Analytic Time Domain Formulation for Acoustic Pressure Gradient Prediction in a Moving Medium

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This paper presents an analytic time domain formulation for acoustic pressure gradient prediction in a moving medium, which has significant application potential in evaluating the acoustic scattering boundary condition. Based on the convective Ffowcs Williams–Hawkings equation, a semi-analytic time domain acoustic pressure gradient formulation with a form involving the observer time differentiation outside the integrals is first developed, and then the desired analytic time domain acoustic pressure gradient formulation is derived. Because the derived formulations are performed directly in the time domain, they are particularly applicable to the moving observer case. Simulation results for a stationary monopole source, a stationary dipole source, as well as a rotating monopole source in a moving medium demonstrate the effectiveness and accuracy of the proposed formulations for both stationary and moving sources with moving observers.

Nomenclature

A	=	amplitude of velocity potential, m^2s^{-1}
c_0	=	speed of sound in undisturbed medium, ms^{-1}
f	=	data surface function
G	=	time domain Green's function in a steady, uniform subsonic flow, m^{-1}
H	=	Heaviside function
$\boldsymbol{L}$	=	source strength of the loading source, Pa

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L_i	=	strength of the loading source components, Pa
$\dot{L}_i$	=	$\partial L_i \;/\; \partial \tau$, Pa·s^{-1}
L_R	=	$L_i \tilde{R}_i$, Pa
$\dot{L}_R$	=	$\dot{L}_i \tilde{R}_i$, Pa·s^{-1}
L_{R^*}	=	$L_i \tilde{R}_i^*$, Pa
L_M	=	$L_i M_i$, Pa
$\boldsymbol{M}$	=	source Mach number vector
M_i	=	components of source Mach number vector
$\dot{M}_i$	=	$\partial M_i \;/\; \partial \tau$
M_R	=	$M_i \tilde{R}_i$
$\dot{M}_R$	=	$\dot{M}_i \tilde{R}_i$
M_{R^*}	=	$M_i \tilde{R}_i^*$
$\boldsymbol{M}_\infty$	=	moving medium Mach number vector
$M_{\infty i}$	=	components of moving medium Mach number vector
$M_{\infty R}$	=	$M_{\infty i} \tilde{R}_i$
$M_{\infty R^*}$	=	$M_{\infty i} \tilde{R}_i^*$
$M_{\infty M}$	=	$M_{\infty i} M_i$
$M_{\infty L}$	=	$M_{\infty i} L_i$, Pa
n_i	=	components of unit vector normal to the data surface
p'	=	sound pressure, Pa
p	=	pressure of local fluid, Pa
p_0	=	pressure of undisturbed medium, Pa
Q	=	source strength of the thickness source, kg·m^{-2}s^{-1}

$\dot{Q}$ = $\partial Q / \partial \tau$, kg·m^{-2}s^{-2}

R^*, R = acoustic radii, m

$\tilde{R}_i^*$ = $\partial R^* / \partial x_i$

$\tilde{R}_i$ = $\partial R / \partial x_i$

$\boldsymbol{r}$ = geometrical vector between source and receiver, $\boldsymbol{x} - \boldsymbol{y}$, m

rs = rotating radius of source, m

S = data surface

T = source period, s

T_{ij} = Lighthill stress tensor, kg·m^{-1}s^{-2}

t = observer time, s

$\boldsymbol{U}_\infty$ = velocity vector of moving medium, ms^{-1}

$U_{\infty n}$ = local normal velocity of moving medium, ms^{-1}

$U_{\infty i}$ = velocity vector components of moving medium, ms^{-1}

$\boldsymbol{u}$ = fluid velocity vector, ms^{-1}

u_i = components of fluid velocity vector, ms^{-1}

u_n = local normal velocity of fluid, ms^{-1}

v_i = components of data surface velocity vector, ms^{-1}

v_n = local normal velocity of data surface, ms^{-1}

$\boldsymbol{x}$ = observer position vector, m

(x_1, x_2, x_3) = Cartesian coordinate for the observer

$\boldsymbol{y}$ = source position vector, m

(y_1, y_2, y_3) = Cartesian coordinate for the source

$\delta(\cdot)$ = Dirac delta function

δ_{ij} = Kronecker delta

$\varphi(\boldsymbol{x}, t)$	=	velocity potential function of source, m^2s^{-1}
ρ	=	local fluid density, $kg{\cdot}m^{-3}$
ρ'	=	density perturbation of fluid, $kg{\cdot}m^{-3}$
ρ_0	=	undisturbed medium density, $kg{\cdot}m^{-3}$
σ_{ij}	=	viscous stress tensor, $kg{\cdot}m^{-1}s^{-2}$
τ	=	source time, s
ω	=	source pulsation angular frequency, $rad{\cdot}s^{-1}$
ω_r	=	source rotating angular speed, $rad{\cdot}s^{-1}$

Subscripts

e	=	calculation at retarded time
L	=	loading source
O	=	observer point
S	=	source point
T	=	thickness source

I. Introduction

THE acoustic scattering effect in many engineering applications, such as the scattering by a fuselage boundary layer [1–6], the rotor noise scattered by the centerbody [7–9], and the noise scattered by a centrifugal volute [10], should not be neglected because it substantially influences the overall noise both in magnitude and directivity [11].

Based on the solutions of the Ffowcs Williams–Hawkings (FW–H) equation [12] or the Kirchhoff formulation [13], numerical methods such as the boundary element method [4, 14, 15] and the equivalent source method [16–18] have been developed to predict the acoustic scattering field in recent years. When solving acoustic scattering problems, the key aspect is obtaining the acoustic velocity on the scattering surface to serve as the boundary condition. Recently, Ghorbaniasl et al. [19] suggested the analytic formulations V1 and V1A for calculating the acoustic velocity directly in the time domain while the counterpart in the frequency domain was proposed by Mao et al. [20]. Given that the direct derivation of the acoustic velocity involved heavy algebraic manipulations, the acoustic pressure gradient can also be used as the boundary condition because it is related to the acoustic velocity through the acoustic velocity potential [21]. However, the direct numerical evaluation of the acoustic pressure gradient for a realistic scattering surface is computationally expensive, and therefore many researches have been done to obtain the analytic pressure gradient formulation. Farassat and Brentner [22] derived a semi-analytic formulation to calculate the acoustic pressure gradient. Lee et al. [23] first presented fully analytical formulation for the acoustic pressure gradient and implemented it into numerical codes. In that paper, the semi-analytical formulation was revisited and named formulation G1 and the fully analytic formulation was named formulation G1A.

It should be noted that the medium is assumed stationary in the above studies. However, convection effects as in a wind tunnel experiment may be important in aeroacoustic calculations. To realize the more complex acoustic scattering prediction for wind tunnel experiments, the convective FW–H equation [24, 25] which explicitly takes into account the presence of the moving medium should be used and the acoustic scattering boundary condition in the moving medium should be calculated as well. Recently, Ghorbaniasl et al. [26] derived an analytic acoustic pressure gradient formulation in the frequency domain which accounts for the effect of a constant uniform flow with arbitrary direction. Considering that the time domain formulation can be useful in some cases of acoustic scattering prediction, for example the moving observer case, an analytic time domain acoustic pressure gradient formulation which explicitly takes into account the presence of the moving medium is developed in the present paper. Inspired by the earlier work of Lee et al. [23], we will use similar names for our analytic formulation in a moving medium in the current paper: G1A-M in which M stands for a moving medium. This formulation can be seen as the extension of formulation G1A to a moving medium case. At the same time, a semi-analytic acoustic pressure gradient formulation G1-M is also given as part of the present study.

This paper is organized as follows. The convective FW–H equation and its time domain solution are first briefly reviewed in Sec. II A, and then the derivation of the formulation with a modified source term for acoustic pressure gradient is described in Sec. II B. Subsequently, three numerical test cases are used to examine the performance of the proposed formulations in Sec. III. Finally, conclusions are drawn in Sec. IV.

II. Theory

A. The Convective FW–H Equation and Its Time Domain Solution

Consider a uniform flow that moves at a constant velocity $\boldsymbol{U}_\infty$ and the direction of the velocity is arbitrary. Reorganizing the continuity and momentum equations that include the constant convective velocity term, the acoustic pressure at the observer $\boldsymbol{x}$ at time t could be described by the convective FW–H equation

$$[\frac{1}{c_0^2}\frac{D^2}{Dt^2} - \nabla^2]\{p'(\boldsymbol{x}, t)H(f)\} = \frac{D}{Dt}[Q\delta(f)] - \frac{\partial}{\partial x_i}[L_i\delta(f)] + \frac{\partial^2}{\partial x_i \partial x_j}[T_{ij}H(f)] \tag{1}$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_{\infty i}\frac{\partial}{\partial x_i} \tag{2}$$

$$Q = \rho_0(v_n - U_{\infty n}) + \rho[u_n - (v_n - U_{\infty n})] \tag{3}$$

$$L_i = [(p - p_0)\delta_{ij} - \sigma_{ij}]n_j + \rho u_i[u_n - (v_n - U_{\infty n})] \tag{4}$$

$$T_{ij} = \rho u_i u_j + [(p - p_0) - c_0^2(\rho - \rho_0)]\delta_{ij} - \sigma_{ij} \tag{5}$$

where $\delta(f)$ is the Dirac delta function and $H(f)$ is the Heaviside function; $f = 0$ denotes the data surface; $U_{\infty n} = U_{\infty i}n_i$ with the local unit outer normal $n_i = \partial f / \partial x_i$ in the direction x_i (i=1, 2, 3); δ_{ij} is the Kronecker delta; ρ_0, p_0, and c_0 are the density, pressure and sound speed in the undisturbed medium, respectively; p is the local fluid pressure, ρ is the local fluid density, p' is the acoustic pressure; the local fluid velocity component is denoted by u_i; the local normal components to the data surface of fluid and the body velocities are u_n and v_n, respectively; T_{ij} is the Lighthill stress tensor; σ_{ij} is the viscous stress tensor.

The first two terms on the right hand side of Eq. (1) are the monopole and dipole source terms which are also known as the thickness and loading sources, respectively. The third term is the quadrupole source term, which is typically small compared to the other two terms when the fluid and moving body's velocities are both small, and thus it is reasonably omitted in the subsonic calculations. By neglecting the quadrupole source term, the integral solution of the convective FW–H equation has been derived by Ghorbaniasl and Lacor in Ref. [27] as

$$p'(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = p'_T(\boldsymbol{x}, t, \boldsymbol{M}_\infty) + p'_L(\boldsymbol{x}, t, \boldsymbol{M}_\infty) \tag{6}$$

with the integral formulations over the data surface S

$$
\begin{aligned}
4\pi p'_T(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = & \int_S \left[\frac{(1 - M_{\infty R})\dot{Q}}{R^*(1 - M_R)^2} \right]_e dS - \int_S \left[\frac{c_0 M_{\infty R^*} Q}{R^{*2}(1 - M_R)} \right]_e dS \\
& + \int_S \left[(1 - M_{\infty R}) Q \, \frac{R^* \dot{M}_R + c_0 (M_{R^*} - |\boldsymbol{M}|^2)}{R^{*2}(1 - M_R)^3} \right]_e dS \\
& - \int_S \left[(1 - M_{\infty R}) Q \, \frac{c_0 M_{R^*} M_R + c_0 \gamma^2 (M_{\infty M}^2 - M_{R^*}^2)}{R^{*2}(1 - M_R)^3} \right]_e dS \\
& - \int_S \left[\frac{c_0 \gamma^2 (M_{\infty R^*} M_{R^*} - M_{\infty M}) Q}{R^{*2}(1 - M_R)^2} \right]_e dS
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
4\pi p'_L(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = & \frac{1}{c_0} \int_S \left[\frac{\dot{L}_R}{R^*(1 - M_R)^2} \right]_e dS + \int_S \left[\frac{L_{R^*} - L_M}{R^{*2}(1 - M_R)^2} \right]_e dS \\
& + \frac{1}{c_0} \int_S \left[L_R \, \frac{R^* \dot{M}_R + c_0 (M_{R^*} - |\boldsymbol{M}|^2)}{R^{*2}(1 - M_R)^3} \right]_e dS \\
& - \int_S \left[L_R \, \frac{M_{R^*} M_R + \gamma^2 (M_{\infty M}^2 - M_{R^*}^2)}{R^{*2}(1 - M_R)^3} \right]_e dS \\
& - \int_S \left[\frac{L_{R^*} M_R + \gamma^2 (M_{\infty M} M_{\infty L} - L_{R^*} M_{R^*})}{R^{*2}(1 - M_R)^2} \right]_e dS
\end{aligned}
\tag{8}
$$

The mathematical background as well as the derivation procedure can be seen in Ref. [27]. In Eqs. (7) and (8), the acoustic radii R^* and R are

$$
R^* = \frac{1}{\gamma} \sqrt{r^2 + \gamma^2 (\boldsymbol{M}_\infty \cdot \boldsymbol{r})^2} \tag{9}
$$

$$
R = \gamma^2 (R^* - \boldsymbol{M}_\infty \cdot \boldsymbol{r}) \tag{10}
$$

where

$$
\gamma = \sqrt{1 / (1 - \boldsymbol{M}_\infty^2)} \tag{11}
$$

$$
\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y} \tag{12}
$$

The quantities in brackets should be evaluated at the retarded time $\tau = t - R / c_0$ and the symbol $\boldsymbol{M}_\infty$ is the flow Mach number vector with $M_{\infty i} = U_{\infty i} / c_0$, where M is the magnitude of $\boldsymbol{M}$ that is the body Mach number vector with $M_i = v_i / c_0$. The other nomenclatures are defined as follows: $M_{\infty R} = M_{\infty i}\tilde{R}_i$, $M_{\infty R^*} = M_{\infty i}\tilde{R}_i^*$, $\tilde{R}_i^* = \partial R^* / \partial x_i$, $\tilde{R}_i = \partial R / \partial x_i$, $M_{\infty M} = M_{\infty i}M_i$, $M_R = M_i\tilde{R}_i$, $\dot{M}_R = \dot{M}_i\tilde{R}_i$, $M_{R^*} = M_i\tilde{R}_i^*$, $M_{\infty L} = M_{\infty i}L_i$, $L_M = L_iM_i$, $L_{R^*} = L_i\tilde{R}_i^*$, $L_R = L_i\tilde{R}_i$, $\dot{L}_R = \dot{L}_i\tilde{R}_i$. The dots on the several quantities denote derivatives with respect to the source time τ and the dots on the main variables do not imply the differentiation of any of the associated vectors implied by the subscripts, for example, $\dot{L}_i = \partial L_i / \partial \tau$ and $\dot{M}_{R^*} = \dot{M}_i\tilde{R}_i^*$.

From a theoretical point of view, the analytic acoustic pressure gradient formulation can be derived by directly calculating the gradients of Eqs. (7) and (8), however, this process requires heavy mathematical operations. An alternative way to derive the analytic acoustic pressure gradient formulation will be presented in the next sub-section and this derivation is easily manipulated by adopting a modification of source term at the beginning of the derivation.

B. Derivation Procedure of Formulations G1-M and G1A-M

Starting from the convective FW–H Eq. (1) and omitting the quadrupole source term, we can obtain a simplified convective FW–H equation as

$$[\frac{1}{c_0^2}\frac{D^2}{Dt^2} - \nabla^2]\{p'(\boldsymbol{x},t)H(f)\} = \frac{D}{Dt}[Q\delta(f)] - \frac{\partial}{\partial x_i}[L_i\delta(f)] \tag{13}$$

Employing Eq. (2) and adopting a modification of the source term recently suggested by Ghorbaniasl et al. [26]

$$F_i = L_i - QU_{\infty i} \tag{14}$$

Equation (13) can be further simplified to the following form

$$[\frac{1}{c_0^2}\frac{D^2}{Dt^2} - \nabla^2]\{p'(\boldsymbol{x},t)H(f)\} = \frac{\partial}{\partial t}[Q\delta(f)] - \frac{\partial}{\partial x_i}[F_i\delta(f)] \tag{15}$$

The Green's function used in a steady, uniform subsonic flow with the Mach number vector $\boldsymbol{M}_\infty$ is

$$G(\boldsymbol{x},t;\boldsymbol{y},\tau) = \frac{\delta(\tau - t + R/c_0)}{4\pi R^*} = \frac{\delta(g)}{4\pi R^*} \tag{16}$$

where

$$g = \tau - t + R / c_0 \tag{17}$$

Using the above Green's function yields the solution of the convective FW–H Eq. (15)

$$p'(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty) + p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty) \tag{18}$$

where

$$4\pi p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = \frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S Q\,\frac{\delta(g)}{R^*}dSd\tau \tag{19}$$

$$4\pi p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty) = -\frac{\partial}{\partial x_i}\int_{-\infty}^{t}\int_S F_i\,\frac{\delta(g)}{R^*}dSd\tau \tag{20}$$

To obtain acoustic pressure gradient formulations, the gradient operation is performed to Eqs. (18–20), yielding

$$\frac{\partial p'(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = \frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} + \frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} \tag{21}$$

$$4\pi\,\frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = \frac{\partial}{\partial t}\frac{\partial}{\partial x_i}\int_{-\infty}^{t}\int_S Q\,\frac{\delta(g)}{R^*}dSd\tau \tag{22}$$

$$4\pi\,\frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = -\frac{\partial^2}{\partial x_i^2}\int_{-\infty}^{t}\int_S F_i\,\frac{\delta(g)}{R^*}dSd\tau \tag{23}$$

Since the integral variables S and τ in Eqs. (22) and (23) are independent of the observer coordinates x_i (i=1, 2, 3), the gradient operators can be moved inside the integrals. Using the following equation that has been derived in Ref. [27]

$$\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{R^*}\right] = -\frac{1}{c_0}\frac{\partial}{\partial t}\left[\frac{\tilde{R}_i\delta(g)}{R^*}\right] - \left[\frac{\tilde{R}_i^*\delta(g)}{R^{*2}}\right] \tag{24}$$

Equations (22) and (23) can be further rewritten as

$$4\pi\,\frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0}\frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S \frac{\partial}{\partial t}\left[\frac{Q\tilde{R}_i\delta(g)}{R^*}\right]dSd\tau - \frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S\left[\frac{Q\tilde{R}_i^*\delta(g)}{R^{*2}}\right]dSd\tau \tag{25}$$

$$4\pi\,\frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = \frac{1}{c_0}\frac{\partial}{\partial t}\int_{-\infty}^{t}\int_S F_j\,\frac{\partial}{\partial x_i}\left[\frac{\tilde{R}_j\delta(g)}{R^*}\right]dSd\tau + \int_{-\infty}^{t}\int_S F_j\,\frac{\partial}{\partial x_i}\left[\frac{\tilde{R}_j^*\delta(g)}{R^{*2}}\right]dSd\tau \tag{26}$$

In order to calculate the integral over $d\tau$, the identity of generalized function [27, 28] should be used

$$\begin{aligned}\int_{-\infty}^{t} h(\tau)\delta(g)d\tau &= \left[\frac{h(\tau)}{\left|\partial g / \partial\tau\right|}\right]_{g=0} \\ &= \left[\frac{h(\tau)}{1 - M_R}\right]_e\end{aligned} \tag{27}$$

With the help of Eq. (27), Eq. (25) can be written as

$$4\pi \frac{\partial p'_\alpha(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = -\frac{\partial}{\partial t} E_1 \tag{28}$$

where

$$E_1 = \frac{1}{c_0}\frac{\partial}{\partial t}\int_S \left[\frac{Q\tilde{R}_i}{R^*(1 - M_R)}\right]_e dS + \int_S \left[\frac{Q\tilde{R}_i^*}{R^{*2}(1 - M_R)}\right]_e dS \tag{29}$$

In order to avoid the numerical evaluation of the observer time differentiation outside the integral in Eq. (29), they are converted to the source time differentiation through the following identity [27]

$$\frac{\partial}{\partial t} = \left[\frac{1}{1 - M_R}\frac{\partial}{\partial\tau}\right]_e \tag{30}$$

E_1 then becomes

$$\begin{aligned}E_1 &= \frac{1}{c_0}\int_S \left[\frac{1}{(1 - M_R)}\right]_e \frac{\partial}{\partial\tau}\left[\frac{Q\tilde{R}_i}{R^*(1 - M_R)}\right]_e dS + \int_S \left[\frac{Q\tilde{R}_i^*}{R^{*2}(1 - M_R)}\right]_e dS \\ &= \frac{1}{c_0}\int_S \left[\frac{\dot{Q}\tilde{R}_i}{R^*(1 - M_R)^2}\right]_e dS + \int_S \left[Q\frac{-M_i + \gamma^2(M_{R^*}\tilde{R}_i^* - M_{\infty M}M_{\infty i})}{R^{*2}(1 - M_R)^2}\right]_e dS \\ &+ \frac{1}{c_0}\int_S \left[Q\tilde{R}_i \frac{R^*\dot{M}_R + c_0(M_{R^*} - M^2)}{R^{*2}(1 - M_R)^3}\right]_e dS \\ &- \frac{1}{c_0}\int_S \left[Q\tilde{R}_i \frac{c_0 M_{R^*}(M_R - \gamma^2 M_{R^*}) + c_0\gamma^2 M_{\infty M}^2}{R^{*2}(1 - M_R)^3}\right]_e dS \\ &+ \int_S \left[\frac{Q\tilde{R}_i^*}{R^{*2}(1 - M_R)}\right]_e dS\end{aligned} \tag{31}$$

Following the same steps used to obtain Eqs. (28) and (31) and employing an extra relation that is used to simplify the equations

$$\frac{\partial}{\partial x_i}\,\delta(g) = \frac{\tilde{R}_i}{c_0}\,\delta'(g) = -\frac{\tilde{R}_i}{c_0}\frac{\partial}{\partial t}\left[\delta(g)\right] \tag{32}$$

Equation (26) would be further written as

$$4\pi\,\frac{\partial p'_\beta(\boldsymbol{x}, t, \boldsymbol{M}_\infty)}{\partial x_i} = \frac{1}{c_0}\frac{\partial}{\partial t}E_2 + \int_S \frac{1/\gamma^2 \cdot F_i + M_{\infty i}M_{\infty F} - 3\tilde{R}_i^* F_{R^*}}{R^{*3}(1 - M_R)}\,dS \tag{33}$$

where

$$\begin{aligned} E_2 = &-\frac{1}{c_0}\int_S\left[\frac{\dot{F}_R\tilde{R}_i}{R^*(1-M_R)^2}\right]_e dS - \int_S\left[\frac{\gamma^2\tilde{R}_i(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F}) - F_M\tilde{R}_i}{R^{*2}(1-M_R)^2}\right]_e dS \\ &-\int_S\left[\frac{\gamma^2F_R(M_{R^*}\tilde{R}_i^* - M_{\infty M}M_{\infty i}) - F_RM_i}{R^{*2}(1-M_R)^2}\right]_e dS \\ &-\frac{1}{c_0}\int_S\left[\frac{F_R\tilde{R}_iR^*\dot{M}_R + c_0F_R\tilde{R}_i(M_{R^*} - M^2)}{R^{*2}(1-M_R)^3}\right]_e dS \\ &+\int_S\left[\frac{F_R\tilde{R}_iM_{R^*}(M_R - \gamma^2M_{R^*}) + \gamma^2F_R\tilde{R}_iM_{\infty M}^2}{R^{*2}(1-M_R)^3}\right]_e dS \\ &+\int_S\left[\frac{F_i + \gamma^2(M_{\infty i}M_{\infty F} - \tilde{R}_i^*F_{R^*}) - \tilde{R}_i^*F_R - \tilde{R}_iF_{R^*}}{R^{*2}(1-M_R)}\right]_e dS \end{aligned} \tag{34}$$

It should be noted that Eqs. (21), (28), and (33) are together called formulation G1-M which can be seen as an extension of formulations G1 to moving medium cases. Comparing formulation G1-M with the acoustic pressure formulation Eqs. (6), (7), and (8), it is found that no more data are needed to calculate the acoustic pressure gradient than those used to predict the acoustic pressure in a moving medium, and therefore the two acoustic variables could be calculated at the same time. The observer time derivatives outside the integrals in Eqs. (28) and (33) can be evaluated numerically with various difference algorithm, such as the forward, backward, and central differences [29]. Compared with the direct numerical evaluation of the acoustic pressure gradient by using the acoustic pressure data of several observers, it is also an advantage that the formulation G1-M does save considerable computing resources because the integral data of only one observer are needed.

The main drawback of formulation G1-M is that it is inconvenient to deal with the cases where the observer is not stationary. If the observer is stationary, the numerical observer time derivatives of the integrals in formulation G1-M are easy to deal with because the time history of the integrals in formulation G1-M can be obtained together with the acoustic pressure data at each observer time step. However, if the observer is moving, several extra evaluations of the integrals are needed to calculate the numerical observer time derivatives at each observer time step. In order to eliminate the numerical observer time derivatives of the acoustic pressure gradient calculation, an analytic formulation called as G1A-M is deduced in the following.

The procedure for eliminating the observer time derivatives is to apply Eq. (30) to formulation G1-M and then evaluate the source time derivatives of the relevant variables. Inspired by the work of Lee [23], some new functions and key source time derivatives are given in the following to make the formulation G1A-M more concise:

$$U(m,n) = \frac{1}{(R^*)^m (1-M_R)^n} \tag{35}$$

$$\begin{aligned} V(m,n) &= \frac{\partial U(m,n)}{\partial \tau} \\ &= \frac{nR^*\dot{M}_R + nc_0\gamma^2(M_{R^*}^2 - M_{\infty M}^2) + mc_0 M_{R^*}(1-M_R) - nc_0 M^2}{(R^*)^{m+1}(1-M_R)^{n+1}} \end{aligned} \tag{36}$$

$$W = R^*\dot{M}_R + c_0(M_{R^*} - M^2) \tag{37}$$

$$\begin{aligned} \dot{W} &= R^*\ddot{M}_R + c_0(\dot{M}_{R^*} - M_{R^*}\dot{M}_R - M_i\dot{M}_i) + c_0\gamma^2(M_{R^*}\dot{M}_{R^*} - M_{\infty M}M_{\infty\dot{M}}) \\ &+ \frac{c_0^2\gamma^2(M_{R^*}^2 - M_{\infty M}^2) - c_0^2 M^2}{\gamma^2 R^*} \end{aligned} \tag{38}$$

$$Z = c_0 M_{R^*}(M_R - \gamma^2 M_{R^*}) + c_0\gamma^2 M_{\infty M}^2 \tag{39}$$

$$\begin{aligned} \dot{Z} &= \frac{(M_R - \gamma^2 M_{R^*})[c_0 R^*\dot{M}_R + c_0^2(\gamma^2 M_{R^*}^2 - M^2) - c_0^2\gamma^2 M_{\infty M}^2]}{R^*} \\ &+ c_0 M_{R^*}(\dot{M}_R - -\gamma^2\dot{M}_{R^*}) + 2c_0\gamma^2 M_{\infty M}M_{\infty\dot{M}} \end{aligned} \tag{40}$$

$$B_i = -M_i + \gamma^2(M_{R^*}\tilde{R}_i^* - M_{\infty M}M_{\infty i}) \tag{41}$$

$$\begin{aligned} \dot{B}_i &= -\dot{M}_i - \gamma^2 M_{\infty\dot{M}}M_{\infty i} + \frac{\gamma^2 R^*\dot{M}_{R^*}\tilde{R}_i^*}{R^*} \\ &+ \frac{c_0\gamma^2(2M_{R^*}^2\tilde{R}_i^* - M_{\infty M}^2\tilde{R}_i^* - M_{R^*}M_{\infty M}M_{\infty i}) - c_0(M^2\tilde{R}_i^* + M_{R^*}M_i)}{R^*} \end{aligned} \tag{42}$$

$$A_i = \tilde{R}_i[\gamma^2(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F}) - F_M] \tag{43}$$

$$\begin{aligned}\dot{A}_i = {} & \frac{\tilde{R}_i}{R^*}[\gamma^2 R^*(F_{R^*}\dot{M}_{R^*} + M_{R^*}\dot{F}_{R^*})] \\ & + \frac{c_0\gamma^2\tilde{R}_i(2M_{R^*}^2F_{R^*} - M_{\infty M}^2F_{R^*} - M_{\infty M}M_{\infty F}M_{R^*})}{R^*} \\ & + \frac{\tilde{R}_i}{R^*}[-c_0(F_{R^*}M^2 + F_M M_{R^*}) - \gamma^2 R^*(M_{\infty\dot{M}}M_{\infty F} + M_{\infty M}M_{\infty\dot{F}})] \\ & + \frac{c_0 B_i}{R^*}[\gamma^2(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F}) - F_M]\end{aligned} \tag{44}$$

$$D = R^*\dot{M}_R + c_0(M_{R^*} - M^2) \tag{45}$$

$$\begin{aligned}\dot{D} = {} & -c_0 M_{R^*}\dot{M}_R + R^*\ddot{M}_R + c_0\gamma^2(M_{R^*}\dot{M}_{R^*} - M_{\infty M}M_{\infty\dot{M}}) - c_0 M_i\dot{M}_i \\ & + \frac{c_0}{\gamma^2 R^*}[\gamma^2 R^*\dot{M}_{R^*} - c_0 M^2 + c_0\gamma^2(M_{R^*}^2 - M_{\infty M}^2) - 2\gamma^2 R^* M\dot{M}]\end{aligned} \tag{46}$$

$$H = M_{R^*}(M_R - \gamma^2 M_{R^*}) + \gamma^2 M_{\infty M}^2 \tag{47}$$

$$\begin{aligned}\dot{H} = {} & \frac{(M_R - 2\gamma^2 M_{R^*})[\gamma^2 R^*\dot{M}_{R^*} - c_0 M^2 + c_0\gamma^2(M_{R^*}^2 - M_{\infty M}^2)]}{\gamma^2 R^*} \\ & + \frac{R^* M_{R^*}\dot{M}_R + c_0(\gamma^2 M_{R^*}^3 - M_{R^*}M^2) - c_0\gamma^2 M_{R^*}M_{\infty M}^2}{R^*} \\ & + 2\gamma^2 M_{\infty M}M_{\infty\dot{M}}\end{aligned} \tag{48}$$

$$K_i = F_i + \gamma^2(M_{\infty i}M_{\infty F} - \tilde{R}_i^* F_{R^*}) - \tilde{R}_i^* F_R - \tilde{R}_i F_{R^*} \tag{49}$$

$$\begin{aligned}\dot{K}_i = {} & \dot{F}_i + \gamma^2 M_{\infty i}M_{\infty\dot{F}} - \dot{F}_{R^*}(\tilde{R}_i + \gamma^2\tilde{R}_i^*) \\ & - \frac{[-c_0 F_M + c_0\gamma^2(M_{R^*}F_{R^*} - M_{\infty M}M_{\infty F})](\tilde{R}_i + 2\gamma^2\tilde{R}_i^*)}{\gamma^2 R^*} \\ & - \frac{c_0 B_i(F_R - 2\gamma^2 F_{R^*})}{\gamma^2 R^*}\end{aligned} \tag{50}$$

It should be noted that the second partial derivative with respect to the source time is denoted by two dots over the quantity and dots over the subscripts mean differentiation of the associated vectors implied by the subscripts, for example, $\ddot{M}_R = \ddot{M}\tilde{R}_i$ and $M_{\infty\dot{M}} = M_{\infty i}\dot{M}_i$. Taking the observer time derivatives inside the integrals of G1-M and using the above definitions, one obtains

$$4\pi\,\frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[\dot{Q}\tilde{R}_i U(1,2)\right]_e dS$$
$$-\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[Q\tilde{R}_i U(2,3)W\right]_e dS$$
$$+\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[Q\tilde{R}_i U(2,3)Z\right]_e dS \tag{51}$$
$$-\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[Q\tilde{R}_i^* U(2,1)\right]_e dS$$
$$-\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[QB_i U(2,2)\right]_e dS$$

and

$$4\pi\,\frac{\partial p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = -\frac{1}{c_0^2}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[\dot{F}_R\tilde{R}_i U(1,2)\right]_e dS$$
$$-\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[A_i U(2,2)\right]_e dS$$
$$-\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[F_R B_i U(2,2)\right]_e dS$$
$$-\frac{1}{c_0^2}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[F_R\tilde{R}_i D U(2,3)\right]_e dS \tag{52}$$
$$+\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[F_R\tilde{R}_i H U(2,3)\right]_e dS$$
$$+\frac{1}{c_0}\int_S\left[\frac{1}{(1-M_R)}\right]_e\frac{\partial}{\partial\tau}\left[K_i U(2,1)\right]_e dS$$
$$+\int_S(1/\gamma^2\cdot F_i + M_{\infty i}M_{\infty F} - 3\tilde{R}_i^* F_{R^*})U(3,1)dS$$

The last step is to further rewrite Eqs. (51) and (52) as

$$4\pi\,\frac{\partial p'_\alpha(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = I_1 + I_2 + I_3 + I_4 + I_5 \tag{53}$$

$$4\pi\,\frac{\partial p'_\beta(\boldsymbol{x},t,\boldsymbol{M}_\infty)}{\partial x_i} = I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} \tag{54}$$

where I_1 to I_{12} correspond to each of the integrals in Eqs. (51) and (52). The forms of I_i are given as follows:

$$I_1 = -\frac{1}{c_0}\int_S\left[\ddot{Q}\tilde{R}_i U(1,3) + \dot{Q}\dot{\tilde{R}}_i U(1,3) + \dot{Q}\tilde{R}_i V(1,2)U(0,1)\right]_e dS \tag{55}$$

$$I_2 = -\frac{1}{c_0}\int_S\left[\dot{Q}\tilde{R}_i U(2,4)W + Q\dot{\tilde{R}}_i U(2,4)W + Q\tilde{R}_i V(2,3)U(0,1)W + Q\tilde{R}_i U(2,4)\dot{W}\right]_e dS \tag{56}$$

$$I_3 = \frac{1}{c_0}\int_S\left[\dot{Q}\tilde{R}_i U(2,4)Z + Q\dot{\tilde{R}}_i U(2,4)Z + Q\tilde{R}_i V(2,3)U(0,1)Z + Q\tilde{R}_i U(2,4)\dot{Z}\right]_e dS \tag{57}$$

$$I_4 = -\int_S\left[\dot{Q}\tilde{R}_i^* U(2,2) + Q\dot{\tilde{R}}_i^* U(2,2) + Q\tilde{R}_i^* V(2,1)U(0,1)\right]_e dS \tag{58}$$

$$I_5 = -\int_S\left[\dot{Q}B_i U(2,3) + Q\dot{B}_i U(2,3) + QB_i V(2,2)U(0,1)\right]_e dS \tag{59}$$

$$I_6 = -\frac{1}{c_0^2}\int_S\left[(\frac{\partial\dot{F}_R}{\partial\tau})\tilde{R}_i U(1,3) + \dot{F}_R\dot{\tilde{R}}_i U(1,3) + \dot{F}_R\tilde{R}_i V(1,2)U(0,1)\right]_e dS \tag{60}$$

$$I_7 = -\frac{1}{c_0}\int_S\left[\dot{A}_i U(2,3) + A_i V(2,2)U(0,1)\right]_e dS \tag{61}$$

$$I_8 = -\frac{1}{c_0}\int_S\left[(\frac{\partial F_R}{\partial\tau})B_i U(2,3) + F_R\dot{B}_i U(2,3) + F_R B_i V(2,2)U(0,1)\right]_e dS \tag{62}$$

$$\begin{aligned} I_9 = &-\frac{1}{c_0^2}\int_S\left[(\frac{\partial F_R}{\partial\tau})\tilde{R}_i DU(2,4) + F_R\dot{\tilde{R}}_i DU(2,4)\right]_e dS \\ &-\frac{1}{c_0^2}\int_S\left[F_R\tilde{R}_i\dot{D}U(2,4) + F_R\tilde{R}_i DV(2,3)U(0,1)\right]_e dS \end{aligned} \tag{63}$$

$$I_{10} = \frac{1}{c_0}\int_S\left[(\frac{\partial F_R}{\partial\tau})\tilde{R}_i HU(2,4) + F_R\dot{\tilde{R}}_i HU(2,4) + F_R\tilde{R}_i\dot{H}U(2,4) + F_R\tilde{R}_i HV(2,3)U(0,1)\right]_e dS \tag{64}$$

$$I_{11} = \frac{1}{c_0}\int_S\left[\dot{K}_i U(2,2) + K_i V(2,1)U(0,1)\right]_e dS \tag{65}$$

$$I_{12} = \int_S(1/\gamma^2 \cdot F_i + M_{\infty i}M_{\infty F} - 3\tilde{R}_i^* F_{R^*})U(3,1)dS \tag{66}$$

It should be noted that $\dot{\tilde{R}}_i^*$, $\dot{\tilde{R}}_i$ and $\frac{\partial F_R}{\partial\tau}$ are defined as

$$\dot{\tilde{R}}_i^* = \frac{\partial\tilde{R}_i^*}{\partial\tau} = \frac{-c_0M_i + c_0\gamma^2(M_{R^*}\tilde{R}_i^* - M_{\infty M}M_{\infty i})}{\gamma^2R^*} \tag{67}$$

$$\dot{\tilde{R}}_i = \frac{\partial\tilde{R}_i}{\partial\tau} = \gamma^2\frac{\partial\tilde{R}_i^*}{\partial\tau} \tag{68}$$

$$\frac{\partial F_R}{\partial \tau} = \dot{F}_R + \frac{-c_0 F_M + c_0 \gamma^2 (M_{R^*} F_{R^*} - M_{\infty M} M_{\infty F})}{R^*} \tag{69}$$

Equations (21), (53) and (54), together with the definitions of I_i are referred to as formulation G1A-M. The formulation G1A-M can be seen as an extension of formulation G1A to a moving medium case as it explicitly takes into account the effects of constant uniform flow. Compared with formulation G1-M, the observer time derivatives of the integrals in formulation G1A-M are no longer needed, and thus it is an advantage that only the time-dependent input data of the flow field or at most, numerical differentiation of them are required. Moreover, the formulation G1A-M is more suitable to obtain the acoustic pressure gradient in cases where the observer is not stationary. However, it should be noted that a disadvantage of the suggested formulation G1A-M is its mathematical complexity, in spite of the fact that some new functions and key source time derivatives are defined to make the expression of G1A-M concise.

III. Numerical Simulations

In this section, numerical simulations of three test cases in a moving medium are presented to validate the time domain acoustic pressure gradient formulations developed in this paper. The first two test cases are the stationary monopole and dipole sources located in a moving medium with moving observers, while the third case consists of a rotating monopole with a moving observer for validating the corresponding moving source and moving observer case.

In the first two test cases, the stationary spherical surfaces are used as the data surfaces. The acoustic pressure gradient time history at the observer is evaluated and compared against the analytic solution. In the third test case, a moving spherical surface enclosing the monopole source is used as the data surface and the predicted acoustic pressure gradient time history at the observer is also compared against the analytic solution. Moreover, the efficiency of formulations G1-M and G1A-M is compared in all three test cases.

In order to avoid any error related to flow field simulation codes, all input flow field data on the data surface are obtained from the analytic solutions of the flow field generated by the sources. In this paper, the two order central difference algorithm is performed to obtain the results from the formulation G1-M.

A. Test Case 1: Monopole Source in a Moving Medium

The first test case is to consider a single-frequency monopole source located at the origin of a Cartesian coordinate system in a uniform flow with an arbitrary orientation.

The velocity potential for the monopole contains the uniform flow with an arbitrary direction is defined as [27]:

$$\varphi(\boldsymbol{x}, t) = \frac{A}{4\pi R^*} \exp[\, \mathrm{i}\omega(t - R / c_0)] \tag{70}$$

where the acoustic radii R^* and R have been defined in Sec. II.A. The acoustic particle velocity can be obtained from the gradient of the velocity potential

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla\varphi(\boldsymbol{x}, t) \tag{71}$$

The induced acoustic pressure and density in a uniformly moving flow with an arbitrary direction are given by the unsteady Bernoulli equation

$$p'(\boldsymbol{x}, t) = -\rho_0 (\frac{\partial\varphi(\boldsymbol{x}, t)}{\partial t} + U_{\infty i} \frac{\partial\varphi(\boldsymbol{x}, t)}{\partial x_i}) \tag{72}$$

and

$$\rho'(\boldsymbol{x}, t) = p'(\boldsymbol{x}, t) / c_0^2 \tag{73}$$

The acoustic pressure gradient in the x_i -direction is given by

$$\frac{\partial p'(\boldsymbol{x}, t)}{\partial x_i} = -\rho_0 (\mathrm{i}\omega \frac{\partial}{\partial x_i} + U_{\infty j} \frac{\partial^2}{\partial x_j \partial x_i}) \varphi(\boldsymbol{x}, t) \tag{74}$$

The analytic solutions of acoustic pressure gradient at an observer point can be obtained through Eq. (74).

In this test case, the velocity potential amplitude of the monopole is $A = 1$ m^2/s. The angular frequency is $\omega = 10\pi$ rad/s. The ambient speed of sound c_0 is chosen as 340 m/s. The free stream flow density ρ_0 is assumed to be 1.234 kg/m^3. Two different mean flow Mach numbers $\boldsymbol{M}_\infty$ = (0, 0, 0) and $\boldsymbol{M}_\infty$ = (0.6, 0.1, 0.5) are considered. The radius of the spherical data surface *S* is 1 m and there are 15292 triangular elements uniformly distributed on *S* for fine enough spatial resolution. There are 30 time points used per source period $(T = 2\pi / \omega)$ to ensure enough temporal resolution.

Figure 1 shows the predicted acoustic pressure gradient time history at the moving observer with different mean flow Mach numbers. The observer is moving along the x_1-axis at a constant velocity v_o = 30 m/s and its initial position is $\boldsymbol{x}$ = (20, 0, 0) m. It can be seen that the predicted results obtained from the formulations G1A-M and G1-M both accurately match the analytic solutions, and thus the accuracy of the proposed time domain formulations for the prediction of the acoustic pressure gradient is confirmed. When using a computer with an i7 CPU and 16 GB memory, the computation time for formulation G1-M is 3.255 s and that for formulation G1A-M is 1.402 s. The reason why the computation time of formulation G1-M is longer than that of formulation G1A-M is that the formulation G1-M needs to calculate several extra integrals at each time step when the observer is moving and the differentiation calculation is also time consuming.

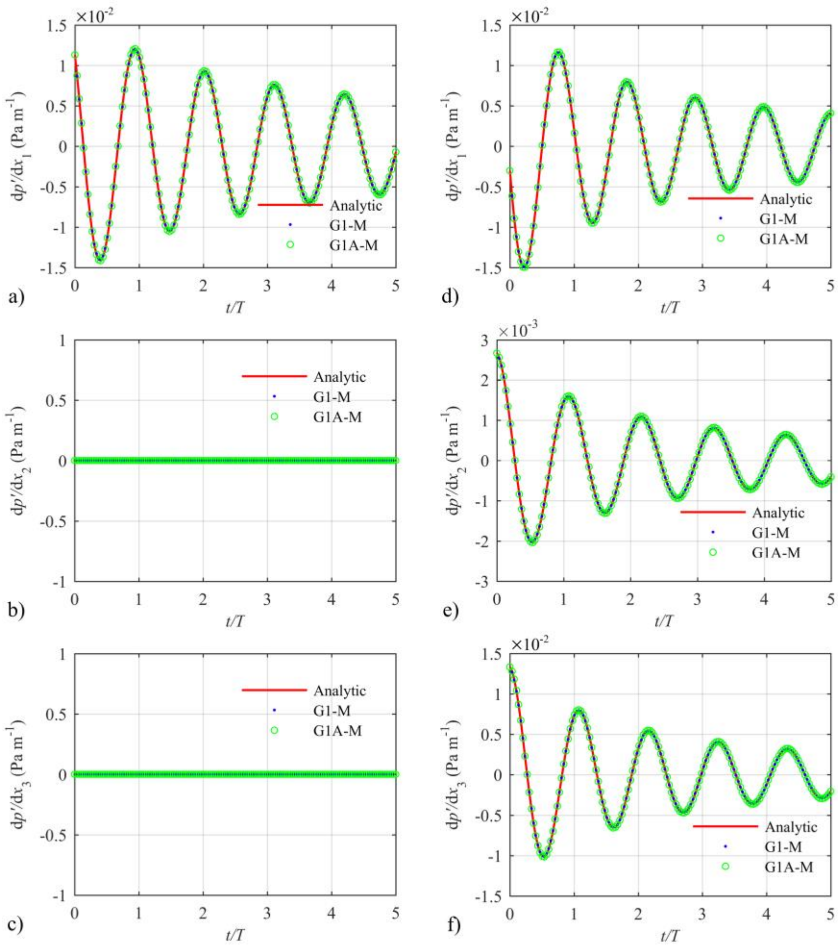

Fig. 1 (Color online) Comparison of the predicted stationary monopole acoustic pressure gradient time history with that of the analytic solution for different mean flow Mach numbers at the moving observer: a–c) $\boldsymbol{M}_\infty$ = (0, 0, 0); d–f) $\boldsymbol{M}_\infty$ = (0.6, 0.1, 0.5).

B. Test Case 2: Dipole Source in a Moving Medium

The second test case is a dipole source located at the origin of a Cartesian coordinate system in a uniform flow with an arbitrary orientation and the dipole axis is aligned with the x_2-axis. The velocity potential for such dipole can be obtained by

$$\varphi(\boldsymbol{x}, t) = \frac{\partial}{\partial x_2}\left\{\frac{A}{4\pi R^*}\exp[\mathrm{i}\omega(t - \frac{R}{c_0})]\right\} \tag{75}$$

The procedure for obtaining the analytic acoustic pressure gradient is similar to that in the monopole source case. The following relations may be used in the calculation:

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_1}) = -\frac{\frac{\partial \tilde{R}_2^*}{\partial x_i}\cdot \tilde{R}_1^* + \frac{\partial \tilde{R}_1^*}{\partial x_i}\cdot \tilde{R}_2^* + \frac{\partial \tilde{R}_2^*}{\partial x_1}\cdot \tilde{R}_i^*}{R^*} \tag{76}$$

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_2}) = -\frac{\frac{\partial \tilde{R}_2^*}{\partial x_i}\cdot 2\tilde{R}_2^* + \frac{\partial \tilde{R}_2^*}{\partial x_2}\cdot \tilde{R}_i^*}{R^*} \tag{77}$$

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_3}) = -\frac{\frac{\partial \tilde{R}_2^*}{\partial x_i}\cdot \tilde{R}_3^* + \frac{\partial \tilde{R}_3^*}{\partial x_i}\cdot \tilde{R}_2^* + \frac{\partial \tilde{R}_2^*}{\partial x_3}\cdot \tilde{R}_i^*}{R^*} \tag{78}$$

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2}{\partial x_1}) = \gamma^2\,\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_1}) \tag{79}$$

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2}{\partial x_2}) = \gamma^2\,\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_2}) \tag{80}$$

$$\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2}{\partial x_3}) = \gamma^2\,\frac{\partial}{\partial x_i}(\frac{\partial \tilde{R}_2^*}{\partial x_3}) \tag{81}$$

where the spatial derivatives on the right hand sides of Eqs. (76–78) have been defined in Sec. II.A.

In this test case, two different mean flow Mach numbers $\boldsymbol{M}_\infty$ = (0, 0, 0) and $\boldsymbol{M}_\infty$ = (0.8, 0.1, 0.4) are considered. The spherical data surface together with its mesh data used here are the same as those in the first test case and the parameters A, ω, c_0, ρ_0 as well as the time sampling points in one source period are also set to the same values as those in the first test case.

Figure 2 shows the predicted acoustic pressure gradient time history at the moving observer which is moving along the x_1-axis at a constant velocity v_o = 30 m/s and its initial position is $\boldsymbol{x}$ = (50, 50, 0) m. The excellent agreement further validates the reliability and accuracy of the proposed formulations G1-M and G1A-M for acoustic pressure gradient prediction in the time domain. Because of the aforementioned reasons, the computation time for formulation G1-M, 3.505 s, is longer than that for formulation G1A-M, 1.541 s.

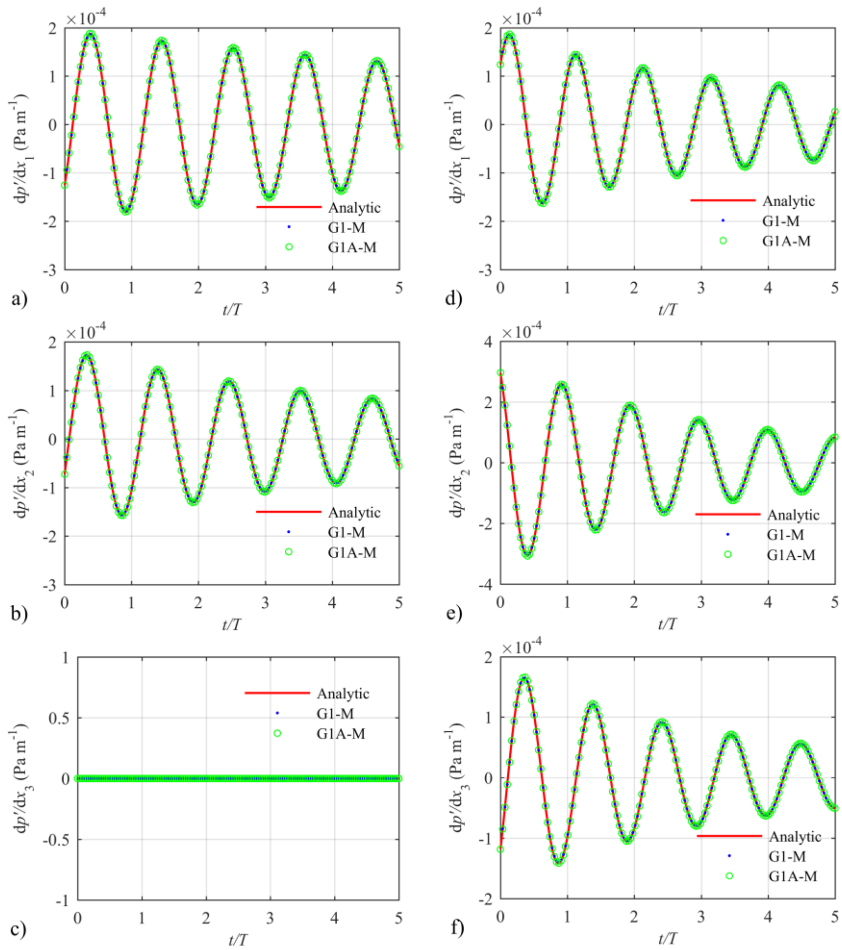

Fig. 2 (Color online) Comparison of the predicted stationary dipole acoustic pressure gradient time history with that of the analytic solution for different mean flow Mach numbers at the moving observer: a–c) $\boldsymbol{M}_\infty$ = (0, 0, 0); d–f) $\boldsymbol{M}_\infty$ = (0.8, 0.1, 0.4).

C. Test Case 3: Rotating Monopole Source in a Moving Medium

In order to show the feasibility and applicability of the proposed formulations for a moving source in a moving medium, a rotating monopole case is considered as shown in Fig. 3.

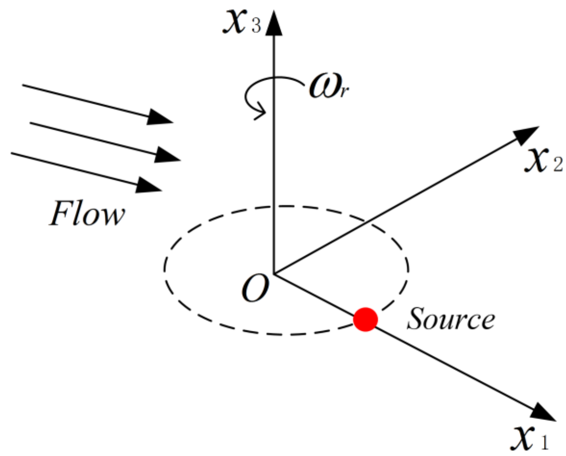

Fig. 3 (Color online) The schematic of a rotating monopole in a moving medium.

The monopole rotates counter-clockwise around the x_3-axis with an angular speed of $\omega_r = 2\pi$ rad/s at radius $rs = 1$ m in the $x_1 x_2$ plane and the initial position is $\boldsymbol{x}_s = (1, 0, 0)$ m when τ =0. The corresponding parameters A, ω, c_0, ρ_0 as well as the time sampling points in one source period are set the same values as those in the stationary monopole case. In this case, a spherical data surface, that is the same as the one used in the previous two test cases enclosing the monopole moving along with the source, is adopted to predict the acoustic pressure gradient time history at the observer.

The observer is moving along the x_1-axis at a constant velocity v_o = 30 m/s and its initial position is $\boldsymbol{x}_o$ = (5, 5, 0) m. In this test case there are two flow Mach numbers considered, corresponding to a medium at rest and a moving medium of $\boldsymbol{M}_\infty$ = (0.4, 0.3, 0.5). Figure 4 depicts the acoustic pressure gradient time history predicted by the formulations G1-M and G1A-M. The excellent agreement between the predicted results and the analytic solutions proves the capability of the derived formulations to predict accurately the acoustic pressure gradient in a moving medium. Here, the computation time for formulation G1-M is 9.410 s and that for formulation G1A-M is 3.947 s.

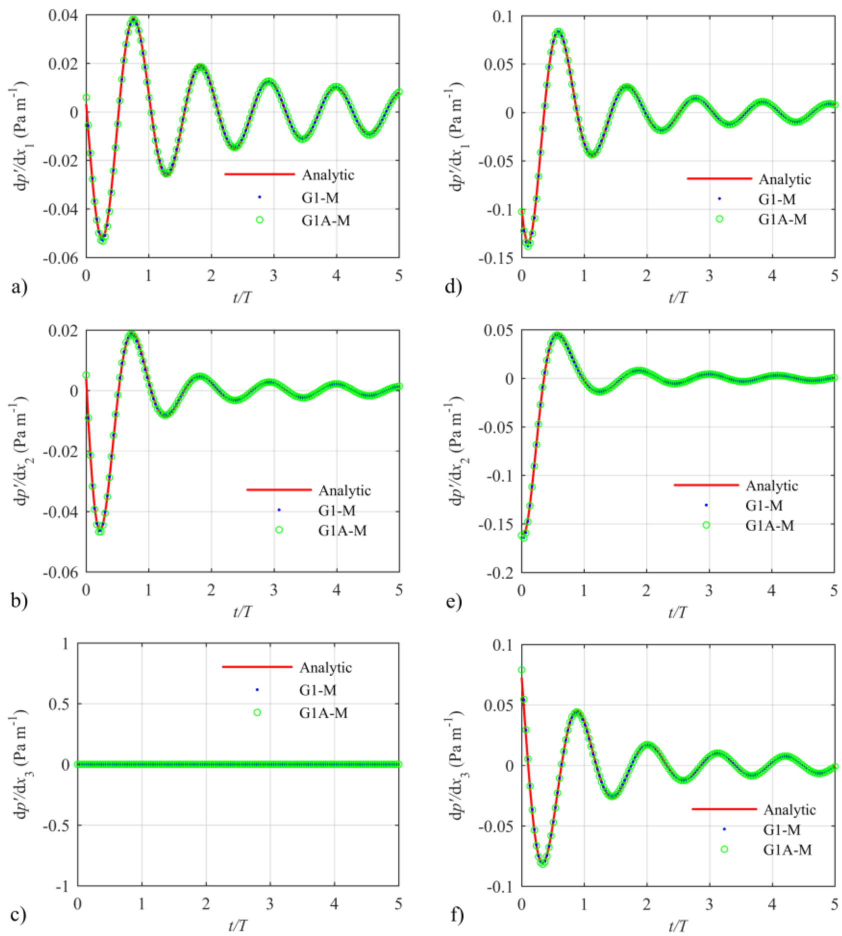

Fig. 4 (Color online) Comparison of the predicted rotating monopole acoustic pressure gradient time history with that of the analytic solution for different mean flow Mach numbers at the moving observer: a–c) M_∞ = (0, 0, 0); d–f) M_∞ = (0.4, 0.3, 0.5).

IV. Conclusions

In this paper, based on the convective FW–H equation, both a semi-analytic time domain formulation G1-M and an analytic time domain formulation G1A-M for the prediction of acoustic pressure gradient in a moving medium were derived. Although a moving medium, for example in a wind tunnel, can be equivalently solved in a stationary medium as well by using a moving observer, which was justified in Ref. [23], the formulations G1-M and G1A-M which explicitly take into account the presence of the uniform flow are more easily interpreted to examine the convective effects.

The validity and applicability of the derived formulations were verified through three computational test cases consisting of a stationary monopole source, a stationary dipole source as well as a rotating monopole source. Different flow configurations were considered to obtain the predicted acoustic pressure gradient data, and the agreement between the predicted results and analytic solutions was excellent. Meanwhile, the computational efficiencies of the formulations G1-M and G1A-M to deal with the moving observer cases were compared, and it has been found that the formulation G1A-M leads to a more efficient calculation than the formulation G1-M because it eliminates the observer time differentiation of the integrals.

The derived formulations G1-M and G1A-M explicitly take into account the presence of the moving medium, and thus can be used to predict the acoustic pressure gradient on the scattering surface which can serve as the boundary condition in the aeroacoustic scattering calculation. In future work the authors will consider aeroacoustic scattering phenomena in the time domain using these formulations.

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