In this chapter we will discuss acoustic energy, its different components and their respective balance equations. Emphasis will be on definition of acoustic intensity. Acoustic intensity is an important quantity describing a sound field. It is the acoustic energy flux past through unit area during unit time. It is straightforward to define the instantaneous acoustic intensity from acoustic energy balance equation. However, in engineering applications most of the concerns are time averaged energy flux. The instantaneous acoustic intensity needs to be simplified for a time steady process. Those terms with zero time average value should be dropped. And those terms that have time average values but don’t contribute to mean acoustic energy should also be dropped. In this way only fluctuation terms with even orders in the instantaneous acoustic intensity are to be kept in the final acoustic intensity definition.

Ideal \((\overline{p}_j = \overline{p}\delta_{ij})\), isentropic medium \((k \frac{\partial T}{\partial x_i} = 0)\) will be assumed. No body force. The unsteady flow is only caused by sound waves. And linear disturbance is assumed.

**Acoustic Energy Balance Equations**

**Decomposition of Variables**

All undisturbed flow variables are subscribed by ‘0’, such as, \(p_0, \overline{p}_0, \overline{u}_0\). With sound waves, any variable in the disturbed flow can be decomposed into a time averaged part and a fluctuation part,

\[
p = \overline{p} + p',
\]

\[
\overline{p} = \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} p dt, \quad p' = p - \overline{p}, \quad \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} p' dt = 0.
\]

Linear acoustic waves are fluctuations around the undisturbed flow, so

\[
p_0 = \overline{p}.
\]

For very strong waves the mean flow may be changed (acoustic streaming), which is out of the scope of this chapter. Only linear analysis is discussed here.

For a product of two (or more) variables, such as \(pu\),
\[ \bar{p}u = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} (p_0 u_0 + p_0 u' + p_0' u_0 + p_0' u') dt = p_0 u_0 + p_0'. \]  

\textit{Euler Equations with Decomposed Variables}

One can rewrite Euler equations using the time averaged and fluctuation variables in Eq.(1).

The continuity equation is:

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \bar{u}_j)}{\partial x_j} = 0, \text{ or,} \]

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} \bar{u}_j}{\partial x_j} (\bar{\rho}_0 u_0 + \bar{\rho}_0' u_0' + \bar{\rho}_0' u_0' + \bar{\rho}_0 u_0) = 0. \]

The momentum equation is:

\[ \frac{\partial (\bar{\rho} \bar{u}_i)}{\partial t} + \frac{\partial (\bar{\rho} \bar{u}_j \bar{u}_i + p \delta_{ij})}{\partial x_j} = 0, \text{ or,} \]

\[ \frac{\partial \bar{\rho} \bar{u}_i}{\partial t} + \frac{\partial \bar{\rho} \bar{u}_j}{\partial x_j} (\bar{\rho}_0 u_0 + \bar{\rho}_0' u_0' + \bar{\rho}_0' u_0' + \bar{\rho}_0 u_0) + \frac{\partial \bar{\rho} \bar{u}_i}{\partial x_j} (\bar{\rho}_0 u_0, \bar{\rho}_0' u_0', \bar{\rho}_0' u_0', \bar{\rho}_0 u_0') = 0. \]

The conservative form of energy equation is:

\[ \frac{\partial e}{\partial t} + \frac{\partial (\bar{\rho} \bar{u}_j)}{\partial x_j} (\bar{\rho} \bar{u}_j + pu_j) = 0, \]

where

\[ e = \bar{\rho} \bar{E} + \frac{1}{2} \bar{\rho} \bar{u}_i \bar{u}_i \]

is the total energy in unit fluid volume. The total energy is composed of two types of energies. One is the \textit{internal energy} \( \bar{\rho} \bar{E} \); the other is the \textit{kinetic energy} \( \frac{1}{2} \bar{\rho} \bar{u}_i \bar{u}_i \). Since the perturbations are only acoustic waves, it has only the translational kinetic energy (no rotational kinetic energy.)

\[ I_j = \bar{\rho} \bar{u}_j + pu_j \]
is the *instantaneous acoustic intensity*. It also includes two parts. One is the energy convection intensity \( e u_j \); the other is the energy production across the fluid element surface, *i.e.* work done by pressure \( p u_j \).

The conservation form of energy equation (8) shows that, the total energy flux through the surface enclosing a fluid volume equals to the total energy increase rate in this volume. For a time steady process, the time average of the total energy in the volume is constant. Therefore the time averaged equation of (8) is:

\[
\frac{\partial I_j}{\partial x_j} = 0, \quad (11)
\]

In a steady process, the time average of total energy flux across the enclosing surface is zero.

Eq.(10) is a general definition of acoustic intensity. But it may have never been used. We will show that the acoustic intensity has many components. Each component has different contributions to the acoustic energy. Some of them can be dropped so that a simplified acoustic intensity formation can be obtained.

*Acoustic Energy Components and Their Equations*

For isentropic flow, pressure changes adiabatically as density:

\[
p = \square^0 \cdot \text{Constant}, \quad \text{or} \quad \partial p / \partial \square = \partial \square / \partial p = a^2, \quad (12)
\]

and the internal energy density is:

\[
\square E = \frac{p}{\square 1}, \quad (13)
\]

For an isentropic process, the continuity and momentum equations (5)&(7) with equation of state (12) is a complete system for solving the sound field. Energy equation (8) is only a linear combination of continuity and momentum equations (5)&(7). It is not needed for solving the sound field. It is only used for energy balance analysis.

With the isentropic relation, continuity equation (5) can be rewritten as:

\[
\frac{\partial}{\partial t} \square \left[ \frac{p}{\square 1} \right] + \frac{\partial}{\partial x_j} \left[ \frac{p}{\square 1} u_j \right] + p \frac{\partial u_j}{\partial x_j} = 0. \quad (14)
\]
This is the equation for internal energy balance. The increase rate of internal energy is equal to the internal energy convection into the fluid element \( \frac{p}{\partial x} u_j \), plus work done by the pressure on the fluid element volume change \( p \frac{\partial u}{\partial x} \).

Multiplying both sides of momentum equation (7) by \( u_i \) and making use of the continuum equation (4), we have:

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \mu_i u_i \right] + \frac{\partial}{\partial x_j} \left[ \mu_i u_i u_j \right] + u_j \frac{\partial p}{\partial x_j} = 0. \tag{15}
\]

This is the kinetic energy balance equation. The increase rate of kinetic energy is the sum of convection of kinetic energy into the fluid \( \frac{1}{2} \mu_i u_i \) and work done by the net force on the movement of fluid element \( u_j \frac{\partial p}{\partial x_j} \).

Adding Eqs.(14) and (15) together one can obtain the total energy balance equation (8).

**Acoustic Intensity in Stationary Ideal Homogeneous Medium**

First we discuss the simplest case: quiescent mean flow \( \bar{u}_0 = 0 \).

**Acoustic Energy Balance Equation**

Since pressure changes adiabatically as density, we have from Eq.(12):

\[
p = p_0 + \frac{\partial p}{\partial \bar{u}_0} \left[ \frac{1}{2} \mu_0^2 \right] \bar{u}_0^2 + \cdots = p_0 + a_0^2 \bar{u}_0^2 + \frac{\partial p}{\partial \bar{u}_0} \left[ \frac{1}{2} \mu_0^2 \right] \bar{u}_0^2 + \cdots, \tag{16}
\]

i.e., \( p' = a_0^2 \bar{u}_0^2 + \frac{\partial p}{\partial \bar{u}_0} \left[ \frac{1}{2} \mu_0^2 \right] \bar{u}_0^2 + \cdots \)

where \( a_0 = \sqrt{\frac{\partial p}{\partial \bar{u}_0}} = \sqrt{\frac{\partial p_0}{\partial \bar{u}_0}} \). Therefore, the internal energy

\[
\mathcal{E} = \frac{p}{\mu} = \frac{1}{\mu} p_0 + a_0^2 \bar{u}_0^2 + \frac{\partial p}{\partial \bar{u}_0} \left[ \frac{1}{2} \mu_0^2 \right] \bar{u}_0^2 + \cdots. \tag{17}
\]

The zero, first, and second order fluctuations of the internal energy are \( \frac{p_0}{\mu} \), \( a_0^2 \bar{u}_0^2 \), and \( \frac{\partial p}{\partial \bar{u}_0} \left[ \frac{1}{2} \mu_0^2 \right] \bar{u}_0^2 \).
The kinetic energy to the second order is:

\[ \frac{1}{2} \square u_i u_i = \frac{1}{2} \square u_i' u_i'. \]  

(18)

The total acoustic energy fluctuation to the second order is:

\[ e' = \frac{1}{\square 1} a_0^2 \square + \frac{1}{2 \square 0} a_0^2 \square + \frac{1}{2} \square u_i' u_i'. \]  

(19)

and the acoustic energy balance equation is:

\[ \frac{\partial}{\partial t} \frac{1}{\square 1} a_0^2 \square + \frac{1}{2 \square 0} a_0^2 \square + \frac{1}{2} \square u_i' u_i' \square + \frac{\partial}{\partial x_j} \frac{\square p_0}{\square 1} u_j' + \frac{a_0^2}{\square 1} \square u_j' + p_0 u_j' + p' u_j' \square = 0. \]  

(20)

**Decomposition of Acoustic Energy Balance Equation and Acoustic Intensity**

From Eq.(20), by dropping first order terms one may define the time averaged acoustic intensity as:

\[ \overline{I}_j = \frac{a_0^2}{\square 1} \square u_j' + p' u_j'. \]  

(21)

We will show that the convection intensity, \( \frac{a_0^2}{\square 1} \square u_j' \), contributes nothing to the total acoustic energy and thus can be dropped.

Energy balance equation (20) can be separated into two equations respectively about internal energy such as Eq.(14) and kinetic energy such as Eq.(15). To facilitate the definition of mean acoustic intensity, here we will decompose the energy equation based on the work done by the pressure on the fluid element:

\[ p u_j = p_0 u_j' + p' u_j'. \]  

(22)

About the first order internal energy, \( \frac{a_0^2}{\square 1} \square u_j' \), from continuity equation (5), we have:

\[ \frac{\partial}{\partial t} \frac{\square a_0^2}{\square 1} \square + \frac{\partial}{\partial x_j} \frac{\square p_0}{\square 1} u_j' + \frac{a_0^2}{\square 1} \square u_j' + p_0 u_j' + p' u_j' \square = 0. \]  

(23)

This shows the rate of the first order internal energy change due to the work done by ambient pressure on the acoustic velocity \( p_0 u_j' \). In this equation the time average of
\( \frac{a_0^2}{\partial_0} \partial u_j \) is not zero. However the time average of energy \( \frac{1}{\partial_0} a_0^2 \partial \) is zero. Which means \( \frac{a_0^2}{\partial_0} \partial u_j \) contributes no mean energy to the sound field and should be excluded from the acoustic intensity.

Subtracting Eq.(23) from (20), we have:

\[
\frac{\partial}{\partial t} \frac{1}{\partial_0} a_0^2 \partial + \frac{1}{2} \partial_0 u_i u_i \partial + \frac{\partial}{\partial x_j} \left( p' u'_j \right) = 0. \quad (24)
\]

This equation shows the changing rate of acoustic energy due to the work done by acoustic pressure on the acoustic velocity, \( p' u'_j \). Of the acoustic energy, \( \frac{1}{2} \partial_0 u_i u_i \) is the acoustic kinetic-energy density, and second order internal energy \( \frac{1}{2} \partial_0 a_0^2 \partial^2 - \frac{1}{2} \partial_0 a_0^2 p^2 \) is the acoustic potential-energy density.

From equation (24), one can define the acoustic intensity as:

\[
\bar{I}_j = \bar{p}' u'_j. \quad (25)
\]

One may see that this energy flux only affects the second order energy fluctuation \( \frac{1}{2} \partial_0 a_0^2 \partial^2 + \frac{1}{2} \partial_0 u_i u_i \) instead of the total acoustic energy fluctuation \( e' \). Since the time average of the first order fluctuation internal energy \( a_0^2 / \left( \partial_0 \right) \) is zero, this definition of acoustic energy flux can only work in the sense of time average. It is widely used for stationary medium.

**Acoustic Intensity in Uniform Mean Flow**

**Acoustic Energy Balance Equation**

The total energy per unit volume of the mean flow is:

\[
e_0 = \frac{p_0}{\partial_0} + \frac{1}{2} \partial_0 u_0 u_0, \quad (26)
\]

and to the second order the acoustic energy fluctuation in unit volume of the fluid is:

\[
e' = \frac{a_0^2}{\partial_0} \partial + \frac{1}{2} u_0 u_0 \partial + \partial_0 u_0 u_0 + u_0 \partial u_0 + \frac{1}{2} \partial_0 a_0^2 \partial^2 + \frac{1}{2} \partial_0 u_i u'_i. \quad (27)
\]
The time averages of first order terms are zero.

The acoustic energy balance equation is:

\[
\frac{\partial e'}{\partial t} + \frac{\partial I_j}{\partial x_j} = 0,
\]  

(28)

where

\[
I_j = \frac{a_0^2}{\sqrt{\gamma}} + \frac{1}{2} u_0 u_0 \cdot u_0_j + \frac{p_0}{\sqrt{\gamma}} + \frac{a_0^2}{\sqrt{\gamma}} \frac{1}{\sqrt{2}} \cdot u_0 \cdot u_0_j + p_0 u_0_j + \frac{1}{2} \left( u_0 \cdot u_0 \right) u_0_j + p_0 u_0_j.
\]  

(29)

**Decomposition of Acoustic Energy**

From continuity equation (5), we can have:

\[
\frac{\partial}{\partial t} \left( \frac{a_0^2}{\sqrt{\gamma}} + \frac{1}{2} u_0 \cdot u_0 \right) + \frac{p_0}{\sqrt{\gamma}} + \frac{a_0^2}{\sqrt{\gamma}} \frac{1}{\sqrt{2}} \left( u_0 \cdot u_0 \right) u_0_j + p_0 u_0_j = \frac{\partial}{\partial x_j} \left( \frac{a_0^2}{\sqrt{\gamma}} + \frac{1}{2} u_0 \cdot u_0 \right) u_0_j + p_0 u_0_j.
\]  

(30)

This part of the first order fluctuation energy balance is due to the work done by the ambient pressure on the acoustic velocity \( p_0 u_0' \). The time average of this part of energy is zero, therefore this part of the energy flux should not be included in the acoustic intensity.

From momentum equation (7) we have the energy balance equation:

\[
\frac{\partial}{\partial t} \left( \frac{a_0^2}{\sqrt{\gamma}} + \frac{1}{2} u_0 \cdot u_0 \right) + \frac{p_0}{\sqrt{\gamma}} + \frac{a_0^2}{\sqrt{\gamma}} \frac{1}{\sqrt{2}} \left( u_0 \cdot u_0 \right) u_0_j + p_0 u_0_j = \frac{\partial}{\partial x_j} \left( \frac{a_0^2}{\sqrt{\gamma}} + \frac{1}{2} u_0 \cdot u_0 \right) u_0_j + p_0 u_0_j.
\]  

(31)

This part of energy balance is due to the work done by the acoustic pressure on the mean velocity \( p' u_{0,j} \). The energy has second order term with nonzero time average.

The equation for the rest of the acoustic energy is:
The energy terms are all second order with nonzero time average. It is due to the work by acoustic pressure and acoustic velocity \( p'u' \).

The three acoustic energy balance equations, (29), (30), (31), are decomposed bases on \( pu_j = p_0u'_j + p'u_{0,j} + p'u' \).

**Acoustic Intensity**

According to the above analyses, the *acoustic intensity* in \( j \) direction can be defined as:

\[
I_j = \frac{1}{2} a_0^2 \nabla \overline{|u'_j|^2} + \frac{1}{2} \overline{\nabla u'_i u'_j} \overline{u_{0,j} + p'u'}. \tag{33}
\]

All first order fluctuation terms are averaged out. Third and higher order terms are ignored. According to Lighthill (p.11), in linear acoustics first order quantities are retained and quadratic terms neglected in linear continuity and momentum equations, while in linear energy equation, first order terms should be absent (not neglected), quadratic terms retained, and cubic terms or above neglected.

A similar definition was proposed by Cantrell and Hart (Cantrell&Hart 1964, Eq.(13)) for isentropic uniform mean flows:

\[
I_j = \frac{1}{2} a_0^2 \nabla \overline{|u'_j|^2} + \frac{1}{2} \overline{\nabla u'_i u'_j} \overline{u_{0,j} + p'u'}. \tag{34}
\]

This definition has been widely accepted (Tester et.al.2004, Morfey1971, Mörhing1971). One can prove from Eq.(15-0) in ch1.doc, that when there is no mean flow, or the wave number in mean flow direction is real, the time mean acoustic potential energy is equal to the acoustic kinetic energy: \( \frac{1}{2} a_0^2 \nabla \overline{|u'_j|^2} = \frac{1}{2} a_0^2 \overline{p^2} = \frac{1}{2} \overline{\nabla u'_i u'_i}. \) With this property, Eq.(34) is the same as Eq.(33).

**Acoustic Intensity in Nonuniform Mean Flow**

The analysis of acoustic energy in nonuniform flows is similar to that in uniform flows. The total mean flow energy per unit volume is described by Eq.(26), and to the second
order the *acoustic energy* in unit volume of the fluid is as in Eq.(27). The acoustic energy balance equation is Eq.(28). However the acoustic intensity in nonuniform flow is

\[
I_j = (e_0 + p_0)u_{0j} + \frac{a_0^2}{2} \left[ \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + u_0 u_{0j} u_{ij} + p_0 u_j' \\
+ \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \]

instead of Eq.(29).

We may drop the odd order fluctuation terms and define the acoustic intensity as:

\[
I_j = (e_0 + p_0)u_{0j} + \frac{a_0^2}{2} \left[ \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \\
+ \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \]

From continuity equation (5), we have:

\[
\frac{\partial}{\partial t} \left[ \frac{a_0^2}{2} \right] + \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \\
+ \left[ \frac{\partial}{\partial x_j} \right] \frac{a_0^2}{2} \left[ \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \\
\]

According to Bernoulli’s theorem, for a frictionless non-conducting flow,

\[
\frac{D}{Dt} \left[ \frac{a_0^2}{2} \right] + \frac{1}{2} u_{0i} u_{0i} \right] = 0. \quad \text{(38)}
\]

It is a constant along a streamline. If the flow energy at upstream is uniform, then Eq.(37) can be written as,

\[
\frac{\partial}{\partial t} \left[ \frac{a_0^2}{2} \right] + \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \\
\]

\[
\frac{\partial}{\partial x_j} \left[ \frac{a_0^2}{2} \right] + \frac{1}{2} u_{0i} u_{0i} \right] u_{0j} + e_0 u_j' + \frac{1}{2} u_{0i} u_{0i} \right] u_{ij} + p_0 u_j' \\
\]

\[
\]
Subtracting this equation from Eq.(28), then the acoustic intensity can be defined as:

\[
I_j = \begin{bmatrix} u_{0i} \cdot u_{i}' + u_0 u_{0j} \cdot u_{i}' + u_{0j} \cdot p'^2 + \frac{1}{2} u_{0i} \cdot u_{i}' + u_0 u_{0j} \cdot u_{i}' + u_{0j} \cdot p' u_{i}' \end{bmatrix},
\]  

(40)

which is exactly the same as Eq.(33).

**Applications of Acoustic Intensity**

In this section we will give an application example of the acoustic intensity (33) or (34) in hard wall circular duct.

From Eq.(11), the time averaged acoustic energy flux across an enclosed surface is:

\[
\vec{f} \cdot d\vec{S} = 0.
\]  

(41)

For a circular duct in Fig.1, we choose the enclosing surface shown by the shaded area. At cross section A, the sound power (sound energy across this section in unit time) is:

\[
W_A = \int_0^R \int_0^{2\pi} r dr d\theta.
\]  

(42)

![Fig1, Control surface in the circular duct.](image)

Sound powers across section B, \( W_B \), and sound power across the duct wall, \( W_w \), can be defined similarly. From Eq.(41), we have:

\[
W_A \cdot W_B = W_w.
\]  

(43)

This equation means from sound power distribution in axial direction one can compute the sound power absorbed by the duct wall. This turns out to be very useful for liner duct analysis.
Sound Power in Axial Direction for Hard Wall Circular Duct

Dimension scales used here are: length: duct diameter \( D \), velocity: sound speed \( a_0 \), density: \( \rho \), pressure: \( \rho a_0^2 \), energy flux: \( \rho a_0^3 \), and sound power: \( \rho a_0^3 D^2 \).

Acoustic intensity in axial direction from (34) (Cantrell&Hart) is:

\[
I_x = \left( 1 + M^2 \right) p'u' + M \left( p'u'^2 + u'^2 \right).
\]  

Assume the next form of the waves in the duct for circular frequency \( \omega \):

\[
p'(x,r,\omega,t) = \text{Real} \left[ \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \hat{p}_{mn}(x) J_m(\sqrt{\rho a_0^2} r) e^{i(m \omega - \omega t)} \right]
\]

\[
u'(x,r,\omega,t) = \text{Real} \left[ \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \hat{u}_{mn}(x) J_m(\sqrt{\rho a_0^2} r) e^{i(m \omega - \omega t)} \right]
\]  

\( m, n \) are respectively the circumferential and radial mode numbers. \( mn \)th mode will be used to indicate the mode with circumferential mode number \( m \) and radial mode number \( n \). \( J_m(\sqrt{\rho a_0^2} r) \) is the first kind Bessel function. For hard wall, \( \sqrt{\rho a_0^2} \) is real.

By means of orthogonality of Bessel function, one can show that:

\[
\int_{0}^{2\pi} \int_{0}^{R} p' u' r drd\theta = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \int_{0}^{2\pi} \frac{1}{T} \int_{t}^{t+T} \text{Real} \left[ \hat{p}_{mn}(x) e^{i(m \omega - \omega t)} \right] \text{Real} \left[ \hat{u}_{mn}(x) e^{i(m \omega - \omega t)} \right] dt
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \text{Real} \left[ \hat{p}_{mn}(x) \hat{u}_{mn}^*(x) \right].
\]  

where * represents complex conjugate, and,

\[
Q_{mn} = \int_{0}^{R} J_m^2(\sqrt{\rho a_0^2} r) r dr = \begin{cases} \frac{1}{2} R^2, & \text{for } m = 0, n = 1; \\ \frac{1}{2} \left( R^2 - m^2 / \sqrt{\rho a_0^2} \right) J_m^2(R \sqrt{\rho a_0^2}), & \text{other } m, n. \end{cases}
\]  

Similar equations hold for \( \int_{0}^{2\pi} \int_{0}^{R} p'^2 r drd\theta \) and \( \int_{0}^{2\pi} \int_{0}^{R} u'^2 r drd\theta \).

Therefore, the total sound power is the sum of all sound powers calculated from each \( mn \)th mode:

\[
W = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} W_{mn},
\]  

11
Next we will discuss the sound power for a single $mn$th mode.

$m nth Mode in One Axial Direction$

\[ \hat{p}_{mn} = A_{mn} e^{\Omega_{mn} x} e^{i \psi_{mn}}, \]

\[ \hat{u}_{mn} = \frac{\Omega_{mn}}{\Omega_{mn} M} \hat{p}_{mn}. \]  

(49)

$A_{mn}$ and $\Omega_{mn}$ are real numbers.

For a cut-on mode,

\[ \Omega \geq \sqrt{1 - M^2 \Omega_{mn}}; \quad \Omega_{mn} = \frac{\sqrt{M^2 \pm \sqrt{M^2 (1 - M^2) \Omega_{mn}^2}}}{1 - M^2} \]

is real;

\[ W_{mn} = \Omega_{mn} A_{mn}^2 \left[ (1 + M^2) \Omega_{mn} - M^2 \right] + \Omega_{mn}^2 \left| \Omega_{mn} M \right|^2 \]  

(50)

For a cut-off mode,

\[ \Omega < \sqrt{1 - M^2 \Omega_{mn}}; \quad \Omega_{mn} = \frac{\sqrt{M^2 \pm i \sqrt{M^2 (1 - M^2) \Omega_{mn}^2}}}{1 - M^2} \]

is complex;

\[ W_{mn} = \Omega_{mn} A_{mn}^2 e^{2 \text{Im} \Omega_{mn}} \left[ (1 + M^2) \text{Real} \frac{\Omega_{mn}}{\Omega_{mn} M} - M^2 \right] + \Omega_{mn}^2 \left| \Omega_{mn} M \right|^2 \]

But since

\[ (1 + M^2) \text{Real} \frac{\Omega_{mn}}{\Omega_{mn} M} - M^2 + \Omega_{mn}^2 \left| \Omega_{mn} M \right|^2 = 0, \]

we have

\[ W_{mn} = 0. \]  

(51)

Wave with cutoff mode is called evanescent wave. An evanescent wave transports no net acoustic energy (in stationary media), for pressure and axial velocity are 90° out of phase.
There is an abrupt change of acoustic intensity from cut-on modes to cut-off modes (Pierce, p316). Physics meaning of evanescent waves (cut-off modes) in duct is: the waves decay very fast away from its source. But the medium is ideal and thus the decay is not due to viscous dissipation. Each fluid element oscillates with sound energy. This energy is established at the unsteady stage when there is transmitted energy across the duct section. After it gets to the steady stage, there is no net transmitted sound across duct section.

**mnth Mode in Both Axial Directions**

\[
\hat{p}_{mn} = A_{mn}^* e^{iQ_{mn}^x} e^{iQ_{mn}^z} + A_{mn} e^{iQ_{mn}^x} e^{iQ_{mn}^z},
\]

\[
\hat{u}_{mn} = B_{mn}^* A_{mn} e^{iQ_{mn}^x} e^{iQ_{mn}^z} + B_{mn} e^{iQ_{mn}^x} e^{iQ_{mn}^z},
\]

\[
B_{mn}^z = \frac{Q_{mn}^z}{M Q_{mn}^z}.
\]

For a cut-on mode,

\[
Q \geq \sqrt{1 - M^2 Q_{mn}^z},
\]

\[
Q_{mn}^z = \frac{\sqrt{Q^2 M^2 (1 - M^2) Q_{mn}^z}}{1 M^2},
\]

\[
W_{mn} = W_{mn}^* + W_{mn}^\parallel + W_{mn}^c,
\]

\[
W_{mn}^c = Q_{mn} (A_{mn}^* \sqrt{1 + M^2}) \frac{Q_{mn}^{z*}}{Q_{mn}^z M} + M^2 M + \left| \frac{Q_{mn}^{z*}}{Q_{mn}^z M} \right|^2.
\]

The cross term is zero:

\[
W_{mn}^c = Q_{mn} A_{mn}^* A_{mn}^\parallel \cos(Q_{mn}^z Q_{mn}^z + Q_{mn}^z + Q_{mn}^z) \left[ (1 + M^2) (B_{mn}^* + B_{mn}^\parallel) + 2 M (1 + B_{mn}^* B_{mn}^\parallel) \right] = 0
\]

Therefore,

\[
W_{mn} = W_{mn}^* + W_{mn}^\parallel.
\]

For a cut-off mode,

\[
Q < \sqrt{1 - M^2 Q_{mn}^z},
\]

\[
Q_{mn}^z = \frac{Q^2 M^2 (1 - M^2) Q_{mn}^z}{1 M^2},
\]

\[
B_{mn}^z = \frac{Q_{mn} Q_{mn}^z \pm i Q^2 M^2 (1 - M^2) Q_{mn}^z}{Q_{mn}^z + M^2 Q_{mn}^z}, \quad \left( B_{mn}^* \right)^z = B_{mn}, \quad \left( B_{mn}^\parallel \right)^z = B_{mn}^*.
\]
\[ W_{mn} = W^*_{mn} + W^\|_{mn} + W^c_{mn}, \]

Since

\[ W^*_{mn} = W^\|_{mn} = 0, \]

we have

\[ W_{mn} = W^c_{mn}. \] \hspace{1cm} (54)

\[ W^c_{mn} = 2 |Q|_{mn} A^*_{mn} A_{mn} \]
\[
\cdot \left\{ (1 + M^2) \text{Real} \left[ B^*_{mn} e^{i \theta_a} \right] + M \left[ \text{cos} (\theta_a) + \text{Real} \left( B^*_{mn} e^{i \theta_a} \right) \right] \right\}^*, \]

**In Summary**

(1) Total sound power is the sum of sound powers calculated from each individual \( mn \)th mode;

(2) For each cut-on \( mn \)th mode, \( W_{mn} \neq 0 \) is constant in axial direction;

(3) For each cut-off \( mn \)th mode, if the wave only ‘propagates’ in one direction, \( W_{mn} = 0 \); if the wave ‘propagates’ in both directions, \( W_{mn} \neq 0 \) is constant in axial direction.