Acoustic Energy and Intensity

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In this section we discuss acoustic energy, acoustic intensity, and their components. Emphasis will be played on the definition of acoustic intensity, which is the acoustic energy flux passing through a unit area in unit time. It is straightforward to define an instantaneous acoustic intensity from the energy equation. However, in practical problems the major concern is time averaged energy flux. Terms with zero time average should be ignored. Terms that make no contribution to mean energy should also be ignored. Only terms with even orders of magnitude in the expansion of instantaneous acoustic intensity are important and should be kept in the definition of the acoustic intensity.

Ideal gas ($\sigma_{ij} = -p\delta_{ij}$) with isentropic process ($k\partial T / \partial x_i = 0$) is assumed. Linear disturbances are only generated by acoustic waves.

Balance of Acoustic Energies

Decomposition of Variables

Variables for undisturbed flows are defined with subscript '0', such as, p_0, ρ_0, $\vec{u}_0$. Any variable in a disturbed flow is decomposed into a time averaged component and a fluctuation component:

$$p = \overline{p} + p', \tag{1}$$

$$\overline{p} = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} p\,dt,$$

$$p' = p - \overline{p}, \ \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} p'\,dt = 0.$$

Generally for linear disturbances,

$$p_0 = \overline{p}. \tag{2}$$

For a product of two variables, such as pu,

$$\overline{pu} = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \left(p_0 u_0 + p_0 u' + p' u_0 + p' u'\right) dt = p_0 u_0 + \overline{p' u'}. \tag{3}$$

Euler Equations

Euler equations and their forms with time averaged and fluctuation variables are as following.

The continuity equations are:

$$\frac{\partial \rho}{\partial t}+\frac{\partial (\rho u_j)}{\partial x_j}=0\,, \tag{4}$$

$$\frac{\partial \rho'}{\partial t}+\frac{\partial}{\partial x_j}(\rho' u_{0j}+\rho_0 u'_j+\rho' u'_j)=0\,. \tag{5}$$

The momentum equations are:

$$\frac{\partial (\rho u_i)}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_i u_j+p\delta_{ij})=0\,, \tag{6}$$

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho' u_{0i}+\rho_0 u'_i+\rho' u'_i)+\frac{\partial}{\partial x_j}(\rho' u_{0i}u_{0j}+\rho_0 u'_i u_{0j}+\rho_0 u_{0i}u'_j\\&+\rho' u'_i u_{0j}+u_{0i}\rho' u'_j+\rho_0 u'_i u'_j+\rho' u'_i u'_j+p_0\delta_{ij}+p'\delta_{ij})=0\end{aligned}\,. \tag{7}$$

The conservative form of the energy equation is:

$$\frac{\partial e}{\partial t}+\frac{\partial}{\partial x_j}(eu_j+pu_j)=0\,. \tag{8}$$

The total energy in unit fluid volume

$$e=\rho E+\frac{1}{2}\rho u_i u_i \tag{9}$$

has two components: the *internal energy* ρE and the *kinetic energy* $\frac{1}{2}\rho u_i u_i$. Since only acoustic perturbations are assumed, the internal energy is the acoustic potential energy, and the kinetic energy is translational, not rotational, kinetic energy. The *instantaneous acoustic intensity*

$$I_j=eu_j+pu_j \tag{10}$$

also has two components: energy convection intensity eu_j and energy production pu_j (work done by pressure).

For a time steady process, the time average of the total energy in the volume is constant. Therefore the time averaged equation of Eq.(8) is:

$$\frac{\partial \bar{I}_j}{\partial x_j} = 0. \tag{11}$$

The time averaged total energy flux across the enclosing surface of a fluid element is zero.

Acoustic intensity Eq.(10) in this form has never been used. We will show that the acoustic intensity has many components. Some of them can be ignored and a simplified form of acoustic intensity can be defined.

Acoustic Energy Components and Their Equations

For an isentropic flow,

$$p = \rho^{\gamma} \cdot \text{Constant}, \text{ or } \partial p / \partial \rho = \gamma p / \rho = a^2, \tag{12}$$

and the internal energy density is:

$$\rho E = \frac{p}{\gamma - 1}. \tag{13}$$

For the isentropic process, the continuity and momentum equations (5)&(7) with equation of state (12) is a complete system for solving the equations. Energy equation (8) can be derived from equations (5)&(7). It is not needed for solving the sound field but very useful for energy balance analyses.

With the isentropic relation, continuity equation (5) can be rewritten as:

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1}\right) + \frac{\partial}{\partial x_j}\left(\frac{p}{\gamma - 1} u_j\right) + p\frac{\partial u_j}{\partial x_j} = 0. \tag{14}$$

This equation shows the balance of the *internal energy*. The time rate of internal energy is equal to the internal energy convected into the fluid element $[p/(\gamma - 1)]u_j$, plus the work $p\partial u_j / \partial x_j$ done by the pressure on the fluid element by changing its volume.

Multiplying both sides of momentum equation (7) by u_i and making use of the continuum equation (4), we have:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u_i u_i u_j\right) + u_j \frac{\partial p}{\partial x_j} = 0. \tag{15}$$

This equation shows the balance of the *kinetic energy*. The time rate of the kinetic energy is the sum of its convection into the fluid ($[\frac{1}{2}\rho u_i u_i]u_j$) and the work done by the net force on the movement of the fluid volume ($u_j \partial p / \partial x_j$).

Adding Eqs.(14) and (15) together one obtains the total energy equation (8).

Acoustic Intensity in Stationary Ideal Homogeneous Medium

We start with the simplest case: quiescent uniform medium ($\vec{u}_0 = 0$).

Acoustic Energy Balance

Since pressure varies adiabatically with density (Eq.(12)), we have

$$p = p_0 + \left.\frac{\partial p}{\partial \rho}\right|_0 \rho' + \frac{1}{2}\left.\frac{\partial^2 p}{\partial \rho^2}\right|_0 \rho'^2 + \cdots = p_0 + a_0^2 \rho' + \frac{\gamma - 1}{2\rho_0} a_0^2 \rho'^2 + \cdots,$$

$$i.e., \;\; p' = a_0^2 \rho' + \frac{\gamma - 1}{2\rho_0} a_0^2 \rho'^2 + \cdots, \tag{16}$$

$$a_0 = \sqrt{\partial p_0 / \partial \rho_0} = \sqrt{\gamma p_0 / \rho_0}\,.$$

Therefore, the internal energy

$$\rho E = \frac{p}{\gamma - 1} = \frac{p_0}{\gamma - 1} + \frac{a_0^2}{\gamma - 1}\rho' + \frac{a_0^2}{2\rho_0}\rho'^2 + \cdots. \tag{17}$$

The zeroth, first, and second orders of the internal energy fluctuations are $p_0/(\gamma - 1)$, $a_0^2 \rho'/(\gamma - 1)$, and $a_0^2 \rho'^2/(2\rho_0)$ respectively.

The kinetic energy to the second order is:

$$\frac{1}{2}\rho u_i u_i = \frac{1}{2}\rho_0 u'_i u'_i. \tag{18}$$

The total acoustic energy fluctuation to the second order is:

$$e' = \frac{a_0^2}{\gamma - 1}\rho' + \frac{a_0^2}{2\rho_0}\rho'^2 + \frac{1}{2}\rho_0 u'_i u'_i\,. \tag{19}$$

The acoustic energy equation to the 2^{nd} order is:

$$\frac{\partial}{\partial t}\left(\frac{a_0^2}{\gamma-1}\rho'+\frac{a_0^2}{2\rho_0}\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i\right)+\frac{\partial}{\partial x_j}\left(\frac{p_0}{\gamma-1}u'_j+\frac{a_0^2}{\gamma-1}\rho' u'_j+p_0 u'_j+p' u'_j\right)=0\,. \quad (20)$$

Decomposition of Acoustic Energy and Acoustic Intensity

First order perturbations contribute nothing to the mean acoustic energy. It seems appropriate to define the time averaged acoustic intensity from Eq.(20) as

$$\bar{I}_j=\frac{a_0^2}{\gamma-1}\overline{\rho' u'_j}+\overline{p' u'_j}\,. \quad (21)$$

We will show that the convection intensity, $[a_0^2/(\gamma-1)]\overline{\rho' u'_j}$, makes no contribute to the mean acoustic energy and thus should be removed from definition.

Energy balance equation (20) can be separated into two equations about the internal energy such as Eq.(14) and the kinetic energy such as Eq.(15). To facilitate the definition of mean acoustic intensity, we decompose the energy equation based on the work done by the pressure on the fluid element:

$$pu_j=p_0 u'_j+p' u'_j\,. \quad (22)$$

With continuity equation (5) and the uniform medium,

$$\frac{\partial}{\partial t}\left(\frac{a_0^2}{\gamma-1}\rho'\right)+\frac{\partial}{\partial x_j}\left(\frac{p_0}{\gamma-1}u'_j+\frac{a_0^2}{\gamma-1}\rho' u'_j+p_0 u'_j\right)=0\,. \quad (23)$$

It is noted Eq.(23) looks similar but different from the perturbation form of internal energy equation (14)

$$\frac{\partial}{\partial t}\left(\frac{a_0^2}{\gamma-1}\rho'+\frac{a_0^2}{2\rho_0}\rho'^2+\cdots\right)+\frac{\partial}{\partial x_j}\left(\frac{p_0}{\gamma-1}u'_j+p_0 u_j\right)=0\,.$$

Eq.(23) is basically the continue equation but it does reveals the balance of the first order of internal energy $a_0^2\rho'/(\gamma-1)$ in a quiescent uniform flow. The time average of $a_0^2\rho'/(\gamma-1)$ is zero. $[a_0^2/(\gamma-1)]\rho' u'_j$ has no contribution to the mean acoustic energy although its time average is not zero. Therefore, it should not be included in the definition of mean acoustic intensity.

Subtracting Eq.(23) from (20), we have:

$$\frac{\partial}{\partial t}\left(\frac{a_0^2}{2\rho_0}\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i\right)+\frac{\partial}{\partial x_j}\left(p'u'_j\right)=0. \tag{24}$$

The time rate of the 2nd order acoustic energy is due to the net work $\partial(p'u'_j)/\partial x_j$ done by acoustic pressure on acoustic velocity. Of the acoustic energy, $\rho'^2 a_0^2/(2\rho_0)$ is the internal energy or *acoustic potential-energy density*, generated by net pressure pushing the flow element $u'_j \partial p'/\partial x_j$. $\frac{1}{2}\rho_0 u'_i u'_i$ is the *acoustic kinetic-energy density*, generated by pressure pressing the fluid element to change its volume $p'\partial u'_j/\partial x_j$.

From equation (24), one can define the *acoustic intensity* as:

$$\boxed{\bar{I}_j=\overline{p'u'_j}}, \tag{25}$$

which is the standard definition for stationary media. Note that this energy flux only affects the second order energy fluctuation $\frac{1}{2\rho_0}a_0^2\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i$ instead of the total acoustic energy fluctuation e'. Since the time average of the first order fluctuation internal energy $a_0^2\rho'/(\gamma-1)$ is zero, this definition makes more sense for energy analyses.

Acoustic Intensity with Mean Flow

Acoustic Energy Balance Equation

The total energy per unit volume of a mean flow is:

$$e_0=\frac{p_0}{\gamma-1}+\frac{1}{2}\rho_0 u_{0i}u_{0i}. \tag{26}$$

To the second order the *acoustic energy fluctuation* is:

$$e'=\frac{a_0^2}{\gamma-1}\rho'+\frac{1}{2}u_{0i}u_{0i}\rho'+\rho_0 u_{0i}u'_i+u_{0i}\rho'u'_i+\frac{a_0^2}{2\rho_0}\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i. \tag{27}$$

The time averages of first order terms are zero.

The acoustic energy balance equation is:

$$\frac{\partial e'}{\partial t}+\frac{\partial I_j}{\partial x_j}=0, \tag{28}$$

where

$$
\begin{aligned}
I_j = &\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u_{0j} + \left(\frac{p_0}{\gamma-1}+\frac{a_0^2}{\gamma-1}\rho' + \frac{1}{2}\rho u_{0i}u_{0i}\right)u'_j + p_0 u'_j \\
&+ \rho u_{0i}u'_i u_{0j} + \rho u_{0i}u'_i u'_j + p' u_{0j} \\
&+ \left(\frac{a_0^2}{2\rho_0}\rho'^2 + \frac{1}{2}\rho_0 u'_i u'_i\right)u_{0j} + p' u'_j
\end{aligned}
\quad . \tag{29}
$$

Decomposition of Acoustic Energy in Uniform Mean Flow

For uniform a_0 and u_{0j}, continuity equation (5) can be rewritten as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{a_0^2}{\gamma-1}\rho' + \frac{1}{2}u_{0i}u_{0i}\rho'\right) + \frac{\partial}{\partial x_j}\left[\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u_{0j}\right. \\
&\left. + \left(\frac{p_0}{\gamma-1}+\frac{1}{2}\rho_0 u_{0i}u_{0i}\right)u'_j + \left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u'_j + p_0 u'_j\right] = 0
\end{aligned}
\quad . \tag{30}
$$

This reveals time rate of two terms of the 1st order energy due to work $p_0 u'_j$ done by the ambient pressure on the acoustic velocity. The time average of this part of the energies is zero. Although the time average of the second order term $\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u'_j$ on the right hand side is not zero, it contributes nothing to the mean acoustic energy. Therefore this part of the energy flux should not be included in the acoustic intensity.

From momentum equation (7), continuity equation (5) and uniform u_{0i}, we have the balance equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_0 u_{0i}u'_i + u_{0i}\rho' u'_i\right) \\
&= -\frac{\partial}{\partial x_j}\left(\rho u_{0i}u'_i u_{0j} + \rho u_{0i}u'_i u'_j + p' u_{0j}\right)
\end{aligned}
\quad . \tag{31}
$$

This part of energy balance is due to work $p' u_{0j}$ done by the acoustic pressure on the mean velocity. The energy has a second order term with nonzero time average.

Subtracting Eq.(29) by (30) and (31), the equation for the rest of the acoustic energies is:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{a_0^2}{2\rho_0}\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i\right)\\
&=-\frac{\partial}{\partial x_j}\left[\left(\frac{a_0^2}{2\rho_0}\rho'^2+\frac{1}{2}\rho_0 u'_i u'_i\right)u_{0j}+p'u'_j\right]
\end{aligned}. \tag{32}
$$

Both energy terms are second order with nonzero time average. It is due to work $p'u'_j$ by the acoustic pressure and the acoustic velocity.

The three acoustic energy balance equations, (30), (31), (32), are separated based on the separation of work $pu_j = p_0 u'_j + p'u_{0j} + p'u'_j$.

Acoustic Intensity in Uniform Mean Flow

Following the same logic as in the previous analyses for quiescent uniform media, based on Eqs.(31)&(32), the *acoustic intensity* in the *j* direction for a uniform mean flow can be defined as:

$$
\boxed{\bar{I}_j = \rho_0 u_{0i}\overline{u'_i u'_j} + u_{0i}u_{0j}\overline{\rho' u'_i} + \left(\frac{a_0^2}{2\rho_0}\overline{\rho'^2}+\frac{1}{2}\rho_0\overline{u'_i u'_i}\right)u_{0j} + \overline{p'u'_j}.} \tag{33}
$$

All first order terms are averaged out. Third and higher order terms are ignored. According to Lighthill (p.11), in linear acoustics first order quantities are retained and quadratic terms neglected in linear continuity and momentum equations, while in energy equations, first order terms should be absent, most quadratic terms retained, and cubic terms or above neglected. The only quadratic term absent is $[a_0^2/(\gamma-1) + u_{0i}u_{0i}/2]\rho'u'_j$ which does not contribute to mean acoustic energy according to Eq.(30).

A similar definition was proposed in Cantrell&Hart 1964 Eq.(13) for isentropic uniform mean flows:

$$
\bar{I}_j = \rho_0 u_{0i}\overline{u'_i u'_j} + u_{0i}u_{0j}\overline{\rho' u'_i} + \frac{1}{\rho_0 a_0^2}u_{0j}\overline{p'^2} + \overline{p'u'_j}. \tag{34}
$$

This definition has been widely accepted (Tester et.al.2004, Morfey1971, Möhring1971). According to Eq.(31) in cht1.doc, the time averaged acoustic potential energy is equal to the acoustic kinetic energy: $\frac{a_0^2}{2\rho_0}\rho'^2 = \frac{1}{2\rho_0 a_0^2}\overline{p'^2} = \frac{1}{2}\rho_0\overline{u'_i u'_i}$, with which, Eq.(34) and Eq.(33) are equivalent. The benefit of using Eq.(34) instead of (33) is that it excludes vortical energy if the disturbances have vortical component.

Acoustic Intensity in Nonuniform Mean Flow

Equations (30), (31) and (32) do not hold for a general nonuniform flow. From continuity equation (5), we have:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{a_0^2}{\gamma-1}\rho'+\frac{1}{2}u_{0i}u_{0i}\rho'\right)\\
&=-\frac{\partial}{\partial x_j}\left[\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u_{0j}+e_0 u'_j+\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\rho' u'_j+p_0 u'_j\right]. \qquad (37)\\
&+\rho u_j\frac{\partial}{\partial x_j}\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)
\end{aligned}
$$

There is an extra term $\frac{\partial}{\partial x_j}\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)$ compared with Eq.(30) for uniform flows.

We may resort to Eq.(29) to define the acoustic intensity for a general mean flow as:

$$
\begin{aligned}
I_j &= \rho_0 u_{0i}\overline{u'_i u'_j}+u_{0i}u_{0j}\overline{\rho' u'_i}+\left(\frac{a_0^2}{2\rho_0}\overline{\rho'^2}+\frac{1}{2}\rho_0\overline{u'_i u'_i}\right)u_{0j}+\overline{p'u'_j}\\
&+\left(\frac{a_0^2}{\gamma-1}+\frac{1}{2}u_{0i}u_{0i}\right)\overline{\rho' u'_j}.
\end{aligned} \qquad (40)
$$

Quadratic term $[a_0^2/(\gamma-1)+u_{0i}u_{0i}/2]\overline{\rho' u'_j}$ can no longer be absent as for uniform flows in Eq.(33). Its contribution to time averaged energy is zero only when the mean flow is uniform.

Applications of Acoustic Intensity

In this section we give an example of the application of acoustic intensity (33) or (34) in hard wall circular duct.

From Eq.(11), the time averaged acoustic energy flux across an enclosed surface is:

$$\int \vec{\bar{I}}\cdot d\vec{S}=0. \qquad (41)$$

For a circular duct in Fig.1, we choose the enclosing surface shown by the shaded area. At cross section A, the sound power (sound energy across this section in unit time) is:

$$W_A=\int_0^R\int_0^{2\pi}\bar{I}_x r\,d\phi\,dr. \qquad (42)$$

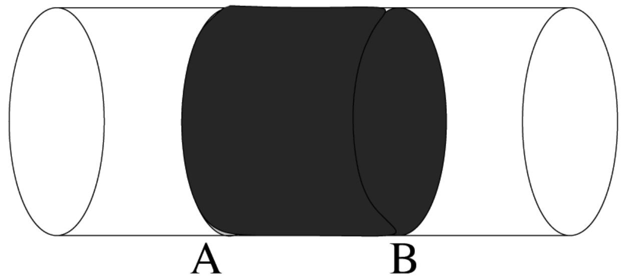

Fig1, Control surface in the circular duct.

Sound powers across section B, W_B, and sound power across the duct wall, W_W, can be defined similarly. From Eq.(41), we have:

$$W_A - W_B = W_W. \tag{43}$$

From the sound power distribution in the axial direction one can calculate the sound power absorbed by the duct wall. This is the base for liner duct analyses.

*Sound Power in Axial Direction for **Hard Wall** Circular Duct*

Dimensions used are: length: duct diameter *D*, velocity: sound speed a_0, density: ρ_0, pressure: $\rho_0 a_0^2$, energy flux: $\rho_0 a_0^3$, and sound power: $\rho_0 a_0^3 D^2$.

Acoustic intensity in the axial direction from (34) (Cantrell&Hart) is:

$$\bar{I}_x = \left(1 + M^2\right)\overline{p'u'} + M\left(\overline{p'^2} + \overline{u'^2}\right). \tag{44}$$

Assume the next form of the waves in the duct for circular frequency ω:

$$\begin{aligned} p'(x,r,\phi,t) &= \mathrm{Real}\left[\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\hat{p}_{mn}(x)J_m(\beta_{mn}r)e^{i(m\phi-\omega t)}\right], \\ u'(x,r,\phi,t) &= \mathrm{Real}\left[\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\hat{u}_{mn}(x)J_m(\beta_{mn}r)e^{i(m\phi-\omega t)}\right]. \end{aligned} \tag{45}$$

m, *n* are respectively the circumferential and radial mode numbers. *mn*th mode will be used to indicate the mode with circumferential mode number *m* and radial mode number *n*. $J_m(\beta_{mn}r)$ is the first kind Bessel function. For hard wall, β_{mn} is real.

By means of orthogonality of Bessel function, one can show that:

$$\int_0^{2\pi}\int_0^R \overline{p'u'}rdrd\phi = \sum_m\sum_n Q_{mn}\frac{1}{T}\int_t^{t+T}\int_0^{2\pi}\text{Real}\left[\hat{p}_{mn}(x)e^{i(m\phi-\omega t)}\right]\text{Real}\left[\hat{u}_{mn}(x)e^{i(m\phi-\omega t)}\right]d\phi dt$$
$$= \pi\sum_m\sum_n Q_{mn}\text{Real}\left[\hat{p}_{mn}(x)\hat{u}^*_{mn}(x)\right]. \tag{46}$$

where * represents complex conjugate, and,

$$Q_{mn} = \int_0^R J_m^2(r\beta_{mn})rdr = \begin{cases} \frac{1}{2}R^2, & \text{for } m=0, n=1; \\ \frac{1}{2}\left(R^2 - m^2/\beta_{mn}^2\right)J_m^2(R\beta_{mn}), & \text{other } m, n. \end{cases} \tag{47}$$

Similar equations hold for $\int_0^{2\pi}\int_0^R \overline{p'^2}rdrd\phi dt$ and $\int_0^{2\pi}\int_0^R \overline{u'^2}rdrd\phi$.

Therefore, the total sound power is the sum of all sound powers calculated from each *mn*th mode:

$$W = \sum_m\sum_n W_{mn}, \tag{48}$$
$$W_{mn} = \pi Q_{mn}\left[\left(1+M^2\right)\text{Real}\left(\hat{p}_{mn}\hat{u}^*_{mn}\right) + M\left(\left|\hat{p}_{mn}\right|^2 + \left|\hat{u}_{mn}\right|^2\right)\right].$$

Note the sound power is the real part of the expression.

Next we will discuss the sound power for a single (*m, n*) mode.

(*m, n*) *Mode in One Axial Direction*

$$\hat{p}_{mn} = A_{mn}e^{i\psi_{mn}}e^{i\alpha_{mn}x},$$
$$\hat{u}_{mn} = \frac{\alpha_{mn}}{\omega - M\alpha_{mn}}\hat{p}_{mn}. \tag{49}$$

A_{mn} and ψ_{mn} are real numbers.

For a cut-on mode,

$$\omega \geq \sqrt{1-M^2}\beta_{mn};\ \alpha^{\pm}_{mn} = \frac{-\omega M \pm \sqrt{\omega^2 - (1-M^2)\beta^2_{mn}}}{1-M^2} \text{ is real;}$$

$$W_{mn} = \pi Q_{mn}A^2_{mn}\left[\left(1+M^2\right)\frac{\alpha_{mn}}{\omega - \alpha_{mn}M} + M\left(1 + \left|\frac{\alpha_{mn}}{\omega - \alpha_{mn}M}\right|^2\right)\right]. \tag{50}$$

Note the sound power doesn't depend on the phase of this mode and the axial location. It depends on the mode amplitude, axial wavenumber, frequency and Mach number.

For a cut-off mode,

$$\omega < \sqrt{1-M^2}\beta_{mn};\ \alpha_{mn}^{\pm} = \frac{-\omega M \pm i\sqrt{(1-M^2)\beta_{mn}^2 - \omega^2}}{1-M^2} \text{ is complex;}$$

$$W_{mn} = \pi Q_{mn} A_{mn}^2 e^{-2x\mathrm{Img}(\alpha_{mn})}\left[\left(1+M^2\right)\mathrm{Real}\left(\frac{\alpha_{mn}}{\omega-\alpha_{mn}M}\right) + M\left(1+\left|\frac{\alpha_{mn}}{\omega-\alpha_{mn}M}\right|^2\right)\right].$$

Since

$$(1+M^2)\mathrm{Real}\left(\frac{\alpha_{mn}}{\omega-\alpha_{mn}M}\right) + M\left(1+\left|\frac{\alpha_{mn}}{\omega-\alpha_{mn}M}\right|^2\right) = 0,$$

we have

$$W_{mn} = 0. \tag{51}$$

A cutoff mode is also called evanescent wave. An evanescent wave transports no net acoustic energy (in stationary media), for pressure and axial velocity are 90^0 out of phase. There is an abrupt change of acoustic intensity from cut-on modes to cut-off modes (Pierce, p316). Physics meaning of evanescent waves (cut-off modes) in duct is: the waves decay very fast away from its source. But the medium is ideal and thus the decay is not due to viscous dissipation. Each fluid element oscillates with sound energy. This energy is established at the unsteady stage when there is transmitted energy across the duct section. After it gets to the steady stage, there is no net transmitted sound across duct section.

(m, n) *Mode in Both Axial Directions*

$$\begin{aligned}\hat{p}_{mn} &= A_{mn}^{+}e^{i\psi_{mn}^{+}}e^{i\alpha_{mn}^{+}x} + A_{mn}^{-}e^{i\psi_{mn}^{-}}e^{i\alpha_{mn}^{-}x},\\ \hat{u}_{mn} &= B_{mn}^{+}A_{mn}^{+}e^{i\psi_{mn}^{+}}e^{i\alpha_{mn}^{+}x} + B_{mn}^{-}A_{mn}^{-}e^{i\psi_{mn}^{-}}e^{i\alpha_{mn}^{-}x},\\ B_{mn}^{\pm} &= \frac{\alpha_{mn}^{\pm}}{\omega - M\alpha_{mn}^{\pm}}.\end{aligned} \tag{52}$$

For a cut-on mode,

$$\omega \geq \sqrt{1-M^2}\beta_{mn},\ \alpha_{mn}^{\pm} = \frac{-\omega M \pm \sqrt{\omega^2 - (1-M^2)\beta_{mn}^2}}{1-M^2},$$

$$W_{mn} = W_{mn}^{+} + W_{mn}^{-} + W_{mn}^{c},$$

$$W_{mn}^{\pm} = \pi Q_{mn}\left(A_{mn}^{\pm}\right)^{2}\left[\left(1+M^{2}\right)\frac{\alpha_{mn}^{\pm}}{\omega-\alpha_{mn}^{\pm}M} + M\left(1+\left|\frac{\alpha_{mn}^{\pm}}{\omega-\alpha_{mn}^{\pm}M}\right|^{2}\right)\right].$$

The cross term is zero:

$$\begin{aligned} W_{mn}^{c} &= \pi Q_{mn}A_{mn}^{+}A_{mn}^{-}\cos(\psi_{mn}^{+}-\psi_{mn}^{-}+\alpha_{mn}^{+}x-\alpha_{mn}^{-}x)\left[\left(1+M^{2}\right)\left(B_{mn}^{+}+B_{mn}^{-}\right)+2M\left(1+B_{mn}^{+}B_{mn}^{-}\right)\right] \\ &= 0 \end{aligned}$$

Therefore,

$$W_{mn} = W_{mn}^{+} + W_{mn}^{-}. \tag{53}$$

For a cut-off mode,

$$\omega < \sqrt{1-M^{2}}\beta_{mn},\ \alpha_{mn}^{\pm} = \frac{-\omega M \pm i\sqrt{(1-M^{2})\beta_{mn}^{2}-\omega^{2}}}{1-M^{2}},$$

$$B_{mn}^{\pm} = \frac{-M\beta_{mn}^{2} \pm i\omega\sqrt{(1-M^{2})\beta_{mn}^{2}-\omega^{2}}}{\omega^{2}+M^{2}\beta_{mn}^{2}},\ \left(B_{mn}^{+}\right)^{*} = B_{mn}^{-},\ \left(B_{mn}^{-}\right)^{*} = B_{mn}^{+}.$$

$$W_{mn} = W_{mn}^{+} + W_{mn}^{-} + W_{mn}^{c},$$

Since

$$W_{mn}^{+} = W_{mn}^{-} = 0,$$

we have

$$W_{mn} = W_{mn}^{c}. \tag{54}$$

$$\begin{aligned} W_{mn}^{c} &= 2\pi Q_{mn}A_{mn}^{+}A_{mn}^{-} \\ &\cdot\left\{\left(1+M^{2}\right)\mathrm{Real}\left[B_{mn}^{+}e^{i(\psi_{mn}^{+}-\psi_{mn}^{-})}\right]+M\left[\cos(\psi_{mn}^{+}-\psi_{mn}^{-})+\mathrm{Real}\left(\left|B_{mn}^{+}\right|^{2}e^{i(\psi_{mn}^{+}-\psi_{mn}^{-})}\right)\right]\right\}, \end{aligned}$$

In Summary

(1)For each cut-on (*m, n*) mode, if the wave propagates only in one direction, $W_{mn} \neq 0$; if the wave propagates in both directions, $W_{mn} = W_{mn}^{+} + W_{mn}^{-} \neq 0$;
(2)For each cut-off (*m, n*) mode, if the wave only 'propagates' in one direction, $W_{mn} = 0$; if the wave 'propagates' in both directions, $W_{mn} \neq 0$;

(3)Total sound power is the sum of sound powers calculated from each individual (*m, n*) mode.