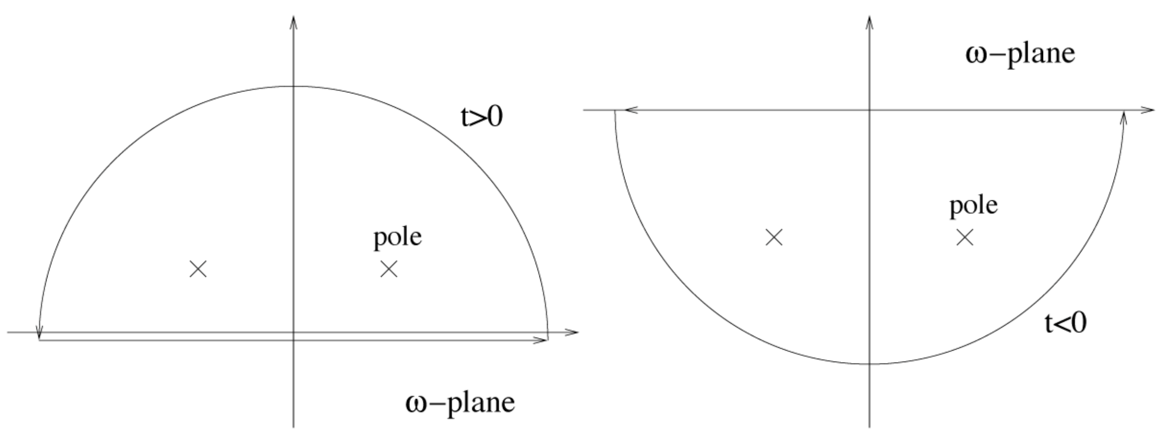

Fig.6, Fourier transform contours.

According to *Jordan's Lemma*, if $\hat{h}(\omega) \to 0$ as $R \to \infty$, then,

$$\lim_{R\to\infty} \int_{C_R} \hat{h}(\omega) e^{i\omega t} d\omega = 0,$$

where C_R is the semicircle in the upper half ω-plane if $t>0$, the lower half ω-plane if $t<0$. Therefore for FT pair with time factor $e^{i\omega t}$, one can form a semicircle below the real ω axis for $t<0$, above the real ω axis for $t>0$, so that the integration on the semicircle is zero. Then the Cauchy's Residue theorem leads to:

$$h(t)=\begin{cases} 2\pi i \sum\limits_{\mathrm{Im}(\omega_k)\geq 0} \mathrm{residue}[\hat{h}(\omega),\omega_k]e^{i\omega_k t}, & t\geq 0,\\ -2\pi i \sum\limits_{\mathrm{Im}(\omega_k)<0} \mathrm{residue}[\hat{h}(\omega),\omega_k]e^{i\omega_k t}, & t<0, \end{cases} \tag{19}$$

if $\hat{h}(\omega)$ only has poles on the finite ω-plane, and $\hat{h}(\omega)\to 0$ as $|\omega|\to\infty$. ω_k is the kth pole of $\hat{h}(\omega)e^{i\omega t}$. Since $e^{i\omega t}$ is analytical, ω_k is only the poles of $\hat{h}(\omega)$. Poles ω_k on or above the real ω axis give the time domain function for $t>0$, the causal response $h(t)$ that decays exponentially, as indicated by $e^{i\omega_k t}$. On the other hand, the poles of the integrand below the real ω axis give the time domain function for $t<0$, corresponding to a noncausal time process that grows exponentially with time. It is infeasible to implement the integral form of TDIBC with noncausal time process. Causality requires that $\hat{h}(\omega)$ has no poles below the real ω axis.

Analytical Solution

We first discuss the reflection of an incident sine wave started at $t=0$ at the boundary characterized by impedance model (18).

The residue theorem requires that integrand $\hat{W}(\omega)$ only has poles on the finite ω-plane, and $\hat{W}(\omega)\to 0$ as $|\omega|\to\infty$. $\hat{W}(\omega)$ in Eq.(6) doesn't satisfy these conditions. We need to rewrite the reflection coefficient as:

$$\hat{W}(\omega)=-1+\hat{\hat{W}}(\omega),\ \hat{\hat{W}}(\omega)=\frac{2}{1+Z}. \tag{20}$$

For a hard wall, $\hat{W}(\omega)=-1$, $\hat{\hat{W}}(\omega)=0$. Therefore $\hat{\hat{W}}(\omega)$ represents the softness of the boundary. The corresponding time process is:

$$u^{-}(t)=-u^{+}(t)+\int_0^{\infty}\tilde{W}(\tau)u^{+}(t-\tau)d\tau,\ \text{or,} \tag{21}$$

$$2u(t)=\begin{cases} 2\pi i \sum\limits_{\mathrm{Im}(\omega_j)\geq 0} \mathrm{Residue}[\hat{\hat{W}}(\omega)\hat{u}^{+}(\omega)e^{i\omega t},\omega_j], & t\geq 0;\\ 0, & t<0. \end{cases} \tag{22}$$

$\text{Residue}[\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)e^{\mathrm{i}\omega t},\omega_j] = (\omega - \omega_j)\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)e^{\mathrm{i}\omega t}\Big|_{\omega=\omega_j}$; ω_j is the *j*th pole of $\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)e^{\mathrm{i}\omega t}$, or $\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)$ since $e^{\mathrm{i}\omega t}$ has no poles. Suppose ω_{Tj} are poles of $\hat{\tilde{W}}(\omega)$, ω_{Sj} are poles of $\hat{u}^{+}(\omega)$. It is assumed that the incident wave u^{+} is only turned on after $t = 0$ therefore all ω_{Sj} are also on the upper half ω -complex plane. All ω_{Tj} must lie on the upper half ω -complex plane for causality.

If ω_{Tj} and ω_{Sj} do not coincide with each other, the response in (22) can be decomposed into two parts (Morse&Ingard1983, p.45):

$$2u(t) = \begin{cases} u_T(t) + u_S(t), & t \geq 0; \\ 0, & t < 0. \end{cases} \tag{23}$$

$$u_T(t) = \mathrm{i}\sum_j \text{Residue}[\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)e^{\mathrm{i}\omega t},\omega_{Tj}],$$

$$u_S(t) = \mathrm{i}\sum_j \text{Residue}[\hat{\tilde{W}}(\omega)\hat{u}^{+}(\omega)e^{\mathrm{i}\omega t},\omega_{Sj}].$$

One can obtain the above solution directly as in the following:

$$\begin{aligned} 2u(t) &= \int_0^{\infty} \tilde{W}(\tau)u^{+}(t-\tau)d\tau \\ &= -\int_0^{\infty}\left[\sum_j \text{Residue}[\hat{\tilde{W}}(\omega)e^{\mathrm{i}\omega\tau},\omega_{Tj}]\right]\left[\sum_k \text{Residue}[\hat{u}^{+}(\omega)e^{\mathrm{i}\omega(t-\tau)},\omega_{Sk}]\right]d\tau \\ ? \quad &= -\int_0^{\infty}\left[\sum_j \text{Residue}[\hat{\tilde{W}}(\omega),\omega_{Tj}]e^{\mathrm{i}\omega_{Tj}\tau}\right]\left[\sum_k \text{Residue}[\hat{u}^{+}(\omega),\omega_{Sk}]e^{\mathrm{i}\omega_{Sk}(t-\tau)}\right]d\tau \\ &= -\sum_j\sum_k \text{Residue}[\hat{\tilde{W}}(\omega),\omega_{Tj}]\text{Residue}[\hat{u}^{+}(\omega),\omega_{Sk}]e^{\mathrm{i}\omega_{Sk}t}\int_0^{\infty} e^{\mathrm{i}(\omega_{Tj}-\omega_{Sk})\tau}d\tau \\ &= -2\pi\sum_j\sum_k \text{Residue}[\hat{\tilde{W}}(\omega),\omega_{Tj}]\text{Residue}[\hat{u}^{+}(\omega),\omega_{Sk}]e^{\mathrm{i}\omega_{Sk}t}\delta(\omega_{Tj}-\omega_{Sk}) \end{aligned}$$

Which leads to the solution (23).

For the three-parameter model,

$$\hat{\hat{W}}(\omega) = \frac{2}{1 + R + i(-K/\omega + M\omega)} = -\frac{2i\omega / M}{(\omega - \omega_1)(\omega - \omega_2)}, \tag{24}$$

where $\omega_{1,2} = \pm\nu + i\alpha$ are two poles of $\hat{\hat{W}}(\omega)$. $\omega_0 = \sqrt{K/M}$ is the natural frequency without damping at which the stiffness reactance and mass reactance have the same magnitude; $\nu = \sqrt{\omega_0^2 - \alpha^2}$ is the natural frequency considering damping; $\alpha = (1+R)/(2M)$ the damping coefficient ($\alpha < \omega_0$ assumed).

Suppose the incident wave is a sine function started at $t = 0$:

$$u^+(t) = H(t)\sin(\Omega t), \; H(t)\text{: Heaviside's step function},$$

then,

$$\hat{u}^+(\omega) = -\frac{\Omega}{\omega^2 - \Omega^2}.$$

One has:

$$\left.\begin{aligned} u_T(t) &= \frac{1}{M_H \nu} e^{-\alpha t}\left[\omega_{T1}\hat{u}^+(\omega_{T1})e^{i\nu t} - \omega_{T2}\hat{u}^+(\omega_{T2})e^{-i\nu t}\right] \\ u_S(t) &= \frac{1}{2} i\left[\hat{W}(-\Omega)e^{-i\Omega t} - \hat{W}(\Omega)e^{i\Omega t}\right] \end{aligned}\right\}. \tag{25}$$

$u_T(t)$ is the transient reflection due to the free oscillation (resonance) of the boundary material excited by the incident waves. The oscillation frequency is the natural frequency of the material ν, the distance of the poles to the imaginary axis on the ω-plane. The amplitude of the oscillation depends on the spectrum of the incident wave at the natural frequency of the material. The transit reflection decays as $t \to \infty$. But it manifests itself in the reflection wave in the medium.

On the other hand, $u_S(t)$ represents a steady-state harmonic reflection. The oscillation frequency is the incident wave frequency. The amplitude depends on the spectrum of the reflection coefficient at the forcing frequency.

In impedance tube experiment, measurements are made after the transient wave $u_T(t)$ fades, *i.e.*, only $u_S(t)$ is measured. A harmonic measurement is used to determine the impedance at a single frequency by measuring $u_S(t)$. From one measurement one can determine $R + iX$ at one frequency. For a damped spring-mass oscillator, measurements at two frequencies are needed to determine all the three parameters. An impulse can also be used to measure the impedance at a broadband frequency range.

Numerical Implementation

To numerically implement (21), we need the time domain softness function $\tilde{W}(t)$:

$$\tilde{W}(t) = \begin{cases} -i \sum_{\mathrm{Im}(\omega_k)>0} \mathrm{Residue}[\hat{\tilde{W}}(\omega), \omega_k] e^{i\omega_k t}, & t \geq 0; \\ 0, & t < 0. \end{cases} \tag{26}$$

For the three-parameter impedance (18),

$$\tilde{W}(t) = H(t)\frac{2}{M}[\cos(\omega t) - \frac{\sigma}{\omega}\sin(\omega t)]e^{-\sigma t}. \tag{27}$$

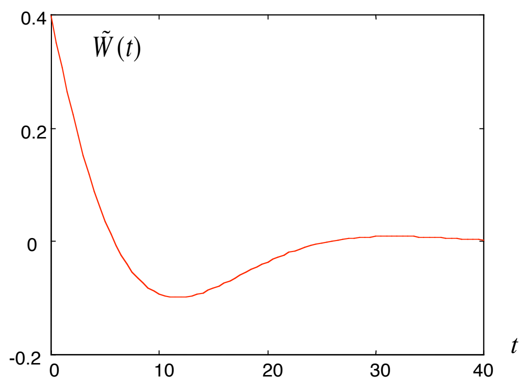

Fig.7, R=0.2, K=13.48, M =0.0739.

A typical $\tilde{W}(t)$ is illustrated in Fig.7. Beyond the range $t \in [0,40]$ $\tilde{W}(t)$ is either zero or significantly small. This figure shows the feasibility of implementing the TDIBC (18) with a finite time history T. A straightforward numerical integration of (21) or (22) is:

$$2u(t) = \sum_{k=0}^{T/\Delta t} \tilde{W}(k\Delta t)u^{+}(t - k\Delta t). \tag{28}$$

The numerical integration is less sensitive to numerical error than numerical differentiation. Storage of $u^{+}(t - k\Delta t)$ depends on T. T is inversely proportional to damping coefficient $|\sigma|$, the imaginary part of the poles. The shorter the T, the more compact the scheme. However T can not be too small as compared to the time step Δt, otherwise the numerical integration will be subject to great error.

There is another way to implement Eqs.(21)&(22). We first rewrite $\tilde{W}(\omega)$ as:

$$\tilde{W}(\tau) = H(\tau)\sum_k A_k e^{i\lambda_k \tau}, \ A_k = \mathrm{i}\cdot\text{Residue}[\hat{\tilde{W}}(\omega), \lambda_k], \tag{29}$$

then,

$$u^{-}(t) = -u^{+}(t) + \sum_k I_k(t), \ I_k(t) = A_k \int_0^{\infty} e^{i\lambda_k \tau} u^{+}(t-\tau)d\tau. \tag{30}$$

One can find

$$I_k(t) = z_k I_k(t-\Delta t) + A_k \int_0^{\Delta t} e^{i\lambda_k \tau} u_k^{+}(t-\tau)d\tau, \ z_k = e^{i\lambda_k \Delta t}. \tag{31}$$

Stability is assured since each subsystem $I_k(t)$ is stable with $|z_k| < 1$. The accuracy of the integral boundary condition hinges solely on the integration within one time step. In numerical schemes, the grid system and time step are carefully chosen to resolve $u^{+}(t)$ and $u^{-}(t)$. Therefore it is sufficiently accurate to assume $u^{+}(t)$ linear within one time step Δt: $u^{+}(\tau) = u^{+}(t-\Delta t) + [u^{+}(t) - u^{+}(t-\Delta t)](\tau - t + \Delta t)/\Delta t$, and then,

$$\int_0^{\Delta t} e^{i\lambda_k \tau} u^{+}(t-\tau)d\tau \approx \left(\frac{z_k - 1}{\lambda_k^2 \Delta t} + \frac{z_k}{i\lambda_k}\right)u^{+}(t-\Delta t) - \left(\frac{z_k - 1}{\lambda_k^2 \Delta t} + \frac{1}{i\lambda_k}\right)u^{+}(t). \tag{32}$$

Compared to the direct numerical integration (28), the numerical scheme by (30), (31) and (32) is compact (only one previous time step storage is needed). And for $e^{i\lambda_k \tau}$, it may have drastic change for very large imaginary part of λ_k. But it poses no problem since this term is handled analytically.

Algebraic Reflection Modeling

Algebraic Modeling and TDIBC in Integral Forms

The three-parameter impedance model (18) has analytical inverse time response function (27). We have used this model as an example to show the effect of the poles on the acoustic reflection, and implementation of TDIBC in integral forms. For impedance other than the three-parameter model, how to get the softness function $\tilde{W}(t)$ and implement the TDIBC is mot straightforward. For experimentally measured impedance such as those in Figs.(2) and (3), we first need to represent the impedance data at discrete frequency using analytical continuous functions. This is called impedance modeling. For analytically rendered impedance models or empirical models, in most of the time the analytical inversion for $\tilde{W}(t)$ is formidable, such as Eq.(7) for a Helmholtz resonator, Eq.(8) for an open end of a pipe, or Eq.(9) for outdoor ground. Even in rare cases the analytical inversions do exist, the causality requirement is not guaranteed. For those impedance models, impedance modeling is also needed.

According to algebra, any functions with only poles as singularities are rational functions (Morse&Feshbach p.382). A rational function is the quotient of two polynomials, $N(s)/D(s)$. If the function has only one pole at infinity, it is a polynomial. If the function has only poles on the finite complex plane, then the degree of $N(s)$ is smaller than the degree of $D(s)$, Deg $N(s)$<Deg $D(s)$. If Deg $N(s)$>Deg $D(s)$, then the function has a pole at infinity and finite number of poles on the finite complex plane, and it can be decomposed into a polynomial $P(s)$ and another rational function $Q(s)/D(s)$ with Deg $Q(s)$<Deg $D(s)$.

As we have discussed, it is very important to control positions of $\hat{\hat{W}}(\omega)$ poles for causality. Therefore it is a natural choice to model $\hat{\hat{W}}(\omega)$ by a rational function with all poles on or above the $\mathrm{Re}\,\omega$ axis in ω-plane:

$$\hat{\hat{W}}(\omega) = \frac{2}{1+Z(\omega)} = \frac{2}{1+R(\omega)+iX(\omega)} = \frac{Q(s)}{D(s)}, \tag{33}$$

where R and X are real, $s=i\omega$, $D(s)$ and $Q(s)$ are polynomials with real coefficients and Deg $Q(s) \le$ Deg $D(s)$. The rational function can be decomposed into a constant C and proper partial fractions:

$$\hat{\hat{W}}(\omega) = C + \sum_{k=1}^{m} \hat{\hat{W}}_k(\omega),\ \hat{\hat{W}}_k(\omega) = \mu_k/(s-\lambda_k),\ \mu_k = Q(\lambda_k)/[\mathrm{d}D(\lambda_k)/\mathrm{d}s]. \tag{34}$$

λ_k ($k = 1,2,\cdots,m$) are zeros of $D(s)$.

The corresponding TDIBC is:

$$u^{-}(t) = -(1-C)u^{+}(t) + \sum_k I_k(t),\ I_k(t) = \mu_k \int_0^{\infty} e^{\lambda_k \tau} u^{+}(t-\tau)d\tau. \tag{35}$$

$I_k(t)$ can be numerically evaluated using the recursive scheme of (31)&(32).

Since $D(s)$ has only real coefficients, λ_k are either real or complex conjugate pairs. For real λ_k, the time-domain reflection impulse, denoted as *Reflection Type*-I:

$$\tilde{W}_k(t) = \mu_k e^{\lambda_k t} H(t), \text{ for } \lambda_k \le 0, \tag{36}$$

corresponds to an overly damped system without oscillation, causal only when $\lambda_k \le 0$. μ_k must be real to give a real reflection function $\tilde{W}_k(t)$. For repeated roots λ_k of order n the corresponding impulse response has the form:

$$\tilde{W}_k(t) = \mu_k t^{n-1} e^{\lambda_k t} H(t)/(n-1)!, \text{ for } \lambda_k \le 0. \tag{37}$$

When one of the roots λ_k of $D(s)$ is complex, its conjugate λ_{k+1} must exist to form the pair:

$$\hat{\hat{W}}_{(k,k+1)}(\omega) \equiv \hat{\hat{W}}_k(\omega) + \hat{\hat{W}}_{k+1}(\omega) = \frac{\mu_k}{s-\lambda_k} + \frac{\mu_{k+1}}{s-\lambda_{k+1}} = \frac{Bs+C}{(s+\alpha)^2+\beta^2}, \tag{38}$$

$\mu_k e^{\lambda_k t}$ and $\mu_{k+1} e^{\lambda_{k+1} t}$ must be conjugate pairs to have a real reflection function. The reflection impulse, *Reflection Type*-II, is:

$$\tilde{W}_{(k,k+1)}(t) = e^{-\alpha t} H(t) \left[B\cos(\beta t) + \frac{C-\alpha B}{\beta}\sin(\beta t) \right], \quad \alpha \geq 0. \tag{39}$$

This is exactly the response of a damped harmonic oscillator of impedance (18) with resistance R, acoustic mass $M=2/B$, stiffness K, damping rate $\alpha = (1+R)/(2M)$, oscillation frequency $\beta = \sqrt{\omega_0^2 - \alpha^2}$, natural frequency $\omega_0 = \sqrt{K/M}$, and an arbitrary phase parameter C.

*TDIBC in Differential Forms*

With algebraic mode $\hat{\hat{W}}(\omega) = Q(s)/D(s)$, we have:

$$2\hat{u} = \frac{Q(s)}{D(s)}\hat{u}^{+}. \tag{40}$$

Since $\hat{\delta}^{(n)}(\omega) = \int_{-\infty}^{\infty} \delta^{(n)}(t) e^{-i\omega t} dt = (i\omega)^n = s^n$, the time domain equivalent is:

$$2D(\frac{\partial}{\partial t})u = Q(\frac{\partial}{\partial t})u^{+}. \tag{41}$$

it is the direct inversion of $(i\omega)^k$ to d^k/dt^k. This is the differential form of the TDIBC. Mathematically it is equivalent to the integral form (35).

Causality is not obvious in the differential form of TDIBC. However it is still implied in the differential equation. If violated, the direct inversion generally results in an unstable system, even though there is no difficulty to numerically implement the differential form TDIBC even for noncausal time process. The way to analyze the causality is to rewrite equation (40) as,

$$\frac{2}{Q(s)}\hat{u} = \frac{1}{D(s)}\hat{u}^{+}. \tag{42}$$

Zeros of both $Q(s)$ and $D(s)$ must be on the upper half ω-plane.

<u>Single Frequency TDIBC</u>

Suppose we know impedance $Z = R + iX$ at $\omega = \Omega$, how do we construct a TDIBC based on the algebraic model?

We may first assume an overly damped system (*Reflection Type*-I, Eq.36):

$$\hat{\tilde{W}}(\square) = C + \frac{\square}{i\square \square \square}, \text{ for which } \frac{2}{1+R+iX} = C + \frac{\square}{i\square \square \square}.$$

$\square$, C, and $\square$ are all real quantities. Then,

$$\square = \frac{\square}{2X}\left\{\left[(1+R)C \square 2\right]^2 + X^2C^2\right\},$$
$$\square = \frac{\square}{2X}\left\{\left[(1+R)^2 + X^2\right]C \square 2(1+R)\right\}.$$

Causality requires $\square \square 0$. Depending on the sign of X, we have two cases.

$\underline{X > 0}$

In this case, we can set $C = 0$ to satisfy the causality:

$$\square = 2\square / X, \ \square = \square(1+R)\square / X.$$

The TDIBC is:

$$u^{\square}(t) = \square u^{+}(t) + \frac{2\square}{X}\int_0^{\square} e^{\square\frac{(1+R)\square}{X}\square} u^{+}(t \square \square)d\square. \tag{43}$$

Physically it is appropriate to represent a positive reactance by a damped oscillator with only mass-like reactance. The corresponding impedance model is: $Z = R + iX\square/\square$. The corresponding TDIBC in differential form is:

$$p = Ru + \frac{X}{\square}\frac{\partial u}{\partial t}. \tag{44}$$

This is exactly the same as proposed by Tam&Auriault1996, Eq.(4).

$\underline{X < 0}$

To satisfy the causality condition,

$$C \geq \frac{2(1+R)}{(1+R)^2 + X^2}.$$

We may just choose

$$C = \frac{2}{1+R},$$

then,

$$\square = 2X\square/(1+R)^2,\ \square = X\square/(1+R).$$

The integral form of TDIBC is:

$$u^{-}(t) = \frac{1-R}{1+R}u^{+}(t) + \frac{2X\square}{(1+R)^2}\int_0^{\square} e^{\frac{X\square}{1+R}\square} u^{+}(t-\square)d\square. \tag{45}$$

For negative reactance, only the damped oscillator with spring-like reactance can well represent the Physics. The corresponding impedance model is $Z = R + iX\square/\square$, and the differential form is:

$$\frac{\partial p}{\partial t} = R\frac{\partial u}{\partial t} - X\square u. \tag{46}$$

This is exactly the same as Eq.(3) of Tam&Auriault1996.

In Tam&Auriault1996, details about the stability of (44) and (46) were given. But there was no mention as to how these equations were proposed. The algebraic model of this work gives a systematic way to construct these two time domain impedance boundary conditions.

We may try a damped harmonic oscillator (*Reflection Type*-II, Eq.(38)) to represent the impedance:

$$Z(\square) = R + \mathrm{i}(\square K/\square + M\square),\ \text{for which}\ \square K/\square + M\square = X.$$

There is a free parameter M (or K) in this model. As we know the transient solution decays with rate inversely proportional to $\square = (1+R)/(2M)$. Small M means more compact scheme and rapid convergence to steady harmonic solution. One may choose M to develop a compact scheme with rapid convergence. Since $\square_0^2 \geq \square^2$ is required in the model, the most compact scheme is when $\square_0^2 = \square^2$, or,

$$M = \frac{1}{2\square}\left[\sqrt{(1+R)^2 + X^2} + X\right],$$

$$K = \frac{\square}{2}\left[\sqrt{(1+R)^2 + X^2} - X\right].$$

It turns out that the compact is very helpful for rapid convergence in simulations.

Sometimes M is chosen for other purposes, such as for numerical stability in liner with soft splices.

The differential form is:

$$M\frac{\partial^2 u}{\partial t^2}+R\frac{\partial u}{\partial t}+Ku=\frac{\partial p}{\partial t}, \tag{47}$$

which is the same as Eq.(27) of Tam&Auriault1996.

<u>Broadband TDIBC</u>

If broadband impedance is modeled by overly damped systems (*Reflection Type*-I), coefficients $□_k$ are usually extremely sparse, which is not suitable for numerical simulations (Fung&Ju2001).

Physically it is more appropriate to model sound absorbing materials by a set of damped harmonic oscillators (*Reflection Type*-II). In Fig.3, two oscillators were used to model the constant depth ceramic tubular liner. Fig.8 shows the model of the outdoor ground impedance, Eq.(9), by three oscillators.

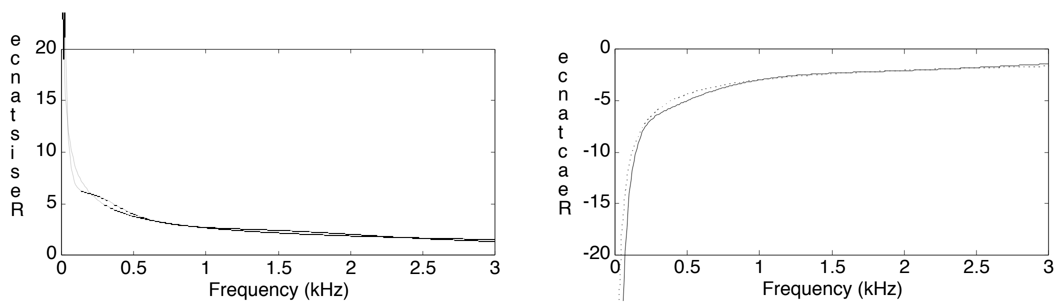

Fig.8, Fitted impedance models for outdoor grounds (with □ in kHz): □, with $□_{0k}$ = 0.2, 1.5, 6 kHz, and □ = 0.85 ; and - - -, Eq. (9).

<u>TDIBC with Mean Flow Effects</u>

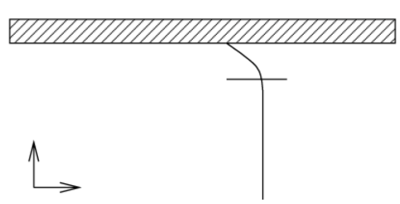

Fig.9, Shear flow near the impedance wall.

The TDIBC established can be directly used for viscous mean flow (Fig.9). However, the high resolution near the boundary is costly, and sometimes the details of the boundary layer and its reactions to the disturbance are of no immediate concern. In those cases Euler equations are preferred, and the impedance model for a plug flow is appropriate.

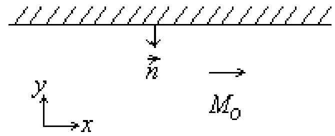

Fig.10, Plug flow parallel to impedance wall.

Myer's B.C

Myer's B.C. in a plug flow is based on the assumption of tangential discontinuity, which leads to *continuity of displacement*.

The plug flow is an ideal model for thin boundary layer. Suppose at the liner surface, the pressure is p_1 and normal velocity into the liner is v_1. And the pressure and velocity at the immediate outside of the boundary are p_2 and v_2. We know $\hat{p}_1 / \hat{v}_1 = Z$, then $\hat{p}_2 / \hat{v}_2 = ?$

Assume displacement ξ at the liner surface and the outside of the boundary layer is continuous, then (assuming $e^{i(kx - \omega t)}$),

$$v_1 = \frac{\partial \xi}{\partial t}, \text{ or, } \hat{v}_1 = -i\omega\hat{\xi},$$

$$v_2 = \frac{\partial \xi}{\partial t} + M\frac{\partial \xi}{\partial x}, \text{ or, } \hat{v}_2 = -i(\omega - Mk)\hat{\xi}$$

And we know across the boundary layer, $\hat{p}_1 = \hat{p}_2$.

Therefore,

$$\frac{\hat{p}_2}{\hat{v}_2} = \frac{\hat{p}_1}{\hat{v}_1} \cdot \frac{\omega}{\omega - Mk} = Z\frac{\omega}{\omega - Mk}, \; i.e., \; \hat{p}_2 - \frac{M}{i\omega}\frac{\partial \hat{p}_2}{\partial x} = Z\hat{v}_2;$$

or by assuming $e^{i(\omega t - kx)}$,

$$\hat{p}_2 + \frac{M}{i\omega}\frac{\partial \hat{p}_2}{\partial x} = Z\hat{v}_2, \tag{48}$$

which is the Myer's B.C.

Effective Plane Wave Impedance in a Plug Flow

Assuming plain waves of the form $e^{i(\omega t - k_x x)}$, then

$$\hat{v} = \hat{p} / Z', \tag{49}$$

where $Z' = Z(1 + M_0 \sin\theta)$ can be seen as the *effective plane-wave impedance* with the effective incident wave angle $\theta = \sin^{-1}[k_x / (\omega - k_x M_0)]$. With the modified reflection coefficient as $\hat{W}' \equiv (1 - Z')/(1 + Z')$, Eq. (27) corresponds to a modified harmonic oscillator of resistance $R_0' = R_0(1 + M_0 \sin\theta)$, stiffness constant $X_{-1}' = X_{-1}(1 + M_0 \sin\theta)$, and mass $X_1' = X_1(1 + M_0 \sin\theta)$. It has the same natural frequency as that of the primary impedance: $\omega_0' = \omega_0$, but the mean flow decreases the damping coefficient and increases the oscillating frequency as a wave moves towards downstream, i.e., $\delta' = [(1 + M_0 \sin\theta)^{-1} + R_0] / 2X_1$, $\omega' = \sqrt{\omega_0^2 - \delta'^2}$, and *vise versa*.

Convection-Modified Impedance in a Plug Flow

For the three-parameter model (18), the convection-modified TDIBC is:

$$M \frac{\partial^2 u}{\partial t^2} + R \frac{\partial u}{\partial t} + Ku = \frac{\partial p}{\partial t} + M \frac{\partial p}{\partial x}. \tag{50}$$