AIAA JOURNAL
Vol. 43, No. 8, August 2005

Fine-Scale Turbulence Noise from Hot Jets

Christopher K. W. Tam* and Nikolai N. Pastouchenko†
Florida State University, Tallahassee, Florida 32306-4510
and
K. Viswanathan‡
The Boeing Company, Seattle, Washington 98124-2207

Experimental measurements indicate that the noise radiated from a jet depends not just on the jet-exit velocity alone, but is significantly affected by the jet temperature. Now, there is evidence to support the proposition that jet mixing noise consists of two principal components. These are the noise from the large turbulence structures of the jet flow and the fine-scale turbulence. The prediction of fine-scale turbulence noise from hot jets is considered. Earlier Tam and Auriault (Tam, C. K. W., and Auriault, L., "Jet Mixing Noise from Fine-Scale Turbulence," *AIAA Journal*, Vol. 37, No. 2, 1999, pp. 145–153) developed a semi-empirical theory capable of predicting the fine-scale turbulence noise from cold to moderate temperature jets. In this work, their semi-empirical theory is extended to high-temperature jets, up to a temperature ratio above that of present day commercial engines. The density gradient present in hot jets promotes the growth of Kelvin–Helmholtz instability in the jet mixing layer. This causes a higher level of turbulent mixing and stronger turbulence fluctuations. In addition, recent experiments reveal that the two-point space–time correlation function of turbulent mixing for hot jets is substantially different from that for cold jets. The eddy decay time is shorter, and the eddy size is slightly reduced. These changes have an appreciable impact on the noise radiated. In the present extended fine-scale turbulence theory, both effects are taken into account. Extensive comparisons between computed noise spectra and measurements for hot jets over the Mach-number range of 0.5–2.0 are reported here. Good agreements are found over inlet angle from 50 to 110 deg. This is the directivity for which fine-scale turbulence noise is dominant.

I. Introduction

JET engines are, invariably, operated at an elevated temperature. As yet, there is no theory or method, other than those entirely dependent on empirical correlations, to predict the noise intensity, directivity, and spectrum of hot jets. Experimental measurements[1,2] have clearly shown that the noise radiated by a turbulent jet is not completely determined by the jet velocity alone. The temperature ratio of a jet (jet total temperature to ambient temperature) also plays an important role. The objective of this investigation is to improve the prediction capability of the semi-empirical fine-scale turbulence noise theory of Tam and Auriault.[3] The Tam and Auriault theory was developed primarily for predicting the noise of cold to moderate temperature jets. The present effort extends its range of applicability to jets at temperature ratios higher than that of the present-day commercial jet engines.

Evidence is now available indicating that jet mixing noise consists of two independent components. One component is generated by the large turbulence structures of the jet flow. At high subsonic or supersonic Mach number, this is the stronger noise component. It radiates primarily in the downstream direction, generally in directions with inlet angle greater than 120 deg. The other component is generated by fine-scale turbulence. It has a fairly uniform directivity. It is the dominant noise component in the sideline and upstream direction. The existence of both large turbulence structures and fine-scale turbulence in jet flows is well established by experimental measurements, especially optical observations.[4,5]

Presented as Paper 2004-0362 at the AIAA 42nd Aerospace Sciences Meeting, Reno, NV, 5–8 January 2004; received 4 February 2004; revision received 1 December 2004; accepted for publication 17 April 2005.

*Robert O. Lawton Distinguished Professor, Department of Mathematics. Fellow AIAA.
†Postdoctoral Research Associate, Department of Mathematics.
‡Engineer, Aeroacoustics and Fluid Mechanics. Associate Fellow AIAA.

In the study of Tam et al.,[1] the noise spectrum of each of the two noise components was found to fit a seemingly universal similarity spectrum. They reasoned that because there was no intrinsic length and timescales in the mixing layer of a high-Reynolds-number jet (up to the end of the core region), not only the mean flow and turbulence statistics must exhibit self-similarity, the same might also be true for the radiated noise. By examining the entire data bank of the Jet Noise Laboratory of NASA Langley Research Center, they were able to identify two similarity noise spectra. One of the spectrum fitted all of the noise spectra radiated in the downstream direction regardless of jet Mach number and temperature. The strong directivity of this noise component is consistent with Mach wave radiation from the large turbulence structures/instability waves of the jet flow.[6,7] The other spectrum fitted all of the noise spectra radiated in the sideline and upstream directions. This is noise from the fine-scale turbulence. More recently, Tam[8] and Tam and Zaman[9] showed that even the noise spectra of nonaxisymmetric jets including jets from rectangular, elliptic, plug, and suppressor nozzles fitted the same two similarity spectra. Dahl and Papamoschou[10] reported that their coaxial jet noise spectra were good fits of the similarity spectra. Viswanathan[11,12] provided further data to confirm the two similarity spectra and the existence of two mixing noise components. He further used the similarity spectra to test the quality of hot jet noise spectra.

For supersonic jets, Laufer et al.[13] provided direct experimental evidence that there were two independent noise sources (see also Ref. 14). They used a spherical mirror to measure the axial noise source distribution of Mach 1.47, 1.97, and 2.47 jets. For the high-Mach-number jets, the measured data revealed that the noise source responsible for radiation in the 90-deg direction was distinctly different from the noise source responsible for radiation in the peak direction at approximately 135- to 150-deg inlet angle. The noise spectra radiated in the 90-deg direction had the same shape as the similarity spectra for the fine-scale turbulence noise. The noise spectra radiated in the peak noise direction had the same shape as the similarity spectrum for large turbulence structures noise.[14]

The existence of large turbulence structures and fine-scale turbulence in a jet is, by now, well established. However, the proposition that the large turbulence structures and fine-scale turbulence both radiate noise but with very different spectral and directional characteristics (in other words, jet mixing noise consists of two components) has not been accepted by the entire jet noise community. A number of investigators, for example, Morris and Farassat,[15] believe that jet noise is generated by quadrupoles, which implies that there is only one noise component. It is possible that their belief is restricted to subsonic jets, for otherwise it would be in conflict with the supersonic jet noise experiment of Laufer et al.[13] and Schlinker.[16] In the past, experimental efforts to identify the sources of jet noise have not been successful because of instrumentation difficulty. Recently, Panda et al.[17] succeeded in measuring directly the correlation of density and velocity fluctuations inside a jet and the radiated noise. They used a Rayleigh scattering technique to measure the fluctuations inside the jet (this avoids the problem of probe interference with the flow) and a microphone to measure the far-field sound. They found good correlation when the microphone was located in the downstream direction (150 deg from the jet inlet axis) but poor correlation at 90 deg. In addition, by decomposing the flow variables into a mean and a random fluctuating part, they found, at inlet angle 150 deg, good correlations between linear fluctuations in the jet flow and far-field sound or good correlation between large turbulence structures/instability waves in the jet flow and radiated sound such as $\langle \rho'_{\text{jet}}, p'_{\text{sound}} \rangle$ (primed variables are fluctuations from the mean value; angle brackets represent correlation). On the other hand, they found correlations of the form $\langle \bar{\rho} u' u'_{\text{jet}}, p'_{\text{sound}} \rangle$ or correlations between quadrupole terms and far-field sound to be poor. They interpreted their measurements of large correlations of linear fluctuations and the radiated sound field in the downstream direction to mean that there was strong noise radiation from the large turbulence structures/instability waves of the jet in this direction. As to the lack of correlation in the 90-deg direction, they interpreted this to mean that the noise source for 90-deg radiation was not organized spatially, consistent with the suggestion that the source of noise was randomly distributed small-scale turbulence. Thus their experiment offers support to the concept of two noise components. Further experimental investigation is underway. It is hoped that their continuing effort would provide a definitive resolution to the question of what really are the sources of jet noise.

In this work, only the prediction of the fine-scale turbulence noise from hot jets is considered. Prediction of the large turbulence structures noise is beyond the scope of the present investigation.

Recently, Tam and Auriault[3] developed a semi-empirical theory for predicting the fine-scale turbulence noise of high-speed jets. This theory is not based on acoustic analogy.[18,19] In formulating their theory, Tam and Auriault incorporated the k–ε turbulence model as a part of the theory. The k–ε model provides turbulence information such as the intensity of turbulence kinetic energy, the length scale of fine-scale turbulence, and decay time of turbulent eddies. They are essential to the noise source model.

Two effects that are crucial to jet noise prediction are incorporated in the Tam and Auriault theory. They are the source convection effect and the mean flow refraction effect. It is well known that a moving noise source emits more sound in the direction of motion. Because all of the noise sources move downstream, more noise is radiated in the downstream direction. Earlier Atvars et al.[20] and Grande[21] demonstrated experimentally that sound radiated from localized sources inside a jet underwent refraction as a result of the presence of mean flow velocity and density gradients. For high-speed jets, especially those at a high temperature, mean flow refraction effectively prevents sound from radiating in the downstream directions close to the jet axis. As a result, there is little fine-scale turbulence noise within a cone enclosing the downstream jet axis. This is often referred to as the cone of silence. Theoretically, the mean flow refraction effect can be adequately taken into account through the use of the linearized Euler equation. To incorporate the mean flow refraction effect into their prediction scheme, Tam and Auriault[3] used an adjoint Green's function of the linearized Euler equations. The adjoint Green's function formulation has proven to be very efficient computationally.

Tam and Auriault[3] demonstrated by comparing with Seiner's NASA data[1] that their theory can predict cold and moderately hot supersonic jet noise spectra well. At a subsonic Mach number, their predictions are in good agreement with the measurements of Norum and Brown[22] and others. In a recent work on jets in forward flight, Tam et al.[23] applied the theory to jets in simulated forward flight. Their computed spectra compared well with the experimental data of Norum and Brown as well as earlier data of Plumblee[24] (Lockheed open wind-tunnel data) over an extended range of flight Mach numbers. Recently, Tam and Pastouchenko[25] extended the original theory to noncircular jets. They found good agreement between theoretical predictions and experimental measurements. The cases studied included a Mach 2 elliptic jet of aspect ratio 3, a Mach 2 rectangular jet of aspect ratio 7.6, an elliptic and rectangular jet of aspect ratio 3 at Mach 0.82, and a Mach 0.82 rectangular jet of aspect ratio 8. The noise fields of large-aspect-ratio rectangular jets, be they supersonic or subsonic, are quite complex. To be able to predict the noise spectra correctly, a theory must have the essential physics.

It turns out the Tam and Auriault theory can be used to calculate the noise source distribution inside a jet. In a recent work, Tam et al.[14] compared the predicted noise source distribution of Mach 1.47, 1.97, and 2.47 jets with the measurements of Laufer et al.[13] and Schlinker.[16] The comparisons included noise source distribution at selected Strouhal numbers as well as noise intensity (integrated over all frequencies). Good agreements were found providing further support for the usefulness of the theory.

In this paper, the principal differences between the mean flow and turbulence noise source characteristics of hot and cold jets will first be examined. They are then incorporated into the Tam and Auriault theory by making appropriate modifications. A noise spectrum formula applicable to both hot and cold jets is derived. The results of extensive comparisons between predictions and experimentally measured noise spectra will be reported. Good agreements are found, indicating that the extended theory is useful for jet noise prediction at elevated jet temperature.

II. Temperature Effect on Jet Mean Flow and Noise Source

When a jet is hot, a large density gradient exists in the jet flow. This density gradient exerts a strong influence on the mean flow and turbulent mixing noise of the jet. The effect of a large density gradient on the mean flow of a jet has recently been studied by Tam and Ganesan.[26] The effect on turbulent mixing noise is the subject of this investigation.

Tam and Ganesan[26] began their study by analyzing the effect of density gradient on the stability of a shear layer. By using a simple vortex sheet shear-layer model, they found that the density difference between a jet and the ambient air tended to increase the spatial growth rate of the intrinsic Kelvin–Helmholtz instability when the lighter fluid was in motion. This means that there would be an enhancement of turbulent mixing when a jet is hot. It follows that there is an increase in the jet spreading rate. To mimic this effect, Tam and Ganesan proposed to modify the k–ε turbulence model by adding a density gradient dependent term to the eddy viscosity. Dimensional reasoning was then invoked to determine the correct form of the added viscosity term. The addition of a density gradient term introduced a new empirical constant. This constant was determined by best fit to the data. Extensive comparisons between computed and measured axial velocity profiles as a function of radial distance were carried out and reported. Good agreements were found for high-temperature jets at both subsonic and supersonic Mach number.

In addition to bringing about a density gradient effect, an increase in jet temperature also affects the convective Mach number. It is known that an increase in convective Mach number tends to stabilize turbulent mixing. This effect has a tendency to counterbalance the density gradient effect. In this work, the Sarkar convective Mach-number correction[27] to the k–ε model is included in the mean flow calculation.

In the present study, the modified k–ε turbulence model will be used for mean flow and turbulence calculation. By using the results of the modified k–ε model, the change in turbulence intensity,

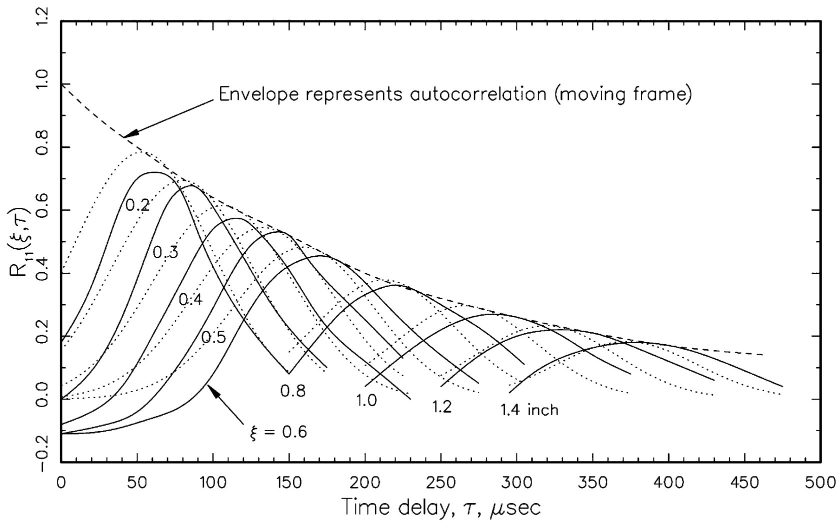

Fig. 1 Two-point space–time correlation of the axial velocity component in a jet: ——, measured data[27] and · · · ·, model function.

length, and decay timescales caused by density gradient are taken into account. However, for noise prediction purposes there is also a change in the two-point space–time noise source correlation function. In other words, there is direct density effect on the noise generation process. This must be incorporated into the theory in order to be able to predict correctly the noise from the fine-scale turbulence of hot jets.

In the Tam and Auriault fine-scale turbulence noise theory,[3] the functional form of the two-point space–time correlation function

$$\left\langle \frac{Dq_s(\boldsymbol{x}_1, t_1)}{Dt_1} \frac{Dq_s(\boldsymbol{x}_2, t_2)}{Dt_2} \right\rangle$$

was modeled after the measurements of Davies et al.[28] Figure 1 shows the measured axial velocity space-time correlation function. One important characteristic is that all of the two-point time correlation curves lie below an overall envelope that decays exponentially with an increase in spatial separation. The model function used by Tam and Auriault is

$$\left\langle \frac{Dq_s(\boldsymbol{x}_1, t_1)}{Dt_1} \frac{Dq_s(\boldsymbol{x}_2, t_2)}{Dt_2} \right\rangle = \frac{\hat{q}_s^2}{c^2\tau_s^2} \exp\left\{ -\frac{|\xi|}{\bar{u}\tau_s} - \frac{\ell n 2}{\ell_s^2}[(\xi - \bar{u}\tau)^2 + \eta^2 + \zeta^2] \right\} \tag{1}$$

where $\xi = x_1 - x_2$, $\eta = y_1 - y_2$, $\zeta = z_1 - z_2$, $\tau = t_1 - t_2$; $\bar{u}$ is the mean flow velocity; c is a constant; $\hat{q}_s^2$, ℓ_s, and τ_s are the three parameters of the model; $\hat{q}_s^2$ is the kinetic energy of fine-scale turbulence per unit volume; ℓ_s is the characteristic size; and τ_s is the characteristic decay time of the fine-scale turbulence. The model parameters are given by the k–ε turbulence model as follows:

$$\hat{q}_s^2 \big/ c^2 = A^2 q^2 \qquad \left(q = \tfrac{2}{3}\rho k\right) \tag{2}$$

$$\ell_s = c_\ell (k^{\frac{3}{2}}/\varepsilon), \qquad \tau_s = c_\tau (k/\varepsilon) \tag{3}$$

The three constants A, c_ℓ, and c_τ were determined empirically. They were assigned the values

$$A = 0.755, \qquad c_\ell = 0.256, \qquad c_\tau = 0.233 \tag{4}$$

In Fig. 1, the dotted curves are the two-point space–time model function (1) with $\ell_s = 0.1758$ in. (0.447 cm), $\tau_s = 447.4$ μs, and $\bar{u} = 3.515 \times 10^{-3}$ in./μs (8.928 cm/μs). As can be seen, the agreement is quite good.

Recently, Doty and McLaughlin[29] measured two-point space–time correlations of density gradient fluctuations of high-speed jets by using optical deflectometry. For cold jets, their measurements were in excellent agreement with the data of Davies et al.[28] However, they found significant differences for simulated hot jets. Their simulated hot jets used a mixture of air and helium. Figure 2 shows their measured two-point correlations at various probe separation distances for a Mach 0.9 jet at jet temperature ratio (T_r/T_a) of 2.0. As can be readily seen, unlike the case of cold jets, the two-point time correlation curves decay rapidly with increase in time and space separations. The rate of decay is so rapid that no overall envelope is possible. Physically, this suggests a rapid decay of turbulent eddies, a process that would greatly affect noise radiation. Figure 3 shows the two-point correlations for a Mach 1.47 jet at a temperature ratio of 1.79 measured by Doty and McLaughlin. This is qualitatively very similar to Fig. 2. Figures 2 and 3 strongly suggest a need to modify the two-point space–time correlation function of the Tam and Auriault theory to account for the change in fine-scale turbulence noise generation process caused by the presence of strong density gradient.

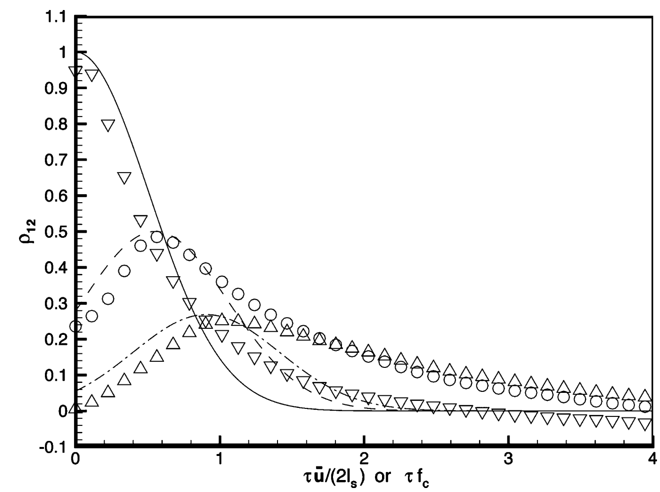

Fig. 2 Space–time correlation of various probe separation distances for a Mach 0.9 jet at temperature ratio (T_r/T_a) = 2.0: for $\xi/u\tau_s$ = 0.0: ∇, experiment and ——, model function (ν = 1); for $\xi/u\tau_s$ = 1.25, ξ/l_s = 0.92: ○, experiment and – – –, model function; and for $\xi/u\tau_s$ = 2.05, ξ/l_s = 1.51: △, experiment and –·–·–, model function.

III. Noise Spectrum for Hot Jets

We will now consider the following more general two-point space–time correlation function as a source model for fine-scale

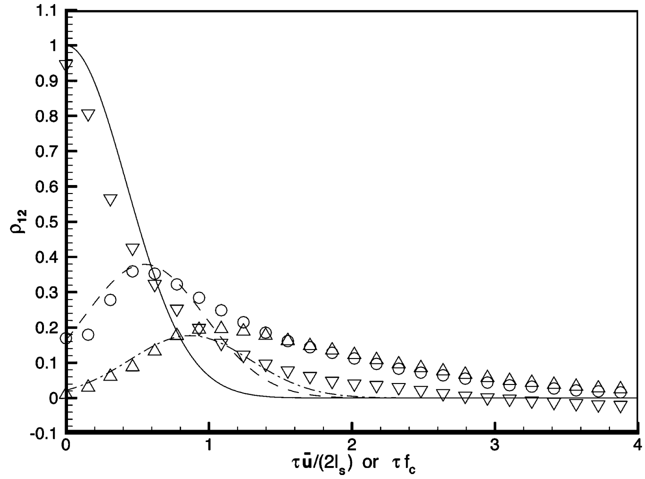

Fig. 3 Space–time correlation of various probe separation distances for a Mach 1.47 jet at temperature ratio (T_r/T_a) = 1.79: for $\xi/u\tau_s = 0.0$: ∇, experiment and ——, model function ($\nu = 1$); for $\xi/u\tau_s = 1.62$, $\xi/l_s = 1.1$: ○, experiment and – – –, model function; and for $\xi/u\tau_s = 2.55$, $\xi/l_s = 1.735$: △, experiment and –·–·–, model function.

turbulence noise of high-speed jets:

$$\left\langle \frac{Dq_s(\boldsymbol{x}_1,t_1)}{Dt_1}\frac{Dq_s(\boldsymbol{x}_2,t_2)}{Dt_2}\right\rangle = \frac{\hat{q}_s^2}{c^2\tau_s^2}\frac{2}{\Gamma(\nu)}\left(\frac{|\xi|}{2\bar{u}\tau_s}\right)^{\nu} K_\nu\left(\frac{|\xi|}{\bar{u}\tau_s}\right) \times \exp\left\{-\frac{\ell n 2}{\ell_s^2}[(\xi-\bar{u}\tau)^2+\eta^2+\zeta^2]\right\} \tag{5}$$

where $K_\nu(z)$ is the modified Bessel function of order ν. The decay rate of the function $z^\nu K_\nu(z)$ is affected by the choice of ν. $\Gamma(\nu)$ is the gamma function, and ν is an additional parameter of the new correlation function. It provides the function an extra degree of freedom to mimic the decay of hot jet correlation function. In the special case of $\nu = \frac{1}{2}$, we note that

$$(2|\xi|/\pi\bar{u}\tau_s)^{\frac{1}{2}} K_{\frac{1}{2}}(|\xi|/\bar{u}\tau_s) = e^{-|\xi|/\bar{u}\tau_s} \tag{6}$$

so that Eq. (5) reduces to Eq. (1), the original two-point space–time correlation function used by Tam and Auriault.[3] In other words, Eq. (5) is a generalization of Eq. (1) with added capability to behave like the two-point space–time correlation function of hot jets.

To test the suitability of using Eq. (5) as hot jet noise source function, we will first check whether this function with properly chosen values for its parameters can fit the measured data of Doty and McLaughlin[29] well. The three curves in Figs. 2 and 3 are the correlation curves of the generalized model function (5). The values of the parameters are given in the legends of the figures. In both cases there are satisfactory agreement if the separation time is not too large. For large time separation the correlation is small. This part of the correlation function would, therefore, contribute little to the radiated noise of a jet; hence, a good fit is not critical.

Now according to the Tam and Auriault theory,[3] the noise spectrum at a far-field point $\boldsymbol{x}$ is given by

$$S(\boldsymbol{x},\omega) = \iint \overset{\infty}{\underset{-\infty}{\cdots}} \iint p_a(\boldsymbol{x}_1,\boldsymbol{x},\omega_1)p_a(\boldsymbol{x}_2,\boldsymbol{x},\omega_2)\left\langle\frac{Dq_s}{Dt_1}\frac{Dq_s}{Dt_2}\right\rangle \times \exp[-i(\omega_1+\omega_2)t + i\omega_1 t_1 + i\omega_2 t_2] \times \delta(\omega-\omega_2)\,d\omega_1\,d\omega_2\,dt_1\,dt_2\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \tag{7}$$

where $p_a(\boldsymbol{x}_1,\boldsymbol{x},\omega)$ is the adjoint Green function with source point at $\boldsymbol{x}$ and observation point at $\boldsymbol{x}_1$. By substitution of Eq. (5) into Eq. (7), it is found that

$$S(\boldsymbol{x},\omega) = \iint \overset{\infty}{\underset{-\infty}{\cdots}} \iint p_a(\boldsymbol{x}_1,\boldsymbol{x},\omega_1)p_a(\boldsymbol{x}_2,\boldsymbol{x},\omega_2)\frac{\hat{q}_s^2}{c^2\tau_s^2}\frac{2}{\Gamma(\nu)} \times \left(\frac{|x_1-x_2|}{2\bar{u}\tau_s}\right)^{\nu} K_\nu\left(\frac{|x_1-x_2|}{\bar{u}\tau_s}\right)\exp\left\{-\frac{\ell n 2}{\ell_s^2}\left[(x_1-x_2 - \bar{u}(t_1-t_2))^2+(y_1-y_2)^2+(z_1-z_2)^2\right]\right\} \times \exp[-i(\omega_1+\omega_2)t+i\omega_1 t_1+i\omega_2 t_2] \times \delta(\omega-\omega_2)\,d\omega_1\,d\omega_2\,dt_1\,dt_2\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \tag{8}$$

It turns out that all of the integrals except the last volume integral over the jet flow can be integrated in closed form. The following steps will facilitate the integration process.

Let us first integrate over t_1. To do so, we will make a change of variable to s defined by

$$s = t_1 - t_2 - (x_1 - x_2)/\bar{u}$$

On separating out the s integral, we find

$$\int_{-\infty}^{\infty}\exp\left[-\frac{(\ell n 2)\bar{u}^2}{\ell_s^2}s^2 + i\omega_1 s\right]ds = \left(\frac{\pi}{\ell n 2}\right)^{\frac{1}{2}}\frac{\ell_s}{\bar{u}}\exp\left[-\frac{\omega_1^2\ell_s^2}{\bar{u}^2 4(\ell n 2)}\right]$$

Next, the t_2 integration can be carried out to yield

$$\int_{-\infty}^{\infty}\exp[i(\omega_1+\omega_2)t_2]\,dt_2 = 2\pi\delta(\omega_1+\omega_2)$$

Because of the presence of the δ functions, the ω_1 and ω_2 integrals can readily be evaluated to give

$$S(\boldsymbol{x},\omega) = \iint \overset{\infty}{\underset{-\infty}{\cdots}} \int \frac{\hat{q}_s^2}{c^2\tau_s^2}\frac{4\pi}{\Gamma(\nu)}\left(\frac{\pi}{\ell n 2}\right)^{\frac{1}{2}}\frac{\ell_s}{2^\nu \bar{u}} \times \exp\left[-\frac{\omega^2\ell_s^2}{\bar{u}^2 4(\ell n 2)}\right]\left(\frac{|x_1-x_2|}{\bar{u}\tau_s}\right)^{\nu}K_\nu\left(\frac{|x_1-x_2|}{\bar{u}\tau_s}\right) \times \exp\left[-\frac{i\omega(x_1-x_2)}{\bar{u}}\right]p_a(\boldsymbol{x}_1,\boldsymbol{x},-\omega)p_a(\boldsymbol{x}_2,\boldsymbol{x},\omega) \times \exp\left\{-\frac{\ell n 2}{\ell_s^2}\left[(y_1-y_2)^2+(z_1-z_2)^2\right]\right\}d\boldsymbol{x}_1\,d\boldsymbol{x}_2 \tag{9}$$

At this point, we invoke the phase factor approximation proposed and justified by Tam and Auriault:

$$p_a(\boldsymbol{x}_1,\boldsymbol{x},-\omega) \cong p_a(\boldsymbol{x}_2,\boldsymbol{x},-\omega)\exp[i(\omega/a_\infty)(x_1-x_2)\cos\Theta] \tag{10}$$

where Θ is the polar angle of $\boldsymbol{x}$ in a spherical coordinate system (R,Θ,ϕ) centered at the nozzle exit with the polar axis pointing in the direction of jet flow. In other words Θ is the exhaust angle. On substituting Eq. (10) into Eq. (9), the $d\boldsymbol{x}_1 = dx_1\,dy_1\,dz_1$ integrals become separated. The dy_1 and dz_1 integrals are standard Gaussian integrals. The dx_1 integral can be evaluated by first casting it into a cosine transform and using a formula given in Ref. 30:

$$\int_{-\infty}^{\infty}\left(\frac{|x_1-x_2|}{\bar{u}\tau_s}\right)^{\nu}K_\nu\left(\frac{|x_1-x_2|}{\bar{u}\tau_s}\right) \times \exp\left[-i\frac{\omega}{\bar{u}}\left(1-\frac{\bar{u}}{a_\infty}\cos\Theta\right)(x_1-x_2)\right]dx_1$$
$$= 2\bar{u}\tau_s\int_0^\infty R_e\left\{\eta^\nu K_\nu(\eta)\exp\left[-i\omega\tau_s\left(1-\frac{\bar{u}}{a_\infty}\cos\Theta\right)\eta\right]\right\}d\eta$$
$$= \frac{\pi^{\frac{1}{2}}2^\nu\bar{u}\tau_s\Gamma\left(\nu+\frac{1}{2}\right)}{\left\{1+\omega^2\tau_s^2[1-(\bar{u}/a_\infty)\cos\Theta]^2\right\}^{\nu+\frac{1}{2}}} \tag{11}$$

Upon integrating over $\mathrm{d}\boldsymbol{x}_1$, the radiated noise spectrum is given by

$$S(\boldsymbol{x},\omega)=\frac{4\pi^3}{(\ell n2)^{\frac{3}{2}}}\iiint\limits_{V_{\text{jet}}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)}\left(\frac{\hat{q}_s^2}{c^2}\right)\frac{\ell_s^3}{\tau_s}\times\frac{|p_a(\boldsymbol{x}_2,\boldsymbol{x},\omega)|^2\exp\left[-\omega^2\ell_s^2/\bar{u}^2 4(\ell n2)\right]}{\left\{1+\omega^2\tau_s^2[1-(\bar{u}/a_\infty)\cos\Theta]^2\right\}^{\nu+\frac{1}{2}}}\,\mathrm{d}\boldsymbol{x}_2 \qquad (12)$$

where V_{jet} is the volume of the jet plume. The spectral density $S(\boldsymbol{x},\omega)$ is per unit angular frequency ω. On converting to decibel per Strouhal number (fD_j/u_j) based on fully expanded jet velocity u_j and diameter D_j the spectral density of the sound field at (R,Θ,ϕ) is

$$S\left(R,\Theta,\phi,\frac{fD_j}{u_j}\right)=10\,\log\left[\frac{4\pi S(\boldsymbol{x},\omega)}{p_{\text{ref}}^2(D_j/u_j)}\right] \qquad (13)$$

where p_{ref} is the reference pressure of the decibel scale. For $\nu=\frac{1}{2}$, Eqs. (12) and (13) reduce to the spectral density formulas of Tam and Auriault.

Now the noise spectrum formula (12) contains four parameters, namely, $\hat{q}_s^2/c^2$, ℓ_s, τ_s, and ν. For cold- to moderate-temperature jets, explicit dependence of these parameters on the k–ε turbulence model is given in Tam and Auriault. For hot axisymmetric jets, the density gradient parameter is $(1/\rho)(\mathrm{d}\rho/\mathrm{d}r)$. We will assume that the density parameter is small so that its first-order effect can be represented by a perturbation term. That is, it can be represented by an additional term that is linear in $(1/\rho)(\mathrm{d}\rho/\mathrm{d}r)$. On balancing the dimensions using k and ε (k and ε are the only two parameters available in the k–ε model for dimensional adjustment), the following formulas can be easily established:

$$\nu=\frac{1}{2}+\begin{cases}c_\eta\dfrac{k^{\frac{3}{2}}}{\varepsilon}\dfrac{1}{\rho}\left|\dfrac{\mathrm{d}\rho}{\mathrm{d}r}\right| & (14a)\\ 0 & (14b)\end{cases}$$

$$\frac{\hat{q}_s^2}{c^2}=A^2q^2+\begin{cases}B\dfrac{k^{\frac{3}{2}}}{\varepsilon}\dfrac{1}{\rho}\left|\dfrac{\mathrm{d}\rho}{\mathrm{d}r}\right|q^2 & (15a)\\ 0 & (15b)\end{cases}$$

$$\ell_s=c_\ell\frac{k^{\frac{3}{2}}}{\varepsilon}+\begin{cases}c_{\ell\rho}\dfrac{k^3}{\varepsilon^2}\dfrac{1}{\rho}\left|\dfrac{\mathrm{d}\rho}{\mathrm{d}r}\right| & (16a)\\ 0 & (16b)\end{cases}$$

$$\tau_s=c_\tau\frac{k}{\varepsilon}+\begin{cases}c_{\tau\rho}\dfrac{k^{\frac{5}{2}}}{\varepsilon^2}\dfrac{1}{\rho}\left|\dfrac{\mathrm{d}\rho}{\mathrm{d}r}\right| & (17a)\\ 0 & (17b)\end{cases}$$

In Eqs. (14–17) the (a) formulas are to be used when $\mathrm{d}\bar{u}/\mathrm{d}r$ and $\mathrm{d}\rho/\mathrm{d}r$ have opposite signs. The (b) formulas are to be used when they have the same sign. The reason for this is because of the effect of density difference on the Kelvin–Helmholtz instability in the shear layer of the jet. Tam and Ganesan[26] have pointed out that enhanced mixing arising from density effect occurs only when the lighter fluid is moving.

For nonaxisymmetric jets, Eqs. (14–17) can be generalized by replacing $|\mathrm{d}\rho/\mathrm{d}r|$ by $|(\nabla\bar{u})\cdot(\nabla\rho)|/|\nabla\bar{u}|$ and that density gradient correction is incorporated only if $(\nabla\bar{u})\cdot(\nabla\rho)$ is negative. Here $\nabla\bar{u}$ is taken as the reference direction as far as density gradient effect is concerned.

Equations (14–17) contained four unknown constants, c_η, B, $c_{\ell\rho}$, and $c_{\tau\rho}$. They are to be determined empirically.

IV. Comparisons with Experiments

In this section, comparisons between the calculated noise spectra based on formulas (12) and (13) and experiments are reported. At the present time, hot jet noise data are not as readily available as they should be. Until the recent work of Viswanathan,[12] there is an absence of high-quality hot jet noise data at a high subsonic Mach number in the literature. In addition to comparing with Viswanathan's measurements, the supersonic hot jet noise data of Seiner (see Ref. 1) will also be used.

Before noise spectrum formulas (12) and (13) can be used, values of the four hot jet noise constants, that is, c_η, B, $c_{\ell\rho}$, and $c_{\tau\rho}$ have to be first determined. In this investigation, these four constants were determined empirically by best fit to the noise data. For this purpose, an iterative improvement procedure was used. First, a sample data set was selected. The mean flow of the jets selected as well as the k and ε distributions were then computed using the mean flow code of Tam and Ganesan.[26] This code incorporated their k–ε turbulence model for hot jets. The next step was to compute the adjoint Green's function using the code of Tam and Auriault.[3] The code computed $|p_a|^2$. Once the computation of the adjoint Green's function was completed, formulas (12) and (13) were used to calculate the radiated noise spectra at a preassigned set of directions. We defined the prediction error for a specific jet Mach number, jet temperature ratio, and direction of radiation as the sum of the squares of the differences between the calculated and measured noise spectrum at a number of chosen Strouhal numbers. The total error was easily calculated by summing over all cases of test data. The values of the four constants were changed incrementally by Newton's iteration method until the total error was minimized. By implementing the preceding procedure, we found the best values of the constants are

$$c_\eta=2.1599,\qquad B=0.806$$

$$c_{\ell\rho}=-0.026,\qquad c_{\tau\rho}=-0.2527$$

Figure 4a shows comparisons between calculated noise spectra and experimental measurements of Viswanathan.[12] The jet Mach numbers are 1.0, 0.8, and 0.6. The jet temperature ratio is 1.8. The direction of radiation (inlet angle) is 50 deg. At this point, a brief explanation about the spectral levels as measured in the experiment is in order. Fine narrowband data, with a bandwidth of 23.4 Hz, were acquired. However, the cutoff frequency of the anechoic chamber is ~200 Hz. The spectral levels in the first seven or eight bands are artificially rolled down because accurate measurements are not possible at these low frequencies. This filtering of the data causes the levels to drop abruptly in the plots shown; this drop should be ignored as this trend is artificially created and does not represent the true spectral behavior at these low frequencies. Figure 4b is the same as Fig. 4a except that it is for jet Mach number 0.9, 0.7, and 0.5. Figures 4c–4h show similar comparisons at inlet angles 70, 90, and 110 deg, respectively. The fine-scale turbulence noise is expected to be the dominant noise component in these directions. Figures 5a–5h show comparisons at a jet temperature ratio of 3.2. We believe the comparisons provided in these two sets of figures, covering the entire high subsonic jet Mach numbers and temperature ratios up to values higher than that operated by present-day commercial jet engines, constitute a vigorous test of the accuracy of the prediction theory. On considering that there are large variations in sound pressure level and spectral shape with jet-exit velocity and temperature ratio over the test range, it is fair to conclude, based on a detailed examination of all of the comparisons, that there is good overall agreement between theory and experiment.

Figures 6–8 are for Mach 2 supersonic jets at three temperature ratios. They are $Tr/Ta=3.28$, 4.08, and 4.89. These are very high-temperature jets. Figure 6 shows comparisons between calculated and measured noise spectra at 83-deg inlet angle. The data are taken from the measurements of Seiner reported in Ref. 1. Figure 7 shows similar comparisons at an inlet angle of 93 deg. Figure 8 shows comparisons at an inlet angle of 107 deg. By and large, there appear to be favorable agreements between computed and measured noise spectra. Based on this observation, we would like to conclude that

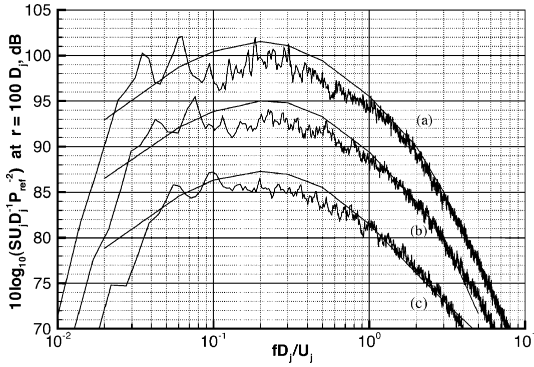

Fig. 4a Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 50 deg: a) $M_j = 1.0$, b) $M_j = 0.8$, and c) $M_j = 0.6$.

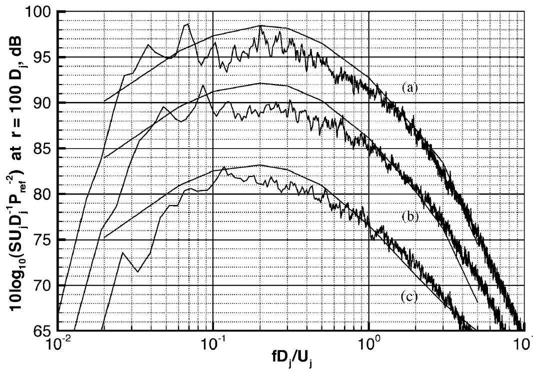

Fig. 4b Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 50 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

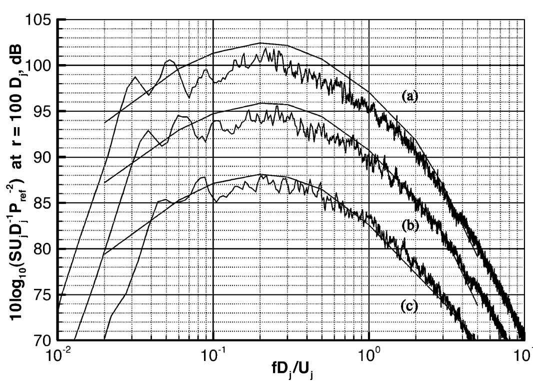

Fig. 4c Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 70 deg: a) $M_j = 1.0$, b) $M_j = 0.8$, and c) $M_j = 0.6$.

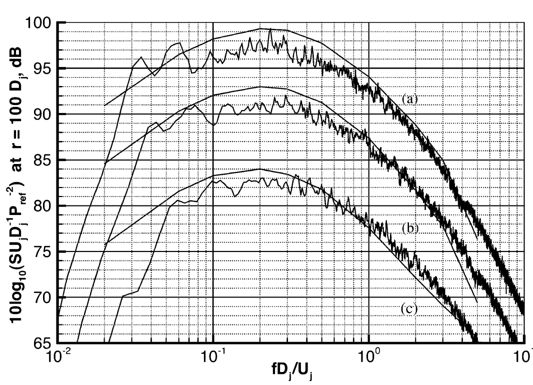

Fig. 4d Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 70 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

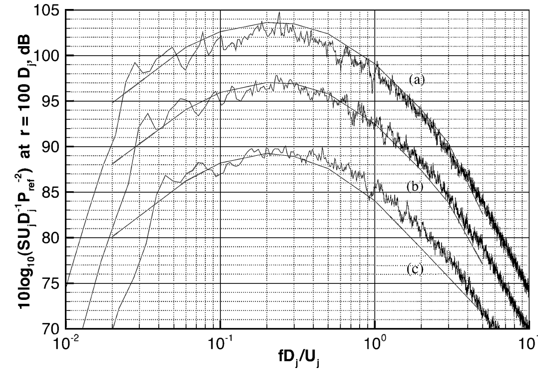

Fig. 4e Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 90 deg: a) $M_j = 1.0$, b) $M_j = 0.8$, and c) $M_j = 0.6$.

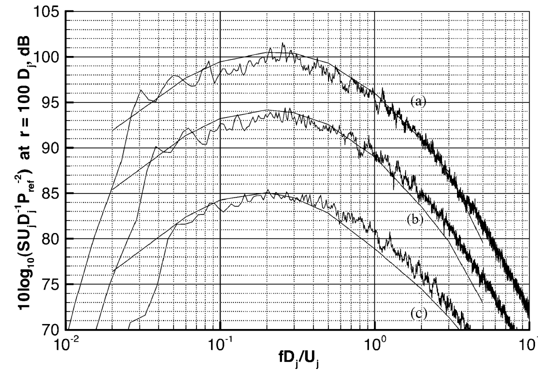

Fig. 4f Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 90 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

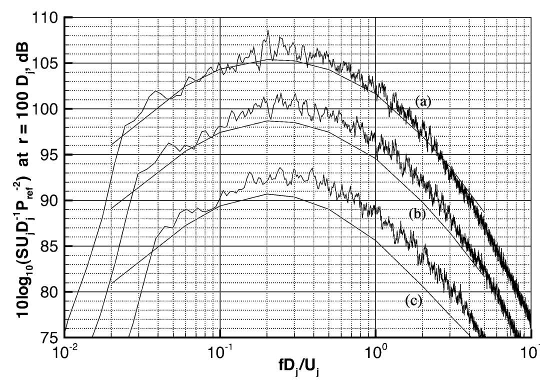

Fig. 4g Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 110 deg: a) $M_j = 1.0$, b) $M_j = 0.8$, and c) $M_j = 0.6$.

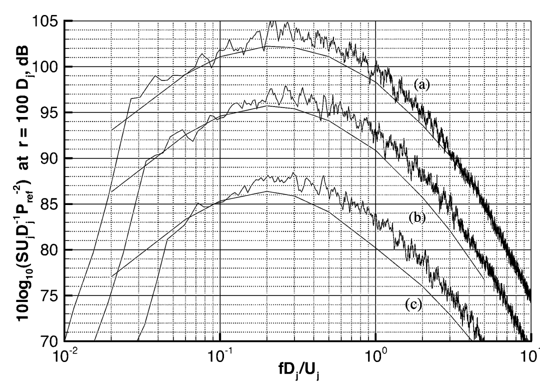

Fig. 4h Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 1.8$ and inlet angle = 110 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

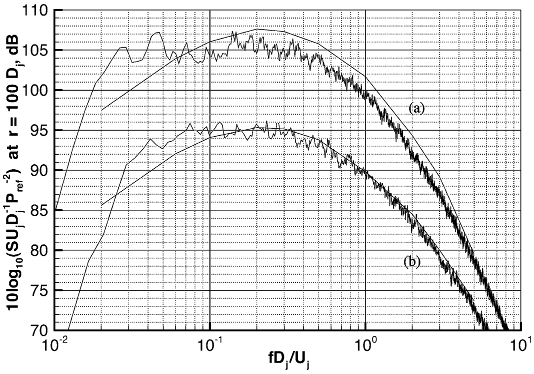

Fig. 5a Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 50 deg: a) $M_j = 1.0$ and b) $M_j = 0.6$.

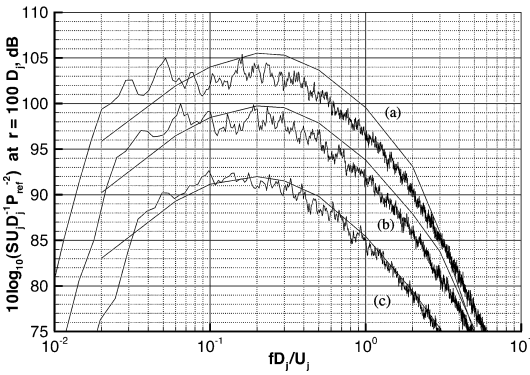

Fig. 5b Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 50 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

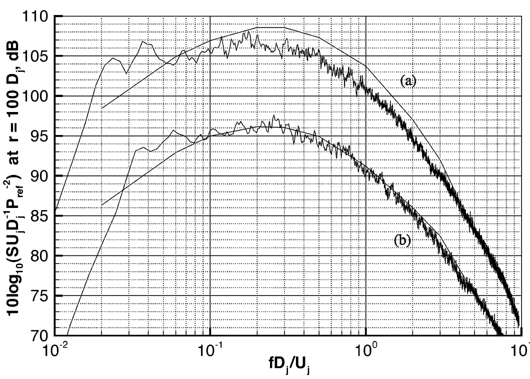

Fig. 5c Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 70 deg: a) $M_j = 1.0$ and b) $M_j = 0.6$.

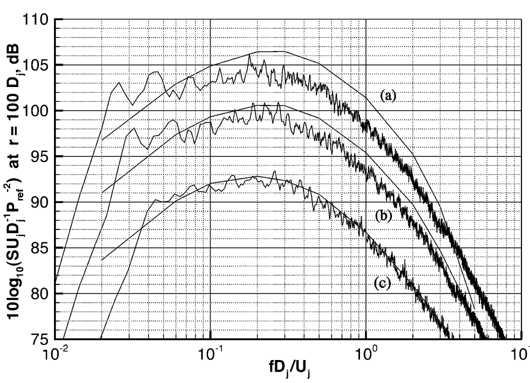

Fig. 5d Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 70 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

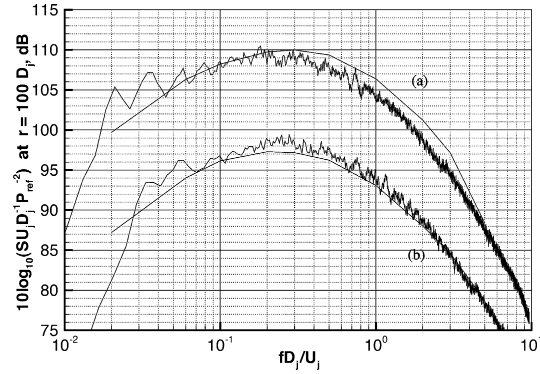

Fig. 5e Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 90 deg: a) $M_j = 1.0$ and b) $M_j = 0.6$.

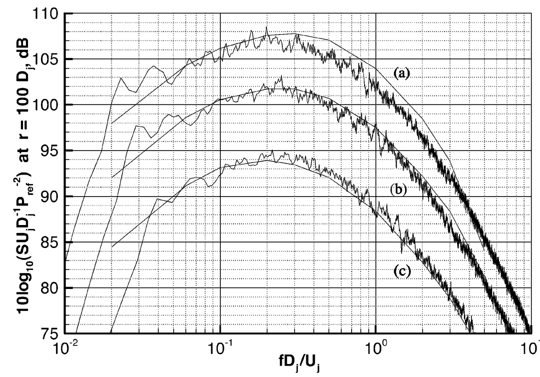

Fig. 5f Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 90 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

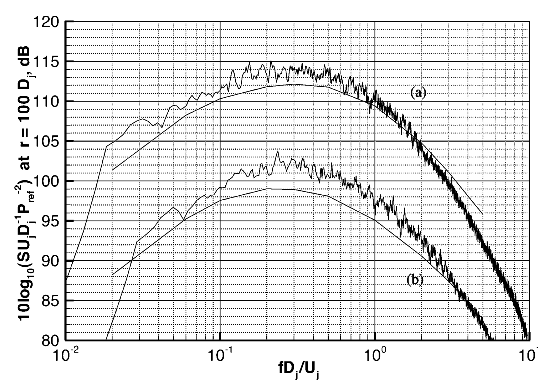

Fig. 5g Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 110 deg: a) $M_j = 1.0$ and b) $M_j = 0.6$.

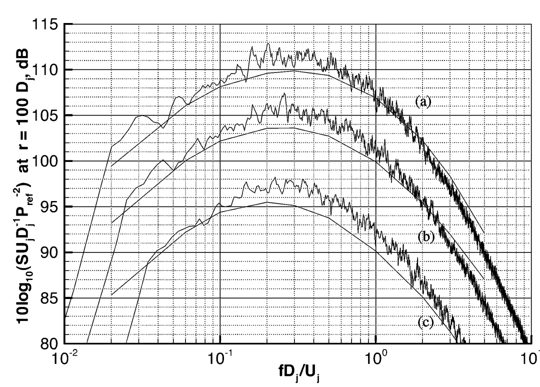

Fig. 5h Comparisons between calculated noise spectra and experiment, where $T_r/T_a = 3.2$ and inlet angle = 110 deg: a) $M_j = 0.9$, b) $M_j = 0.7$, and c) $M_j = 0.5$.

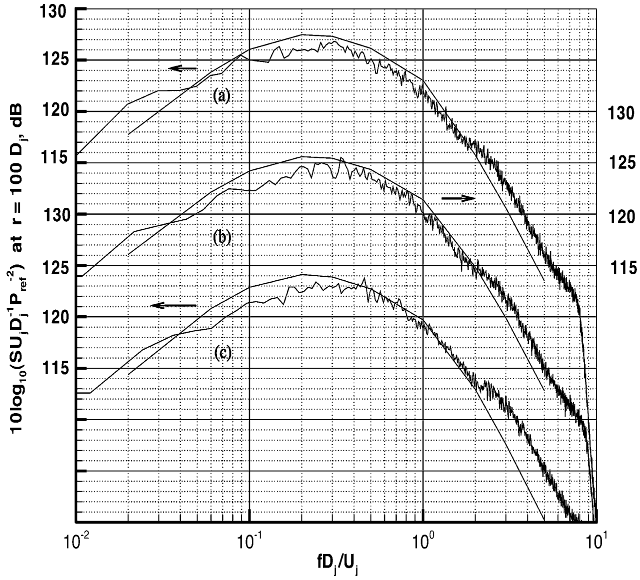

Fig. 6 Comparisons between calculated noise spectra and experiment, where $M_j = 2.0$ and inlet angle = 83 deg: a) $T_r/T_a = 4.89$, b) $T_r/T_a = 4.08$, and c) $T_r/T_a = 3.28$.

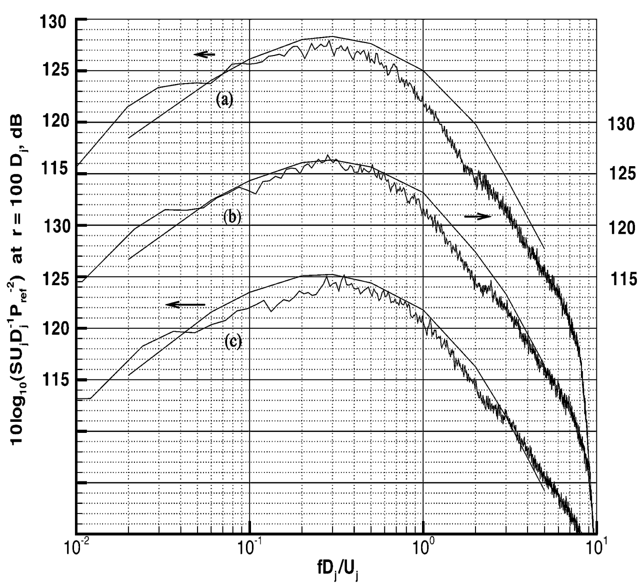

Fig. 7 Comparisons between calculated noise spectra and experiment, where $M_j = 2.0$ and inlet angle = 93 deg: a) $T_r/T_a = 4.89$, b) $T_r/T_a = 4.08$, and c) $T_r/T_a = 3.28$.

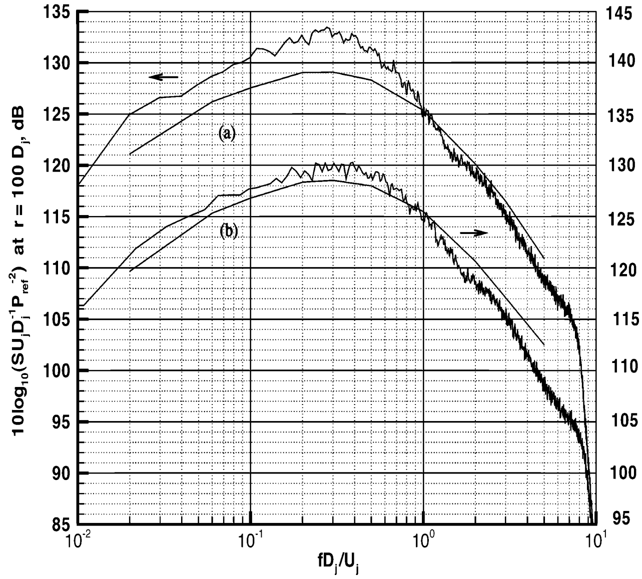

Fig. 8 Comparisons between calculated noise spectra and experiment, where $M_j = 2.0$ and inlet angle = 107 deg: a) $T_r/T_a = 4.89$ and b) $T_r/T_a = 4.08$.

the present modified jet noise theory is applicable to supersonic as well as subsonic jets up to very high jet temperature.

Figures 4–8 consistently indicate that the best comparisons between theory and experiment are found for noise radiation to the sideline around inlet angle 90 deg. At 90 deg, there is little mean flow refraction as well as source convection effects. The accuracy of prediction depends almost entirely on the noise source model. In other words, the good agreement at and around 90 deg suggests that the noise source model of Tam and Auriault,[3] extended in this work, must be essentially correct. At least, it has the correct turbulence noise generation physics. Also, overall there is better agreement at inlet angle 50 deg than at 110 deg. One possible reason for the slight loss of accuracy at 110 deg is that this angle might be too close to the cone of silence.[20,21] For hot jets, the cone of silence for fine-scale turbulence noise is quite large. In the prediction theory, the mean flow refraction effect is taken into account through the adjoint Green's function. To calculate the adjoint Green's function, we followed Tam and Auriault[3] and used a locally parallel flow approximation. The locally parallel flow approximation is very accurate outside the cone of silence especially in the upstream direction. However, it is inaccurate inside and close to the cone of silence. We believe that this might be a principal reason for the loss of accuracy in predicting the noise radiated at 110 deg.

V. Conclusions

Experimental measurements indicate that the mean flow and jet noise are affected by the temperature of the jet through the associated density gradient. Hot jets tend to be dynamically more unstable leading to an increase in turbulent kinetic energy per unit mass. However, this is countered by a decrease in gas density, which is a stronger effect. The overall result is a decrease in turbulent kinetic energy per unit volume. Thus, for a given jet-exit velocity an increase in jet temperature leads to a decrease in radiated noise despite an increase in turbulent mixing intensity (except for very low-Mach-number jets); see the experimental data in Fig. 9 of Ref. 1. There is also a decrease in thrust because of the decrease in density. On the other hand, if one is interested in the effect of an increase in jet temperature on jet noise at a fixed Mach number, then the decrease in jet noise caused by a decrease in turbulent kinetic energy is counteracted by an increase associated with an increase in jet velocity. It turns out that the increase in noise arising from an increase in jet velocity overwhelms the effect of decrease in turbulent kinetic energy per unit volume. This conclusion can easily be seen in the data of Figs. 6–8. Thus the temperature of a jet exerts a complex influence over its noise. Therefore, a general statement on the effect of an increase in jet temperature on its noise cannot be made without a clear specification of the constraints.

The Tam and Auriault theory[3] of noise from fine-scale turbulence, although known to provide good predictions for cold- to moderate-temperature jets, has to be modified slightly for accurate prediction of the noise from hot jets. The first modification is to take into account the increase in turbulent mixing and the change in mean flow profile. This is incorporated into the theory by using the modified k–ε turbulence model as proposed by Tam and Ganesan.[26] The second modification is based on the two-point space–time correlation measurements of Doty and McLaughlin.[29] Their optical deflectometry data suggest that eddies in hot jets decay faster. In this paper a generalized two-point space–time correlation function suitable for noise prediction from hot jets is adopted. For isothermal and colder jets, the model is designed to yield identical far field noise spectrum as the Tam and Auriault theory.

Extensive comparisons between calculated noise spectra and experimental measurements have been carried out over the jet

Mach-number range of from 0.5 to 2.0 and jet temperature ratio to as high as 4.9. Good overall agreements are found for inlet angle 50 to 110 deg. Beyond 110 deg, especially for very high-temperature jets, there is significant contribution of radiated noise from the large turbulence structures of the jet flow. Prediction of the noise from large turbulence structures is beyond the scope of this investigation.

Preliminary computations of the noise spectra from dual-stream jets using the present modified noise spectrum formula and identical model constants appear to compare well with measurements. This suggests that the present extended model might be applicable to jet flows other than single-stream circular jets.

Acknowledgment

Two of the authors (Tam and Pastouchenko) were supported by NASA Contract NAS1-02045.

References

[1]Tam, C. K. W., Golebiowski, M., and Seiner, J. M., "Two Components of Turbulent Mixing Noise from Supersonic Jets," AIAA Paper 96-1716, May 1996.

[2]Tam, C. K. W., "Jet Noise: Since 1952," *Theoretical and Computational Fluid Dynamics*, Vol. 10, No. 10, 1998, pp. 393–405.

[3]Tam, C. K. W., and Auriault, L., "Jet Mixing Noise from Fine-Scale Turbulence," *AIAA Journal*, Vol. 37, No. 2, 1999, pp. 145–153.

[4]Crow, S. C., and Champagne, F. H., "Orderly Structures in Jet Turbulence," *Journal of Fluid Mechanics*, Vol. 48, 1971, pp. 547–591.

[5]Brown, G. L., and Roshko, A., "On Density Effects and Large Structures in Turbulent Mixing Layers," *Journal of Fluid Mechanics*, Vol. 64, 1974, pp. 775–816.

[6]Tam, C. K. W., and Burton, D. E., "Sound Generated by Instability Waves of Supersonic Flows: Part 1, Two-Dimensional Mixing Layers; Part 2, Axisymmetric Jets," *Journal of Fluid Mechanics*, Vol. 138, 1984, pp. 249–295.

[7]Tam, C. K. W., and Chen, P., "Turbulent Mixing Noise from Supersonic Jets," *AIAA Journal*, Vol. 32, No. 9, 1994, pp. 1774–1780.

[8]Tam, C. K. W., "Influence of Nozzle Geometry on the Noise of High Speed Jets," *AIAA Journal*, Vol. 36, No. 8, 1998, pp. 1396–1400.

[9]Tam, C. K. W., and Zaman, K. B. M. Q., "Subsonic Jet Noise from Non-Axisymmetric and Tabbed Nozzles," *AIAA Journal*, Vol. 38, No. 4, 2000, pp. 592–599.

[10]Dahl, M. D., and Papamoschou, D., "Analytical Predictions and Measurements of the Noise Radiated from Supersonic Coaxial Jets," *AIAA Journal*, Vol. 38, No. 4, 2000, pp. 584–591.

[11]Viswanathan, K., "Analysis of the Two Similarity Components of Turbulent Mixing Noise," *AIAA Journal*, Vol. 40, No. 9, 2002, pp. 1735–1744.

[12]Viswanathan, K., "Aeroacoustics of Hot Jets," *Journal of Fluid Mechanics*, Vol. 516, 2004, pp. 39–82.

[13]Laufer, J., Schlinker, R. H., and Kaplan, R. E., "Experiments on Supersonic Jet Noise," *AIAA Journal*, Vol. 14, No. 4, 1976, pp. 489–497.

[14]Tam, C. K. W., Pastouchenko, N. N., and Schlinker, R. H., "On the Two Sources of Supersonic Jet Noise," AIAA Paper 2003-3163, May 2003; also *Journal of Sound and Vibration* (to be published).

[15]Morris, P. J., and Farassat, F., "Acoustic Analogy and Alternative Theories for Jet Noise Prediction," *AIAA Journal*, Vol. 40, No. 4, 2002, pp. 671–680.

[16]Schlinker, R. H., "Supersonic Jet Noise Experiments," Ph.D. Dissertation, Dept. of Aerospace Engineering, Univ. of Southern California, Los Angeles, Aug. 1975.

[17]Panda, J., Seasholtz, R. G., and Elam, K. A., "Further Progress in Noise Source Identification in High Speed Jets via Causality Principle," AIAA Paper 2003-3126, May 2003.

[18]Lighthill, M. J., "On Sound Generated Aerodynamically: I. General Theory," *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, Vol. 211, 1952, pp. 564–581.

[19]Lighthill, M. J., "On Sound Generated Aerodynamically: II. Turbulence as a Source of Sound," *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, Vol. 222, 1954, pp. 1–32.

[20]Atvars, J., Schubert, L. K., and Ribner, H. S., "Refraction of Sound from a Paint Source Placed in an Air Jet," *Journal of the Acoustical Society of America*, Vol. 37, 1965, pp. 168–170.

[21]Grande, E., "Refraction of Injected Sound by a Very Cold Nitrogen Jet," *Journal of the Acoustical Society of America*, Vol. 38, 1965, pp. 1063, 1064.

[22]Norum, T. D., and Brown, M. C., "Simulated High Speed Flight Effects on Supersonic Jet Noise," AIAA Paper 93-4388, Oct. 1993.

[23]Tam, C. K. W., Pastouchenko, N., and Auriault, L., "Effects of Forward Flight on Jet Mixing Noise from Fine-Scale Turbulence," *AIAA Journal*, Vol. 39, No. 7, 2001, pp. 1261–1269.

[24]Plumblee, H. E., "Effects of Forward Velocity on Turbulent Jet Mixing Noise," NASA CR-2702, 1976.

[25]Tam, C. K. W., and Pastouchenko, N. N., "Noise from Fine Scale Turbulence of Nonaxisymmetric Jets," *AIAA Journal*, Vol. 40, No. 2, 2002, pp. 456–464.

[26]Tam, C. K. W., and Ganesan, A., "A Modified k–ε Turbulence Model for Calculating the Mean Flow and Noise of Hot Jets," *AIAA Journal*, Vol. 42, No. 1, 2004, pp. 26–34.

[27]Sarkar, S., and Lakshmanan, B., "Application of a Reynolds Stress Turbulence Model to the Compressible Shear Layer," *AIAA Journal*, Vol. 29, No. 5, 1991, pp. 743–749.

[28]Davies, P. O. A. L., Fisher, M. J., and Barratt, M. J., "The Characteristics of the Turbulence in the Mixing Region of a Round Jet," *Journal of Fluid Mechanics*, Vol. 15, 1963, pp. 337–367.

[29]Doty, M. J., and McLaughlin, D. K., "Two-Point Correlations of Density Gradient Fluctuations in High Speed Jets Using Optical Deflectometry," AIAA Paper 2002-0367, Jan. 2002.

[30]Erdelyi, A., *Tables of Integral Transforms*, Bateman Manuscript Project, Vol. 1, McGraw–Hill, New York, 1954, p. 49.

S. Mahalingam
Associate Editor