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THIRD ORDER A-HYPERGEOMETRIC FUNCTIONS

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To my parents.

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LIST OF SYMBOLS

The following symbols are used throughout this thesis.

K	Differential field, most time represents $\mathbb{C}(x)$ or $\mathbb{C}(x,y)$
∂	Derivation $\frac{d}{dx}$
L	Differential operator in $K[\partial]$
$V(L)$	Solution space of the differential operator L
$\xrightarrow{(i),f}$	Change of variables transformation which sends $y(x) \longmapsto y(f)$
$\xrightarrow{(ii),r}$	Exp-product transformation which sends $y \longmapsto y \cdot \exp \int r$
$\xrightarrow{(iii),G}$	Gauge transformation which sends $y \longmapsto G(y)$
$\exp(\int r)$	A nonzero solution of $\partial - r$
Ⓢ	The symmetric product sign of two differential operators
$\mathbb{P}^1$	$\mathbb{C} \cup \infty$
e_p	An exponent at $x = p \in \mathbb{P}^1$
$\sim$	The sign to represent two exponents (sets) being equivalent
${}_pF_q$	The generalized hypergeometric series
$\mathrm{Hom}_D(M, M')$	Homomorphisms between M and M' as D-modules
$\otimes$	Tensor product
$F_1^D(a, b_1, b_2, c \mid x, y)$	The D-module of Appell $F_1(a, b_1, b_2, c \mid x, y)$ with $D = \mathbb{C}(x,y)[\partial_x, \partial_y]$

ABSTRACT

To solve globally bounded order 3 linear differential equations with rational function coefficients, this thesis introduces a partial ${}_3F_2$-solver (Section 3.2) and F_1-solver (Chapter 4) where ${}_3F_2$ is the hypergeometric function ${}_3F_2(a_1, a_2, a_3; b_1, b_2 \,|\, x)$ and F_1 is the Appell's $F_1(a, b_1, b_2, c \,|\, x, y)$. To investigate the relations among order 3 multivariate hypergeometric functions, this thesis presents two multivariate tools: compute homomorphisms (Algorithm 5.3.10) of two D-modules where D is a multivariate differential ring, and compute projective homomorphisms (Algorithm 5.4.5) using the tensor product module and Algorithm 5.3.10. As an application, all irreducible order 2 subsystems from reducible order 3 systems turn out to come from Gauss hypergeometric function ${}_2F_1(a, b; c \,|\, x)$ (Chapter 6).

CHAPTER 1

INTRODUCTION

1.1 Motivation

Homogeneous linear differential equations with rational function coefficients are used in many fields.

Definition 1.1.1 *A function $y = y(x)$ is* D-finite *of order n if it satisfies an order n ($a_n \neq 0$) linear differential equation (1.1):*

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \;\text{ with } a_0, \ldots, a_n \in \mathbb{C}(x). \tag{1.1}$$

Let $S(x)$ be a D-finite function. Suppose we want to know if (1.1) can be solved "in terms of" S. To make this more precise: *S-type expressions* should allow:

- S
- field operators ($+$, $-$, $\times$, $\div$)
- algebraic functions
- exp and log
- composition
- differentiation and integration.

Not all S-type expressions are relevant for solving (1.1). For example, $S(\exp(x))$ is not D-finite for most D-finite S, which leads to a question: which S-type expressions are D-finite?

D-finite to D-finite operations:

- Operations that do not increase the order (more details in Section 2.1.8):
  - (i) $S(x) \mapsto S(f)$ for some $f \in \mathbb{C}(x) - \mathbb{C}$.
  - (ii) $S \mapsto \exp(\int r) \cdot S$ with $r \in \mathbb{C}(x)$.

– (iii) $S \mapsto r_0 S + r_1 S' + \ldots + r_{n-1} S^{(n-1)}$ with $r_0, r_1, \ldots, r_{n-1} \in \mathbb{C}(x)$.

- Operations that can increase the order:
  - (iv) Same as (i)(ii)(iii) but with algebraic functions f, r_i, r.
  - (v) $S_1, S_2 \mapsto S_1 + S_2$. From order n_1, n_2 to order $\leq n_1 + n_2$.
  - (vi) $S_1, S_2 \mapsto S_1 \cdot S_2$. From order n_1, n_2 to order $\leq n_1 \cdot n_2$.

Order preserving transformations (i)(ii)(iii) are relevant to solving differential equations of any order, while order increasing transforamtions (iv)(v)(vi) are relevant for solving equations of order > 2.

Remark 1.1.2 *Suppose S is D-finite of order n. The non-zero expression*

$$\exp(\int r\mathrm{d}x)(r_0 S(f) + r_1 S(f)' + \ldots + r_{n-1} S(f)^{(n-1)}) \tag{1.2}$$

is the most general S-type expression under (i)(ii)(iii). It is D-finite of order $\leq n$.

Section 3.3 will give an algorithm to decide if two equations of order 3 are connected under transformations (ii)+(iii) (there already was an algorithm for order 2). With it, finding a solution of form (1.2) reduces to finding the parameters in S and the *pullback function* f. What makes this task nontrivial is that to solve L we need to compute f by only using data from L invariant under (ii)+(iii).

Quan Yuan's thesis [29, 32] did this for $n = 2$ for many special functions S, such as Bessel, Airy and Kummer, that satisfy an order 2 equation, with one important exception: the Gauss hypergeometric ${}_2F_1$ function.

Definition 1.1.3 *The ${}_2F_1$ function, also called Gauss hypergeometric function, is defined by*

$${}_2F_1(a, b; c \,|\, x) := \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{(c)_k \cdot k!} x^k$$

where $(\lambda)_k$ denotes the Pochhammer symbol

$$(\lambda)_k = \begin{cases} 1 & k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & k \neq 0. \end{cases}$$

This ${}_2F_1(a,b;c\,|\,x)$ function satisfies the Gauss Hypergeometric Equation (GHE):

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0. \tag{1.3}$$

We can write (1.3) as $L(y)=0$ where $L = x(1-x)\partial^2 + (c-(a+b+1)x)\partial - ab$ (with $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$) is a differential operator.

Vijay Kunwar's [22, 23] and Erdal Imamoglu's [19–21, 28] developed several algorithms to find ${}_2F_1$-type solutions (form (1.2) with $S(x) = {}_2F_1(a,b;c\,|\,x)$ for some a,b,c) of order 2 differential equations.

Definition 1.1.4 *[11] Let $y \in \mathbb{C}[[x]] - \{0\}$, if y has a positive radius of convergence and there exist $c_1, c_2 \in \mathbb{C} - \{0\}$ such that $c_1 \cdot y(c_2 x) \in \mathbb{Z}[[x]]$, then $y(x)$ is called* globally bounded. *If an irreducible operator L has a globally bounded solution, then L is called globally bounded.*

Globally bounded order 2 equations are very common, and so far they all turn out to have ${}_2F_1$-type solutions [3, 9, 10, 33], which motivates Conjecture 1 below.

Conjecture 1 *Let $L = a_2\partial^2 + a_1\partial + a_0$ be a linear differential operator of order 2 with $a_0, a_1, a_2 \in \mathbb{C}(x)$. If $L(y) = 0$ and y is globally bounded, then one of these cases holds:*

- *y is an* algebraic *function, $y \in \overline{\mathbb{C}(x)}$, or,*
- *y can be written in form (1.2) with $n = 2$, $f, r, r_0, r_1 \in \overline{\mathbb{C}(x)}$, $S(x)$ being ${}_2F_1(a,b;c\,|\,x)$ for some $a,b,c \in \mathbb{Q}$ and $c \in \{1,2,\ldots\}$*[1].

Conjecture 1 says that closed form solutions should be very common in many applications, so a good ${}_2F_1$-solver is useful from a practical point of view. With programs from Imamoglu, Kunwar, Fang [17, 18] and van Hoeij [27], this conjecture was tested on hundreds of equations from OEIS (Online Enyclopedia of Integer Sequences [1]).

Example 1.1.5 *Consider the differential operator (Example 1.3 in [21])*

$$L = \partial^2 - \frac{512x^5 + 384x^4 - 64x^3 - 88x^2 - 10x - 1}{x(4x-1)(4x+1)(16x^3+24x^2+5x+1)}\partial + \frac{512x^5 + 64x^4 - 128x^3 - 60x^2 - 8x - 1}{x^2(4x-1)(4x+1)(16x^3+24x^2+5x+1)}.$$

The following ${}_2F_1$-type solution of L is obtained by the algorithms in [21]:

$$(4x^3 + x^2 + \frac{x}{2})\,{}_2F_1(\frac{1}{2},\frac{1}{2};1\,|\,16x^2) + (32x^5 - 2x^3)\,{}_2F_1(\frac{3}{2},\frac{3}{2};2\,|\,16x^2).$$

[1] This condition corresponds to L having at least one logarithmic singularity.

1.2 Univariate Hypergeometric Functions

Definition 1.2.1 *The hypergeometric series ${}_pF_q$ defined by*

$$ {}_pF_q\left(\begin{array}{c}\alpha_1 \ldots \alpha_p\\ \beta_1 \ldots \beta_q\end{array} \,|x\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k k!} x^k $$

is a generalization of the Gauss hypergeometric ${}_2F_1$ function (Definition 1.1.3). It is also denoted as ${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q \,|\, x)$.

The minimal operator of ${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q \,|\, x)$ in $\mathbb{C}(x)[\partial]$ is

$$ \delta(\delta + \beta_1 - 1) \cdots (\delta + \beta_q - 1) - x(\delta + \alpha_1) \cdots (\delta + \alpha_p) \tag{1.4} $$

with $\delta = x\frac{\mathrm{d}}{\mathrm{d}x} = x\partial$. Using $\partial \cdot a = a \cdot \partial + a'$, $\delta\delta = x\partial x\partial = x(x\partial + 1)\partial = x^2\partial^2 + x\partial$, one can check that (1.3) is a special case of (1.4).

Remark 1.2.2 *Hypergeometric ${}_pF_q$ functions with $p + 1 \neq q$ are not globally bounded.*

Globally bounded ${}_pF_q$-type expressions which are D-finite of order 3 are ${}_3F_2$ and the square of ${}_2F_1$, which motivates Question 1, the analogue of Conjecture 1 for univariate operators of order 3.

Question 1 *Let $L = a_3\partial^3 + a_2\partial^2 + a_1\partial + a_0$ be a linear differential operator of order 3 with $a_0, a_1, a_2, a_3 \in \mathbb{C}(x)$. If $L(y) = 0$ and y is globally bounded, must one of these cases hold?*

- *y can be written as in Conjecture 1 or,*
- *y can be written in form (1.2) with $n = 3$, f, r, r_0, r_1, $r_2 \in \overline{\mathbb{C}(x)}$ and S being a ${}_3F_2$ function or the square of a ${}_2F_1$ function.*

Remark 1.2.3 *The analogue of Conjecture 1 for order 4 operators is false and there are many counter examples in the Calabi-Yau database (a large database with hundreds of order 4 equations [2]). But for order 3, we found no counter example in the literature or in [1].*

To test Question 1, we developed a partial ${}_3F_2$-solver (Chapter 3). Next we constructed a counter example of Question 1 by substitution in Appell's F_1 function (Chapter 4), a multivariate hypergeometric function. The resulting globally bounded function is a solution of:

Table 1.1: Hypergeometric Functions of Order 2, 3, 4

Order	Univariate Regular Singular	Multivariate Regular Singular	Irregular Singular
2	${}_2F_1$ solvers [17, 19, 22, 27]	Remark 1.3.1	${}_0F_1$ (Bessel, Airy) ${}_1F_1$ (Kummer, Whittaker) solvers [29, 32]
3	${}_3F_2$ solver in Section 3.2	F_1, G_1, G_2, G_3, H_3, H_6 F_1-solver in Chapter 4	${}_0F_2$, ${}_1F_2$, ${}_2F_2, \ldots$ solvers [30] in progress
4	${}_4F_3$	F_2, F_3, $F_4, \ldots$	${}_0F_3$, ${}_1F_3, \ldots$

$$\begin{aligned} L = \partial^3 &+ \frac{5891x^3 + 9388x^2 - 11890x + 3000}{15x(x+4)(43x-20)(2x-1)}\partial^2 \\ &+ \frac{235296x^3 + 30775x^2 - 191300x + 36000}{900x^2(x+4)(43x-20)(2x-1)}\partial \\ &+ \frac{3096x^2 - 5005x - 1900}{900x^2(x+4)(43x-20)(2x-1)}. \end{aligned} \tag{1.5}$$

This operator L is globally bounded but not ${}_3F_2$-solvable (Appendix). So a ${}_2F_1$-solver and a ${}_3F_2$-solver are not enough to solve globally bounded order 3 equations. We need, among other things, an F_1-solver (Chapter 4) as well.

1.3 Multivariate Hypergeometric Functions

In light of Conjecture 1, we could ask if globally bounded equations of higher order are also solvable in terms of hypergeometric functions. If we aim to solve univariate equations, then it makes sense to consider univariate hypergeometric functions, leading to Question 1. However, example (1.5) (for order 3) and examples (for order 4) in Calabi-Yau database show that univariate hypergeometric functions are not sufficient to solve globally bounded order 3 or 4 equations. But what about multivariate hypergeometric functions? This leads to Question 2:

Question 2 *Are globally bounded operators solvable in terms of A-hypergeometric functions?*

There are many generalizations of ${}_2F_1$ in the literature, see Table 1.1. Fortunately, they can be classified in one framework called A-hypergeometric functions [4–8]. These functions are classified by polytopes (Chapter 7).

Remark 1.3.1 *All multivariate regular singular hypergeometric functions of order* 2 *we encountered so far are special cases of* ${}_2F_1$*. But for higher order, the number of cases grows quickly.*

To test Question 2, we need to develop algorithms to solve *univariate* equations in terms of *multivariate* functions:

- There are many useful tools in [17–23,27–29,32] for *univariate* order 2 operators including one which can recover transformations (ii)+(iii) in Section 1.1. We developed Algorithm 5.3.10 to recover transformation (iii) and Algorithm 5.4.5 for (ii)+(iii) for *multivariate* systems of any order.
- For transformation (i), one need to recover the *pullback function* f. There are several pullback functions instead of one in the multivariate case. So it is not obvious how to generalize the univariate tool to multivariate.
- There are many hypergeometric functions of order greater than 2, so it will be a lot of work if we develop a solver for each of them. First, we developed a solver for one case, F_1 in Chapter 4. Next, to reduce the amount of work, we used Algorithms 5.3.10 and 5.4.5 to investigate the relations among them – to reduce the number of functions for which we need solvers.

Conjecture 2 *Let* L *be a linear differential operator with rational function coefficients of order* 3*. If* $L(y) = 0$ *and* y *is globally bounded, then one of these cases holds:*

- *L has a solution as in Conjecture 1 or,*
- *L has a solution that can be written in form* (1.2) *with* $n = 3$, f, r, r_0, r_1, $r_2 \in \overline{\mathbb{C}(x)}$ *and* S *being a* ${}_3F_2$ *function or Appell* F_1 *function (Chapter 4) or Horn* G_3 *function (Section 6.3) or a square of a* ${}_2F_1$ *function.*

We could ask if in Question 2 we can restrict to irreducible A-hypergeometric systems, or, if reducible systems need to be considered as well. This question leads to:

Question 3 *Given a reducible A-hypergeometric system with order* n *(*$n \geq 3$*), say it has a factor* L *of order* $m < n$*, is* L *solvable in terms of A-hypergeometric functions of order* m*?*

We verified Question 3 for $m = 2$ and $n = 3$ for the regular singular order 3 A-hypergeometric systems in Table 1.1. To study Question 3 for $m = 3$ and $n = 4$, we developed algorithms to compute solutions in terms of order 3 A-hypergeometric functions (so far ${}_3F_2$-solver in Chapter 3 and F_1-solver in Chapter 4).

CHAPTER 2

PRELIMINARIES

This chapter introduces exponents, transformations and relations between them. The reason why we need (generalized) exponents is as follows:

- We have an algorithm (DEtools[Homomorphisms] in Maple) that can recover transformation (iii). We need an algorithm that can recover (ii)+(iii). Then we should compute r in (ii) using only data ("exponents mod $\mathbb{Z}$") that is invariant under (iii).

- Suppose we have an algorithm for (ii)+(iii) and we want an algorithm for (i)+(ii)+(iii). Then we should compute f in (i) using only data ("exponent differences mod $\mathbb{Z}$") invariant under (ii)+(iii).

This explains why the rather technical (generalized) exponents are important for finding solutions of form (1.2).

2.1 Differential Operators and Transformations

Definition 2.1.1 *Let K be a ring. A derivation of K is a linear map $\partial : K \to K$ such that all $a, b \in K$ satisfy the product rule:*

$$\partial(ab) = a \cdot \partial(b) + b \cdot \partial(a).$$

A ring K with a derivation ∂ is called a differential ring.

Let $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$. Then $K = \mathbb{C}(x)$ with ∂ is a differential field.

Definition 2.1.2 *Let $a_i \in K$ and $L = \sum_{i=0}^{n} a_i \partial^i$. The operator L can be considered as a map $L : K \to K$. If $a_n \neq 0$, then n is the order of L. Composition of operators is multiplication in the ring $K[\partial] = \{\sum_{i=0}^{n} a_i \partial^i \mid a_i \in K\}$. So if $a \in K$, then $\partial \cdot a = a\partial + a'$.*

Definition 2.1.3 *A universal extension of $K = \mathbb{C}(x)$ is a commutative differential ring Ω with:*

- $K \subseteq \Omega$.

- *Ω is a $K[\partial]$-module.*

- *For any $L \in K[\partial]$, $\mathrm{Ker}(L : \Omega \to \Omega)$ is a $\mathbb{C}$-vector space of dimension order(L). Denote it as $V(L)$, the solution space of L.*

- *Every $y \in \Omega$ is a solution of some nonzero operator $L \in K[\partial]$.*

[25] shows that such universal extension Ω exists for any differential field K with algebraically closed field of constants, moreover, it is unique up to isomorphism.

Remark 2.1.4 *For the ring $K[\partial]$, one can perform right division with remainder. As a consequence, every left*[1] *ideal of $K[\partial]$ is principal. In fact, $K[\partial]$ has all properties of a Euclidean domain except commutativity.*

Definition 2.1.5 *The least common left multiple $\mathbf{LCLM}(L_1, L_2)$ is the unique monic generator of $K[\partial]L_1 \cap K[\partial]L_2$. The greatest common right divisor $\mathbf{GCRD}(L_1, L_2)$ is the unique monic generator of $K[\partial]L_1 + K[\partial]L_2$. Their solutions are as follows.*

$$V(\mathbf{LCLM}(L_1, L_2)) = V(L_1) + V(L_2). \tag{2.1}$$

$$V(\mathbf{GCRD}(L_1, L_2)) = V(L_1) \cap V(L_2).$$

Note that (2.1) *shows item (v) in Section 1.1 (D-finite plus D-finite is D-finite).*

Definition 2.1.6 *(Item (ii) in Section 1.1). If $L = \sum_{i=0}^{n} a_i \partial^i$, then $L \circledS (\partial - r)$ denotes $\sum_{i=0}^{n} a_i (\partial - r)^i$.*

Remark 2.1.7 *The map $L \mapsto L \circledS (\partial - r)$ is an automorphism of $K[\partial]$. If y is a solution of L, then $y \cdot \exp(\int r)$ is a solution of $L \circledS (\partial - r)$. In this thesis, $\exp(\int r)$ denotes a nonzero solution of $\partial - r$ in Ω.*

Suppose L has a non-zero solution $y(x)$ and

$$y_1 = y(x^2),$$

$$y_2 = e^x \cdot y,$$

[1] We only use left ideals (and right division), but the same is true for right ideals (using left division).

$$y_3 = xy' + (x+1)y$$

are solution of operators L_1, L_2, L_3. If y is a closed form solution of L (Remark 1.1.2), then y_1, y_2, y_3 are closed form solutions of L_1, L_2, L_3 respectively. Note that y_1, y_2, y_3 are examples for transformations (i)(ii)(iii) in Section 1.1. Here we give these transformations in more detail:

Definition 2.1.8 *Let $L = \sum_{i=0}^{n} a_i \partial^i$ be a differential operator of order n. Consider the following transformations. Here $y = y(x)$ denotes an arbitrary solution of L.*

(i) Change of variables transformation: $y(x) \longmapsto y(f)$, or equivalently,

$$L \longmapsto \sum_{i=0}^{n} a_i(f) \partial_f^i$$

where $\partial_f = \frac{\mathrm{d}}{\mathrm{d}f} = \frac{1}{f'}\partial$. Here f is called the pullback function*, and f' must be nonzero.*

(ii) Exp-product transformation: $y \longmapsto \exp(\int r) y$, or equivalently,

$$L \longmapsto L \circledS (\partial - r).$$

(iii) Gauge transformation: $y \longmapsto G(y)$, here $G = r_0 + r_1\partial + r_2\partial^2 + \ldots + r_{n-1}\partial^{n-1}$ or equivalently,

$$L \longmapsto \widetilde{L}, \text{ where } \widetilde{L} \cdot G = \mathbf{LCLM}(L, G).$$

We only allow G with $\mathbf{GCRD}(L, G) = 1$, which is equivalent to $ord(L) = ord(\widetilde{L})$, and G giving a bijection from $V(L)$ to $V(\widetilde{L})$.

Remark 2.1.9 *If a combination of $(i), (ii), (iii)$ in Definition 2.1.8 sends operator L_1 to L_2, then solutions of L_2 can be expressed in form (1.2) with $S(x)$ being solutions of L_1.*

If $L_1 \in K[\partial]$ and the parameters (f in (i), r in (ii) and G in (iii)) are over K as well, then L_2 is also in $K[\partial]$. For the resulting operator $L_2 \in K[\partial]$, use $L_1 \xrightarrow{(i), f} L_2$, $L_1 \xrightarrow{(ii), r} L_2$ and $L_1 \xrightarrow{(iii), G} L_2$ to denote transformations (i), (ii) and (iii) respectively.

Definition 2.1.10 *Let $L_1, L_2 \in K[\partial]$.*

- $L_1 \xrightarrow{(i)} L_2$ *means $\exists\, f \in K$ such that $L_1 \xrightarrow{(i), f} L_2$.*

- $L_1 \xrightarrow{(ii)} L_2$ *means* $\exists\, r \in K$ *such that* $L_1 \xrightarrow{(ii),\, r} L_2$.
- $L_1 \xrightarrow{(iii)} L_2$ *means* $\exists\, G \in K$ *such that* $L_1 \xrightarrow{(iii),\, G} L_2$.
- $L_1 \xrightarrow{(ii),(iii)} L_2$ *means* L_1 *transforms to* L_2 *under transformations* (ii) *and* (iii).
- $L_1 \longrightarrow L_2$ *means* L_1 *transforms to* L_2 *under transformations* $(i), (ii), (iii)$.

Proposition 2.1.11 *[14]. The transformations* $\xrightarrow{(ii)}$, $\xrightarrow{(iii)}$ *and* $\xrightarrow{(ii),\,(iii)}$ *define equivalence relation. The following are equivalent and such operators* L_1, L_2 *are called* projectively equivalent.

- $L_1 \xrightarrow{(ii)}\xrightarrow{(iii)} L_2$,
- $L_1 \xrightarrow{(iii)}\xrightarrow{(ii)} L_2$,
- $L_1 \xrightarrow{(ii),\,(iii)} L_2$.

Proposition 2.1.12 *[14, 15]. If* $L_1, L_2 \in K[\partial]$ *and* $L_1 \longrightarrow L_2$, *then there exists an operator* $M \in K[\partial]$ *such that* $L_1 \xrightarrow{(i)} M \xrightarrow{(ii),\,(iii)} L_2$.

Definition 2.1.13 *Let* $L \in \mathbb{C}(x)[\partial]$. *Clearing denominators means replacing* L *by* $a \cdot L$ *for some* $a \in \mathbb{C}(x)$ *such that* $a \cdot L \in \mathbb{C}[x][\partial]$ *and the greatest common divisor of all coefficients of* $a \cdot L$ *in* $\mathbb{C}[x]$ *is* 1. *After clearing denominators, if* $p \in \mathbb{C}$ *is a root of leading coefficient, then* p *is called a* singularity *of* L. *If* $p = 0$ *is a singularity of* $\widetilde{L}$ *with* $L \xrightarrow{(i),\, \frac{1}{x}} \widetilde{L}$ *then* $p = \infty$ *is a singularity of* L. *All other points are called regular points.*

Definition 2.1.14 *A singularity* $x = p$ *of* L *is a* non-removable singularity *if it remains singular under any* $\xrightarrow{(ii),\,(iii)}$ *transformation, otherwise it is a* removable singularity. *The beginning of this chapter explains why only* non-removable singularities *are relevant for finding the pullback function* f *– we compute* f *from data invariant under* $\xrightarrow{(ii),\,(iii)}$ *transformations.*

Example 2.1.15 $L = (x^2 - x)\partial^2 + \frac{x^2-5x+2}{x-2}\partial + \frac{x+2}{(x-2)^2}$ *has singularities at* $x = 0$, $x = 1$, $x = 2$ *and* $x = \infty$. *Among them,* $x = 1$ *and* $x = 2$ *are removable since* $x = 1$ *is a regular point of* $L \circledS (\partial - \frac{1}{x-1})$ *and* $x = 2$ *is a regular point of* $L \circledS (\partial + \frac{1}{x-2})$. *The other singularities are non-removable: they stay singular under any* $\xrightarrow{(ii),\,(iii)}$ *transformations.*

Definition 2.1.16 *A singularity* $x = p \in \mathbb{C}$ *of* $L \in K[\partial]$ *is called*

- apparent singular *if all solutions of L are analytic at p.*
- regular singular *if there exists some positive integer N such that $(x-p)^N \cdot y$ converges to* 0 *as $x \to p$ for any $y \in V(L)$.*
- irregular singular *otherwise.*

The singularity $x = \infty$ of L is apparent (regular, irregular) singular if $x = 0$ is apparent (regular, irregular) singular of $\widetilde{L}$ with $L \xrightarrow{(i),\, \frac{1}{x}} \widetilde{L}$.

2.2 D-modules

Let $K = \mathbb{C}(x)$ and $D = K[\partial]$ and let $y \in \Omega$ (Definition 2.1.3) with $y \neq 0$. By Remark 2.1.4, there is a unique monic $L \in D$ of minimal order with $L(y) = 0$ and call L the *minimal operator* of y. Let $Dy := \{L(y) \,|\, L \in D\}$. Now $Dy \subseteq \Omega$ is a left D-module that is isomorphic to D/DL. We only consider D-modules that are finitely dimensional K-vector spaces. Any $n \times n$ matrix A over K defines a D-module K^n by letting:

$$\partial \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} := A \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \partial(a_1) \\ \vdots \\ \partial(a_n) \end{bmatrix}.$$

Remark 2.2.1 *Every D-module M is cyclic (cyclic vector theorem [25]): there exists $y \in M$ with $y, \partial(y), \ldots, \partial^{n-1}(y)$ a K-basis of M. That means $M \cong D/DL$ where L is the minimal operator of y. If L is irreducible, so is M.*

Remark 2.2.2 *If $L = L_1 L_2$, then $DL_2/DL \cong D/DL_1$ is a submodule of M and D/DL_2 is a quotient module of M.*

Remark 2.2.3 *Let $L_1 \in D$. The corresponding D-module is $M_1 := D/DL_1$. Note that $dim_K(M_1)$ is the order of L_1. Let M_2 be the D-module for L_2, then $L_1 \xrightarrow{(iii),\, G} L_2$ is equivalent to saying that M_1 and M_2 are isomorphic as D-modules. Any gauge transformation G sends the solution of L_1, y, to the solution of L_2, $G(y)$, giving an isomorphism from D/DL_1 to D/DL_2.*

Definition 2.2.4 *Let M_1, M_2 be the D-modules of L_1 and L_2, then M_1 and M_2 are* projectively equivalent *($M_1 \cong M_2 \otimes I$ for a 1-dimensional module I) if L_1 and L_2 are projectively equivalent.*[2]

[2]We will generalize this definition to multivariate case in chapter 5.

2.3 Exponents

Exponents will be needed in ${}_3F_2$-solver (Chapter 3) and F_1-solver (Chapter 4).

Definition 2.3.1 *Let $L \in D := K[\partial]$, then $e \in \mathbb{C}$ is an* exponent *of L at $x = 0$ if and only if there exists a solution of L at $x = 0$, say y, such that*

$$y = x^e \cdot S, \quad S \in R_0 := \mathbb{C}[[x]][\ln(x)] \text{ and } S \notin x \cdot R_0. \tag{2.2}$$

Definition 2.3.2 *Let $L \in D$, $p \in \mathbb{P}^1$ and*

$$t_p = \begin{cases} x - p, & p \in \mathbb{C} \\ \frac{1}{x}, & p = \infty. \end{cases}$$

Then $e \in \mathbb{C}$ is an exponent *at $x = p$ if and only if L has a solution of form* (2.2) *with x replaced by t_p.*

Definition 2.3.3 *Let $L \in D$ and $e \in \mathbb{C}[x^{-\frac{1}{n}}]$ for some $n \in \mathbb{N}$. Such e is a* generalized exponent *of L at $x = 0$ if and only if there exists a solution of L at $x = 0$, say y, such that*

$$y = \exp(\int \frac{e}{x} \mathrm{d}x) \cdot S, \quad S \in R_{0,n} := \mathbb{C}[[x^{\frac{1}{n}}]][\ln(x)] \text{ and } S \notin x^{\frac{1}{n}} \cdot R_{0,n}. \tag{2.3}$$

Remark 2.3.4 *Generalized exponents will only be used in Section 3.3.*

Definition 2.3.5 *As in Definition 2.3.2, Definition 2.3.3 can be extended to $x = p$ ($p \in \mathbb{P}^1$) in the case of $e \in \mathbb{C}[t_p^{-\frac{1}{n}}]$.*

Remark 2.3.6 *If $e \in \mathbb{C}$, then Definition 2.3.1 and 2.3.3 coincide since $\exp(\int \frac{e}{x} \mathrm{d}x)$ is just x^e in that case.*

2.3.1 Case of Order 1

In this section, we explain generalized exponents of order 1 operators (in this case n is always 1 in Definition 2.3.3). This will be used to recover the exp-product transformation $\xrightarrow{(ii)}$. Let $L = \partial - r$ with $r \in \mathbb{C}(x)$.

Example 2.3.7 *The solution of $L = \partial - \frac{1}{x^2}$ at $x = 0$ is*

$$y = \exp(\int \frac{1}{x^2} \mathrm{d}x) = \exp(\int \frac{\frac{1}{x}}{x} \mathrm{d}x).$$

Then the generalized exponent of L at $x = 0$ is $\frac{1}{x} \in \mathbb{C}[x^{-1}]$.

Remark 2.3.8 *Let e be the exponent of L at $x = 0$. Then:*

- *L is* regular singular *(Definition 2.1.16) at $x = 0$ if $e \in \mathbb{C}$, then we usually write x^e for* $\exp \int \frac{e}{x}$.
- *L is* nonsingular *at $x = 0$ if $e = 0$.*
- *y is* meromorphic *at $x = 0 \Longleftrightarrow e \in \mathbb{Z}$.*

Definition 2.3.9 *Two (generalized) exponents $e_1, e_2 \in \mathbb{C}[t_p^{-1}]$ at $x = p \in \mathbb{P}^1$ are called* equivalent *if they differ by an integer, i.e.,*

$$e_1 \sim e_2 \Longleftrightarrow e_1 - e_2 \in \mathbb{Z}.$$

One can classify the solution $\exp(\int r)$ *of operator $L = \partial - r$ up to a meromorphic factor by taking the image of e in $\mathbb{C}[t_p^{-1}]/\mathbb{Z}$.*

Remark 2.3.10 *[13]*

- *Fuchs' relation: $L = \partial - r$ and the set of all singularities of L is $\{p_1, \ldots, p_n\} \subseteq \mathbb{P}^1$. Then*

$$\sum_{i=1}^{n} \mathrm{ConstTerm}(e_i) = 0, \tag{2.4}$$

where e_i is the generalized exponent of L at $x = p_i$. This is to say that the sum of residues of r is 0.

- *If the e_i are only given up to equivalence, then*

$$\sum_{i=1}^{n} \mathrm{ConstTerm}(e_i) \in \mathbb{Z}. \tag{2.5}$$

Partial Fraction Decomposition. This subsection discusses the relation between the partial fraction decomposition of r and the (generalized) exponents of $L = \partial - r$ at singularities.

Let $L = \partial - r$ with $r \in \mathbb{C}(x)$, then $y = \exp(\int r)$ is the solution of L. Rewrite the partial fraction decomposition

$$r = P(x) + \sum \sum \frac{a_{ij}}{(x - p_i)^j}$$

as

$$r = P(x) + \sum \frac{e_i}{t_i}$$

with $t_i = x - p_i$, $p_i \in \mathbb{C}$ and $e_i \in \mathbb{C}[t_i^{-1}]$. Here $\frac{e_i}{t_i}$ is the polar part of r at $x = p_i$ and describes the asymptotic behavior of y near p_i. For example, y is meromorphic at $x = p_i$ if and only if $e_i \in \mathbb{Z}$. As another example, if $e_i = \frac{5}{4}$, then y behaves as $(x - p_i)^{\frac{5}{4}} \cdot S(x)$ where $S(x)$ is analytic at $x = p_i$ with $S(p_i) \neq 0$. If e_i is not a constant, e.g, $e_i = \frac{1}{t_i}$, then y has an essential singularity at $x = p_i$ and $x = p_i$ is an irregular singularity (Definition 2.1.16) of L.

Remark 2.3.11 *For $p_i \in \mathbb{C}$, the generalized exponent e_i of $\partial - r$ at $x = p_i$ represents the polar part of r at $x = p_i$.*

Example 2.3.12 *If the (generalized) exponents of $\partial - r$ at singularities in $\mathbb{C}$ are:*

Points	*(Generalized) Exponents*
0	$1/2$
1	$3 + 1/t_1$
2	$5 + 1/t_2^2$

then $r = P(x) + \frac{1/2}{x} + \frac{3}{x-1} + \frac{1}{(x-1)^2} + \frac{5}{x-2} + \frac{1}{(x-2)^3}$, *here $P(x)$ is a polynomial.*

Now turn to the generalized exponent of $\partial - r$ at $x = \infty$. Suppose $P(x) = \sum_{i=0}^{n} a_i x^i$, then the solution at infinity is

$$\exp(\int r) = \exp(\sum_{i=0}^{n} \frac{a_i x^{i+1}}{i+1}) = \exp(\sum_{i=0}^{n} \frac{a_i}{(i+1)t_\infty^{i+1}}).$$

By the definition of the generalized exponent, $\exp(\sum_{i=0}^{n} \frac{a_i}{(i+1)t_\infty^{i+1}}) = \exp(\int \frac{e}{t_\infty} dt_\infty)$. So we can solve for $e_\infty = \sum_{i=0}^{n} \frac{-a_i}{t_\infty^{i+1}} + c$, here c is a constant and can be obtained by Fuchs' relation (2.4).

Example 2.3.13 *If $L = \partial - r$ with $r = \frac{3}{x} + \frac{5}{x^2} + \frac{7}{x-1} + \frac{5}{(x-1)^2} + 3 + 5x + 7x^2$, then the generalized exponents of L at $x = 0$, $x = 1$ and $x = \infty$ are:*

$$e_0 = 3 + \frac{5}{x},$$

$$e_1 = 7 + \frac{5}{x-1}.$$

$$e_\infty = -10 - 3x - 5x^2 - 7x^3.$$

The constant -10 in e_∞ is obtained by Fuchs' relation (2.4), or by:

Remark 2.3.14 *Another way to compute e_∞ is by applying the change of variables transformation with $x \mapsto \frac{1}{x}$ on L, then compute the generalized exponent of the new operator in $\mathbb{C}[x^{-1}]$ at $x = 0$, and then replace x by t_∞.*

Remark 2.3.15 *Conversely, if the generalized exponent of $\partial - r$ at $x = \infty$ is $\sum_{i=0}^{k} \frac{b_i}{t^i}$, then the polar part at $x = \infty$ is $\sum_{i=1}^{k} -b_i x^{i-1}$. If the generalized exponent at $x = p_i \in \mathbb{C}$ is e_i, then the polar part of r at $x = p_i$ is $\frac{e_i}{t_i}$ with $t_i = x - p_i$. So if given generalized exponents of $\partial - r$ at all singularities in $\mathbb{P}^1$, then r can be obtained by adding all polar parts at its singularities. (The exponent of $\partial - r$ at regular points is 0, therefore the polar part is also 0.)*

2.3.2 Case of Higher Order

For higher order operators, we focus on *regular singular operators* (all exponents in $\mathbb{C}$) because this is the case of the globally bounded operators.

Definition 2.3.16 *If the order of L_1 is greater than 1, then it has more than one exponents at every point $p \in \mathbb{P}^1$. Exponents of L_1 at $x = p$ (say set A) are equivalent to exponents of L_2 at $x = p$ (say set B) if there exists a one-to-one map $f : A \to B$ such that for any $a \in A$, $a \sim f(a) \in B$ by Definition 2.3.9. Use $\sim$ to denote this equivalence.*

Example 2.3.17 *Let $L_1 = \partial^3 + \frac{4(110x^2-21x-3)}{x(40x^2-37x-3)}\partial^2 + \frac{4}{9}\frac{2640x^2+142x-15}{(40x^2-37x-3)x^2)}\partial + \frac{880}{27}\frac{20x+3}{(40x^2-37x-3)x^2}$ and $L_2 = 9x^2(x-1)\partial^3 + (-18x + 81x^2)\partial^2 + (-2 + 162x)\partial + 54$. Exponents of L_1 and L_2 at $x = 0$ are:*

Operators	*Exponents at $x = 0$*
L_1	$0, -1/3, -2/3$
L_2	$0, 2/3, 1/3$

Then $\{0, -\frac{1}{3}, -\frac{2}{3}\} \sim \{0, \frac{2}{3}, \frac{1}{3}\}$ since $0 \sim 0$, $-\frac{1}{3} \sim \frac{2}{3}$ and $-\frac{2}{3} \sim \frac{1}{3}$.

2.4 Exponents and Transformations

This section discusses the relation between exponents (Definition 2.3.2) and transformations (Definition 2.1.8).

2.4.1 Exponents and Change of Variables Transformation

Let L_1 be a regular singular operator of order 3 and its exponents are:

Singularities	Exponents
p	$e_{p,1}, e_{p,2}, e_{p,3}$
q	$e_{q,1}, e_{q,2}, e_{q,3}$
r	$e_{r,1}, e_{r,2}, e_{r,3}$

Let the pullback function be $f = \frac{ax+b}{cx+d} \in \mathbb{C}(x)$. Say f sends $(\widetilde{p}, \widetilde{q}, \widetilde{r})$ to (p, q, r) if $f(\widetilde{p}) = p$, $f(\widetilde{q}) = q$, $f(\widetilde{r}) = r$. Let L_2 be the new operator: $L_1 \xrightarrow{(i),\, f} L_2$. Then by definition of change of variables ((i) in Definition 2.1.8), exponents of L_2 should be:

Singularities	Exponents
$\widetilde{p}$	$e_{p,1}, e_{p,2}, e_{p,3}$
$\widetilde{q}$	$e_{q,1}, e_{q,2}, e_{q,3}$
$\widetilde{r}$	$e_{r,1}, e_{r,2}, e_{r,3}$

Remark 2.4.1 *Given any two operators $L_1, L_2 \in D$, we can test if $L_1 \xrightarrow{(i)} L_2$ and obtain pullback function f of degree 1 (if exists) from their exponents.*

Example 2.4.2 *Suppose $L_1 \xrightarrow{(i),\, f} L_2$ with some degree 1 pullback function f and exponents of L_1 and L_2 are:*

Singularities of L_1	*Exponents*	*Singularities of L_2*
0	$0,\ 1-b_1,\ 1-b_2$	2
1	$0,\ 1,\ b_1+b_2-a_1-a_2-a_3$	∞
∞	$a_1,\ a_2,\ a_3$	-3

So f sends $(2, \infty, -3)$ to $(0, 1, \infty)$, which implies $f = \frac{x-2}{x+3}$.

2.4.2 Exponents and Exp-product Transformation

Suppose $L \in D$ is of order n. Let $p \in \mathbb{P}^1$ and $e_{p,1}, \ldots, e_{p,n}$ be exponent(s) of L at $x = p$. Let $d_p \in \mathbb{C}[t_p^{-1}]$ be the (generalized) exponent of $\partial - r$ at $x = p$ with $r \in \mathbb{C}(x)$. By (ii) in Definition 2.1.8, exp-product transformation $\xrightarrow{(ii),\, r}$ sends y (solution of L) to $y \cdot \exp(\int r)$, so exponents of $L \circledS (\partial - r)$ at $x = p$ are $e_{p,1} + d_p, \ldots, e_{p,n} + d_p$.

Remark 2.4.3 *If $L_1 \xrightarrow{(ii)} L_2$, then by comparing exponents of L_1 and L_2 at their singularities, we can obtain the exponents of $\partial - r$ at those points, therefore obtain r as described in Remark 2.3.15.*

Example 2.4.4 *The (generalized) exponents of L_1 and L_2 at their singularities are:*

Singularities	*Exponents of* L_1	*(Generalized) Exponents of* L_2
0	0, 0	2, 2
1	0, −1	0, −1
2	1, 2	$1+1/(x-2),\ 2+1/(x-2)$
∞	0, 0	−2, −2

Try to find $r \in \mathbb{C}(x)$ *such that* $L_1 \xrightarrow{(ii),r} L_2$.

- *At* $x = 0$, $[0,0]$ *and* $[2,2] \Rightarrow d_0 = 2 \Rightarrow r_0 = \frac{2}{x}$.
- *At* $x = 1$, $[0,-1]$ *and* $[0,-1] \Rightarrow d_1 = 0 \Rightarrow r_1 = 0$.
- *At* $x = 2$, $[1,2]$ *and* $[1+\frac{1}{x-2}, 2+\frac{1}{x-2}] \Rightarrow d_2 = \frac{1}{x-2} \Rightarrow r_2 = \frac{1}{(x-2)^2}$.
- *At* $x = \infty$, $[0,0]$ *and* $[-2,-2] \Rightarrow d_\infty = -2 \Rightarrow r_\infty = 0$.
- $[d_0, d_1, d_2, d_\infty] = [2, 0, \frac{1}{x-2}, -2]$ *satisfies Fuchs' relation* (2.4). *So* $r = \frac{2}{x} + \frac{1}{(x-2)^2}$.

2.4.3 Exponents and Gauge Transformation

Remark 2.4.5 *[32] Operators* L_1 *and* L_2 *are gauge equivalent* $\Longrightarrow$ *Exponents of* L_1 *and* L_2 *are equivalent at any* $p \in \mathbb{P}^1$.

Remark 2.4.6 *In maple, "Homomorphisms" command in DEtools package checks if two operators are gauge equivalent and return gauge transformations (*$\xrightarrow{(iii)}$*) if exist(s).*

CHAPTER 3

COMPUTE ${}_3F_2$-TYPE SOLUTIONS WITH PULLBACK FUNCTIONS OF DEGREE ONE

For an order 2 operator L, to compute its ${}_2F_1$-type solution containing ${}_2F_1(a, b; c \,|\, f)$ is equivalent to:

- Task 1. Compute a, b, c and the pullback function f in transformation $\xrightarrow{(i)}$ (Definition 2.1.8).
- Task 2. Let M be the minimal operator of $y = {}_2F_1(a, b; c \,|\, f)$, compute $M \xrightarrow{(ii),(iii)} L$ which means to compute $r \in \mathbb{C}(x)$, $G \in \mathbb{C}(x)[\partial]$ such that $\exp(\int r) \cdot G(y) \in V(L)$ for all $y \in V(M)$.

We have Maple program for task 2 on order 2 operators [27] which deals with transformations (ii)+(iii), so the key remaining task is task 1. But for order 3, we do have code for (ii) or (iii) (2.4.6), but not (ii)+(iii). So for ${}_3F_2$-solver in this chapter, we have to start with algorithm for (ii)+(iii). That means to perform task 2, we need a program to find $\exp(\int r)$ for order 3 (Section 3.1). The hypergeometric function is ${}_3F_2(a_1, a_2, a_3; b_1, b_2 \,|\, f)$. So task 1 is to find a_1, a_2, a_3, b_1, b_2 and f (Section 3.2).

Let L_1 be the minimal operator of ${}_3F_2(a_1, a_2, a_3; b_1, b_2 \,|\, x)$ and L_2 be an order 3 operator with ${}_3F_2$-type solutions. Following table shows that (generalized) exponents of L_2 at any $x = p \in \mathbb{P}^1$ are in $\mathbb{C}[t_p^{-1}]$.

(Resulting) Operators	L_1	$L_1 \xrightarrow{(i),f}$	$L_1 \xrightarrow{(i),f} \xrightarrow{(ii),r}$	$L_1 \xrightarrow{(i),f} \xrightarrow{(ii),(iii)}$
(Generalized) Exponents at $x = p \in \mathbb{P}^1$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}[t_p^{-1}]$	$\mathbb{C}[t_p^{-1}]$

So to find r in task 2 (Section 3.1), we restrict operators with generalized exponents in $\mathbb{C}[t_p^{-1}]$. We also implemented a general algorithm (Section 3.3) to deal with other cases: there exist generalized exponents in $\mathbb{C}[t_p^{-\frac{1}{2}}]$ or $\mathbb{C}[t_p^{-\frac{1}{3}}]$.

3.1 Find r in $L_1 \xrightarrow{(ii),r} \xrightarrow{(iii)} L_2$ in ${}_3F_2$-solver

This section deals with the case when all (generalized) exponents of L_1 and L_2 are in $\mathbb{C}[t_p^{-1}]$ at any $p \in \mathbb{P}^1$. Suppose at $x = p \in \mathbb{P}^1$, the (generalized) exponent set of L_1 is A_p and that of L_2 is B_p. Then by Section 2.4.2 and 2.4.3,

$$L_1 \xrightarrow{(ii),(iii)} L_2 \Longleftrightarrow \exists\, d_p \in \mathbb{C}[t_p^{-1}] \text{ s.t. } A_p + d_p \sim B_p \text{ at any } p \in \mathbb{P}^1.$$

Algorithm 3.1.1 *Find Difference(s) d_p at $x = p$.*

- *Input: two (generalized) exponent sets $A_p = \{e_{p,1}, e_{p,2}, e_{p,3}\}$ and $B_p = \{\widetilde{e_{p,1}}, \widetilde{e_{p,2}}, \widetilde{e_{p,3}}\}$.*
- *Output: the set of all differences $d_p \bmod \mathbb{Z}$ ($d_p \in \mathbb{C}[t_p^{-1}]$) such that $A_p + d_p \sim B_p$ if exists.*
- *Steps: for each candidate $d_p \in \{\widetilde{e_{p,1}} - e_{p,1},\ \widetilde{e_{p,1}} - e_{p,2},\ \widetilde{e_{p,1}} - e_{p,3}\}$, check if $A_p + d_p \sim B_p$. Return all such $d_p \bmod \mathbb{Z}$ as a set.*
- *Comment: focus on $d_p \bmod \mathbb{Z}$ rather than d_p since $\xrightarrow{(iii)}$ shifts exponents by integers (Section 2.4.3).*

Algorithm 3.1.2 *Find r in $L_1 \xrightarrow{(ii),r} \xrightarrow{(iii)} L_2$.*

- *Input: two operators $L_1, L_2 \in K[\partial]$.*
- *Output: the set of candidates $r \in \mathbb{C}(x)$ such that $L_1 \xrightarrow{(ii),r} \xrightarrow{(iii)} L_2$.*
- *Steps.*
  - *1. Find all singularities of L_1 and L_2 : $S = \{p_1, p_2, \ldots, p_n\}$.*
  - *2. For each $p_i \in S$, use Algorithm 3.1.1 to find $d_{p_i} \in \mathbb{C}[t_{p_i}^{-1}]$ between (generalized) exponents of L_1 and L_2. Denote the set of such difference(s) as D_{p_i}. Now we have $D_{p_1}, \ldots, D_{p_n}$.*
  - *3. For each $(d_1, \ldots, d_n) \in D_{p_1} \times \ldots \times D_{p_n}$ do: if $(d_1, \ldots, d_n)$ satisfies Fuchs' Relation (2.5), then the sum of all polar parts corresponding to $(d_1, \ldots, d_n)$ is a candidate for r as in Remark 2.3.15.*
- *Comment: command "gen-exp" under DEtools package in Maple returns the (generalized) exponents of the given operator at the given point $p \in \mathbb{P}^1$.*

Example 3.1.3 *(Generalized) Exponents of L_1 and L_2 at their singularities are:*

Singularities	*Exponents of L_1*	*(Generalized) Exponents of L_2*
0	0, 1/2, 2/3	$-4 + 3/x,\ -7/2 + 3/x,\ -10/3 + 3/x$
1	0, 1, −31/6	0, 1, −43/6
2	0, 1, 2	−3/2, −1/2, 1/2
∞	1, 2, 3	1/2, 3/2, 5/2

Table 3.1: Exponents at Singularities of L_1

Singularities	Exponents	Logarithmic[1]
0	$0,\ 1-b_1,\ 1-b_2$	when $\{b_1, b_2, b_1-b_2\} \cap \mathbb{Z} \neq \emptyset$
1	$0,\ 1,\ b_1+b_2-a_1-a_2-a_3$	when $b_1+b_2-a_1-a_2-a_3 \in \mathbb{Z}$
∞	$a_1,\ a_2,\ a_3$	when $\{a_1-a_2, a_1-a_3, a_2-a_3\} \cap \mathbb{Z} \neq \emptyset$

At $x = 0$,

$$\{0, \frac{1}{2}, \frac{2}{3}\} + \frac{3}{x} \sim \{-4 + \frac{3}{x}, -\frac{7}{2} + \frac{3}{x}, -\frac{10}{3} + \frac{3}{x}\}$$

so $d_0 = \frac{3}{x} \bmod \mathbb{Z}$. Likewise, $d_1 = 0 \bmod \mathbb{Z}$, $d_2 = \frac{1}{2} \bmod \mathbb{Z}$, $d_\infty = \frac{1}{2} \bmod \mathbb{Z}$. Now $(d_0, d_1, d_2, d_\infty) = (\frac{3}{x}, 0, \frac{1}{2}, \frac{1}{2})$ satisfies Fuchs' Relation (2.5), so $\frac{1/2}{x-2} + \frac{3}{x^2}$ is one candidate for r.

3.2 Compute ${}_3F_2$-type Solutions with Pullback Functions of Degree One

Recall that our task is to compute ${}_3F_2$-type solutions of any irreducible order 3 operator L_2,

$$L_1 \xrightarrow{(i),\, f} \xrightarrow{(ii),\,(iii)} L_2, \tag{3.1}$$

where L_1 is the minimal operator of ${}_3F_2(a_1, a_2, a_3; b_1, b_2 \,|\, x)$. Now with Algorithm 3.1.2 and command "Homomorphisms" in Maple (Remark 2.4.6) which deals with the second part in (3.1), our task is reduced to finding M, or equivalently, L_1 and the pullback function f. We start with the case that the degree of f is 1.

Table 3.1 gives exponents at all non-removable singularities of L_1.

To compute ${}_3F_2$-type solutions, the essential part is how three types of transformations affect (generalized) exponents[2], which was discussed in Section 2.4.

Remark 3.2.1

- *Change of variables transformation: suppose the pullback function $f = \frac{ax+b}{cx+d}$ sends (p, q, r) to $(0, 1, \infty)$ with different $p, q, r \in \mathbb{P}^1$, i.e., $f(p) = 0$, $f(q) = 1$ and $f(r) = \infty$, then the table of $L_1 \xrightarrow{(i),\, f}$ is:*

Singularities	*Exponents*	*Logarithmic*
p	$0,\ 1-b_1,\ 1-b_2$	*when* $\{b_1, b_2, b_1-b_2\} \cap \mathbb{Z} \neq \emptyset$
q	$0,\ 1,\ b_1+b_2-a_1-a_2-a_3$	*when* $b_1+b_2-a_1-a_2-a_3 \in \mathbb{Z}$
r	$a_1,\ a_2,\ a_3$	*when* $\{a_1-a_2, a_1-a_3, a_2-a_3\} \cap \mathbb{Z} \neq \emptyset$

[2] If a generalized exponent is a constant then it is called an exponent

- *Exp-product transformation* $\xrightarrow{(ii),r}$ *adds the same* $d_p \in \mathbb{C}[t_p^{-1}]$ *to each exponent at* $x = p \in \mathbb{P}^1$, *but this does not change exponent differences. The table of* $L_1 \xrightarrow{(i),f} \xrightarrow{(ii),r}$ *is:*

Singularities	*(Generalized) Exponents*
p	$(d_p, d_p, d_p) + (0, 1-b_1, 1-b_2)$
q	$(d_q, d_q, d_q) + (0, 1, b_1+b_2-a_1-a_2-a_3)$
r	$(d_r, d_r, d_r) + (a_1, a_2, a_3)$

- *Gauge transformation* $\xrightarrow{(iii),G}$ *only shifts generalized exponents in* $\mathbb{C}[t_p^{-1}]$ *by integers. So the table of* $L_1 \xrightarrow{(i),f} \xrightarrow{(ii),r} \to (iii), G$ *is:*

Singularities	*(Generalized) Exponents*
p	$(d_p, d_p, d_p) + (0, 1-b_1, 1-b_2) + (n_1, n_2, n_3)$
q	$(d_q, d_q, d_q) + (0, 1, b_1+b_2-a_1-a_2-a_3) + (n_4, n_5, n_6)$
r	$(d_r, d_r, d_r) + (a_1, a_2, a_3) + (n_7, n_8, n_9)$

Here $n_1, \ldots, n_9$ *could be any integer.*

Example 3.2.2 *Let* $L_2 = -6(2x+1)^2(x+6)(x-5)^5\partial^3 + (-72x^9 + 1332x^8 - 6642x^7 - 17927x^6 + 266074x^5 - 771517x^4 + 423090x^3 + 485575x^2 + 914750x + 445625)\partial^2 + (-72x^{10} + 1368x^9 - 7524x^8 - 9232x^7 + (450971/2)x^6 - (1425263/2)x^5 + 641162x^4 - 277031x^3 + (1137955/2)x^2 + (5503725/2)x + 1243875)\partial + (-24x^{11} + 468x^{10} - 2832x^9 + 448x^8 + (113859/2)x^7 - (817465/4)x^6 + 278133x^5 - (1152739/4)x^4 - (854729/2)x^3 + (10070009/4)x^2 + 757190x + 4259475/4)$, *its exponents at non-removable singularities are shown below. Does it have* ${}_3F_2$*-type solutions?*

Singularities	Exponents	Logarithmic
-6	0, 1, $-31/6$	No
$-1/2$	0, 1/2, 2/3	No
5	1, 2, 3	Yes

First, find all possible (p, q, r) such that $f(p) = 0$, $f(q) = 1$ and $f(r) = \infty$ with some pullback function f. In the following discussion, we say a singularity s is a candidate for p (q, r) if it is possible that $f(s) = 0$ $(1, \infty)$. From Table 3.1, exponents at $x = 1$ are $(0, 1, b_1 + b_2 - a_1 - a_2 - a_3)$, so exponents at candidate for q must be $(e_1, e_1 + n, e_2)$ with some integer n. Any singularity is a candidate for p and r.

Now exponents of L_2 at -6 and 5 can be written as $(e_1, e_1 + n, e_2)$. So they may be candidates for q. But -6 (without logarithmic solution) can only come from 1 because in Table 3.1, an integer exponent difference at $x = 0$ ($x = \infty$) implies a logarithmic solution. So all possible (p, q, r) are $(-\frac{1}{2}, -6, 5)$ and $(5, -6, -\frac{1}{2})$ and their corresponding pullback functions are $\frac{2x+1}{x-5}$ and $\frac{x-5}{2x+1}$ respectively. Next, compute all parameters a_1, a_2, a_3, b_1 and b_2 for each candidate.

Algorithm 3.2.3 *Compute Parameters corresponding to* (p, q, r).

- *Input: one candidate of* (p, q, r).
- *Output: corresponding parameters* a_1, a_2, a_3, b_1 *and* b_2.
- *Steps.*
  - *1. Find exponents at these singularities, say:*

Singularities	*Exponents*
p	$e_{p,1}, e_{p,2}, e_{p,3}$
q	$e_{q,1}, e_{q,2}, e_{q,3}$
r	$e_{r,1}, e_{r,2}, e_{r,3}$

    Here $e_{q,2} = e_{q,1} + n$ *for some integer* n.

  - *2. Subtract* $d_q := e_{q,1}$ *from exponents at* $x = q$*:*

$$[e_{q,1}, e_{q,2}, e_{q,3}] \xrightarrow{-d_q} [0, n, e_{q,3} - e_{q,1}].$$

    Subtract $d_p := e_{p,1}$ *(*d_p *could also be* $e_{p,2}$ *or* $e_{p,3}$*) from exponents at* $x = p$*:*

$$[e_{p,1}, e_{p,2}, e_{p,3}] \xrightarrow{-d_p} [0, e_{p,2} - e_{p,1}, e_{p,3} - e_{p,1}].$$

    Since the sum of all exponents is a constant, so $d_p + d_q$ *needs to be added to exponents at* $x = r$*:*

$$[e_{r,1}, e_{r,2}, e_{r,3}] \xrightarrow{d_p+d_q} [e_{r,1} + e_{p,1} + e_{q,1}, e_{r,2} + e_{p,1} + e_{q,1}, e_{r,3} + e_{p,1} + e_{q,1}].$$

  - *3. Now compare new exponents with those of* L_1 *in Table 3.1, then* $\{a_1, a_2, a_3\} = \{e_{r,1} + e_{p,1} + e_{q,1}, e_{r,2} + e_{p,1} + e_{q,1}, e_{r,3} + e_{p,1} + e_{q,1}\}$ *and* b_1, b_2 *can be obtained by solving* $\{1 - b_1, 1 - b_2\} = \{e_{p,2} - e_{p,1}, e_{p,3} - e_{p,1}\}$.

Continue with Example 3.2.2. Here we just deal with one candidate $(-\frac{1}{2}, -6, 5)$. The pullback function is $f = \frac{2x+1}{x-5}$. There are three cases (other cases are equivalent to these up to gauge transformations) in parameters for this candidate.

Details of first case are shown below. So $\{a_1, a_2, a_3\} = \{1, 2, 3\}$ and $\{b_1, b_2\} = \{\frac{1}{2}, \frac{1}{3}\}$.

Singularities	Exponents	Differences	New Exponents
$-1/2$	$0, 2/3, 1/2$	0	$0, 2/3, 1/2$
-6	$0, 1, -31/6$	0	$0, 1, -31/6$
5	$1, 2, 3$	0	$1, 2, 3$

Likewise for the second case: $\{a_1, a_2, a_3\} = \{\frac{5}{3}, \frac{8}{3}, \frac{11}{3}\}$ and $\{b_1, b_2\} = \{\frac{5}{3}, \frac{7}{6}\}$.

Singularities	Exponents	Differences	New Exponents
$-1/2$	$0, 2/3, 1/2$	$-2/3$	$-2/3, 0, -1/6$
-6	$0, 1, -31/6$	0	$0, 1, -31/6$
5	$1, 2, 3$	$2/3$	$5/3, 8/3, 11/3$

Likewise for the third case: $\{a_1, a_2, a_3\} = \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}\}$ and $\{b_1, b_2\} = \{\frac{3}{2}, \frac{5}{6}\}$.

Singularities	Exponents	Differences	New Exponents
$-1/2$	$0, 2/3, 1/2$	$-1/2$	$-1/2, 1/6, 0$
-6	$0, 1, -31/6$	0	$0, 1, -31/6$
5	$1, 2, 3$	$1/2$	$3/2, 5/2, 7/2$

Now the minimal operator L_1 corresponding to each candidate $[a_1, a_2, a_3; b_1, b_2]$ can be obtained. After applying on L_1 a change of variables transformation with the pullback function f, use Algorithm 3.1.2 to find r and "Homomorphisms" command in Maple to find G in $L_1 \xrightarrow{(i), f} \xrightarrow{(ii), r} \xrightarrow{(iii), G} L_2$. Now in Example 3.2.2, we obtain

$$[[1, 2, 3, \frac{1}{3}, \frac{1}{2}], \frac{2x+1}{x-5}, \frac{1}{x+6} + \frac{2}{x-5} + \frac{1}{2} - x, \frac{1}{x^3 - 4x^2 - 35x + 150}],$$

where the first entry consists of parameters in ${}_3F_2$ function, the second is the pullback function f, the third is the parameter r in the exp-product transformation and the last one is the parameter G in the gauge transformation.

We could generalize ${}_3F_2$-solver to f of degree 2 or 3 as in Vijay's thesis [23]. However, it is still not enough for order 3 since there exist many other hypergeometric functions.

3.3 Find r in $L_1 \xrightarrow{(ii), r} \xrightarrow{(iii)} L_2$ for any Order 3 Operators

Recall that a generalized exponent at $x = p \in \mathbb{P}^1$ can be any element in $\mathbb{C}[t_p^{-\frac{1}{n}}]$ with $n \in \mathbb{N}$ (Definition 2.3.5).

Definition 3.3.1 *Let* $E := \bigcup_{n \geq 1} \mathbb{C}[x^{-\frac{1}{n}}]$.

- *For* $e \in E$, *define the* ramification *of* e *as* $\min\{n \mid e \in \mathbb{C}[x^{-\frac{1}{n}}]\}$ *and denote it as* $ram(e)$. *It is unramified if* $ram(e) = 1$. *Define*
$$V_e = \exp(\int \frac{e}{x})\mathbb{C}((x))[e, \ln(x)] = \exp(\int \frac{e}{x})\mathbb{C}((x^{\frac{1}{n}}))[\ln(x)],$$
where $n = ram(e)$.

- *Let* e_1, $e_2 \in E$, *then* e_1 *is* equivalent *to* e_2 ($e_1 \sim e_2$) *if* $V_{e_1} = V_{e_2}$. *This holds if and only if* $e_1 - e_2 \in \frac{1}{n}\mathbb{Z}$ *where* $n = ram(e_1)$. *Note that this coincides with Definition 2.3.9 when* e_1, $e_2 \in \mathbb{C}[t_p^{-1}]$, *i.e., ramification of them is* 1.

Remark 3.3.2

- V_e *is a* $\mathbb{C}(x)[\partial]$*-module, which is to say,* $G(V_e) \subseteq V_e$ *for any* $G \in \mathbb{C}(x)[\partial]$. *So a gauge transformation sends a generalized exponent to its equivalent generalized exponent.*

- *Let* $V = \bigoplus_{e \in E/\sim} V_e$. *One can show that this is the universal extension of* $\mathbb{C}((x))$ *in the sense of Definition 2.1.3. Thus we can take* $V(L) := \mathrm{Ker}(L : V \to V)$ *from Definition 2.1.3. Denote* $V_e(L) := \mathrm{Ker}(L : V_e \to V_e) = V(L) \cap V_e$, *then* $V(L) = \bigoplus_{e \in E/\sim} V_e(L)$.

Example 3.3.3 *At* $x = \infty$, *generalized exponents of* L_1 *are* $A := \{0, \frac{1}{\sqrt{-t_\infty}} - \frac{3}{4}, -\frac{1}{\sqrt{-t_\infty}} - \frac{3}{4}\}$ *and those of* L_2 *are* $B := \{1, \frac{1}{\sqrt{-t_\infty}} - \frac{1}{4}, -\frac{1}{\sqrt{-t_\infty}} - \frac{1}{4}\}$. *By Definition 2.3.16,* $A \sim B$ *since*
$$0 \sim 1$$
$$\frac{1}{\sqrt{-t_\infty}} - \frac{3}{4} \sim \frac{1}{\sqrt{-t_\infty}} - \frac{1}{4}$$
$$-\frac{1}{\sqrt{-t_\infty}} - \frac{3}{4} \sim -\frac{1}{\sqrt{-t_\infty}} - \frac{1}{4}.$$
The last two equivalences hold because both differences is $\frac{1}{2}$ *and the ramification of them is* 2.

The task of this section is to find r in
$$L_1 \xrightarrow{(ii),\, r} \xrightarrow{(iii)} L_2$$
where L_1 and L_2 have some generalized exponents with ramification greater than 1. Example 3.3.3 shows that finding difference between ramified generalized exponent sets is different from that of unramified sets (Algorithm 3.1.1). So the algorithm that parallels with Algorithm 3.1.1 is needed: denote generalized exponent set of L_1 at $x = p \in \mathbb{P}^1$ as A and that of L_2 as B, there are two cases to consider:

- Two elements in A (B) have ramification 2 (Section 3.3.1).
- All elements in A (B) have ramification 3 (Section 3.3.2).

3.3.1 Case with Ramification 2

In this case there are two generalized exponents of L_1 at $x = p$ with ramification 2 and the other one unramified. Let $e_1 \in \mathbb{C}[t_p^{-1}]$ and $e_2, e_3 \in \mathbb{C}[t_p^{-\frac{1}{2}}] - \mathbb{C}[t_p^{-1}]$. Denote this case as of type $(1, 2)$. In this case, the output of "gen-exp" command in Maple can not be used directly since the notations are inconsistent (Example 3.3.4 below).

Example 3.3.4 *At $x = \infty$, "gen-exp" command in Maple gives the generalized exponents of $L_1 = (x-1)^2\partial^3 + x\partial$ and $L_2 = (x-1)^2\partial^3 + (3x^7 - 6x^6 + 3x^5)\partial^2 + (3x^{12} - 6x^{11} + 3x^{10} + 15x^6 + x - 30x^5 + 15x^4)\partial + (x^{17} - 2x^{16} + x^{15} + 15x^{11} - 30x^{10} + 15x^9 + x^6 + 20x^5 - 40x^4 + 20x^3)$ as shown below.*

	$T = t_\infty = 1/x$	$-T^2 = t_\infty = 1/x$
L_1	0	$1/T - 3/4$
L_2	$1/T^6$	$1/T - 3/4 + 1/T^{12}$

Let $A := \{e_1, e_2, e_3\}$ be the generalized exponents of L_1 at $x = \infty$ and $B := \{\widetilde{e_1}, \widetilde{e_2}, \widetilde{e_3}\}$ be the generalized exponents of L_2 at $x = \infty$ with e_1, $\widetilde{e_1}$ unramified. Our goal is to find the difference between them. Since the ramification is invariant under $\xrightarrow{(ii),(iii)}$ transformations, so to find difference between A and B is equivalent to computing $\widetilde{e_1} - e_1 \bmod \mathbb{Z}$ and $\widetilde{e_2} - e_2 \bmod \frac{1}{2}\mathbb{Z}$. Now $p = \infty$, $e_1 = 0$, $\widetilde{e_1} = \frac{1}{T^6}$ and $\widetilde{e_1} - e_1 \bmod \mathbb{Z} = \frac{1}{T^6} \bmod \mathbb{Z}$, so

$$d_p := \widetilde{e_1} - e_1 = \frac{1}{T^6} = \frac{1}{t_\infty^6}.$$

But note that T in the third column is not the same as T in the second column. So for further computation, we need to make them consistent. Compute the minpoly of $\frac{1}{T} - \frac{3}{4}$ (denoted as m_2) and minpoly of $\frac{1}{T} - \frac{3}{4} + \frac{1}{T^{12}}$ (denoted as $\widetilde{m_2}$) in the field extension from $\mathbb{C}(t_\infty)$ to $\mathbb{C}(T)$, here $-T^2 = t_\infty$.

$$m_2 = X^2 + \frac{3}{2}X + \frac{9}{16} + \frac{1}{t_\infty},$$

$$\widetilde{m_2} = X^2 + (-\frac{2}{t_\infty^6} + \frac{3}{2})X + (\frac{9}{16} + \frac{1}{t_\infty} - \frac{3}{2t_\infty^6} + \frac{1}{t_\infty^{12}}).$$

Now compute the difference between e_2 and $\widetilde{e_2}$ from the coefficients of two minpolys with respect to X,

$$d_p' := \frac{\frac{3}{2} - (-\frac{2}{t_\infty^6} + \frac{3}{2})}{2} = \frac{1}{t_\infty^6} = d_p.$$

And $\widetilde{m_2}(X + d_p') = m_2$, so the difference is $d_p = \frac{1}{t_\infty^6}$.

Algorithm 3.3.5 *Compute the Difference at $x = p$ with type $(1, 2)$.*

- *Input: two irreducible order 3 operators L_1, $L_2 \in K[\partial]$ and one singularity $x = p$ of L_1 or L_2 of type $(1, 2)$.*
- *Output: the difference d_p between the generalized exponents of L_1 and L_2 at $x = p$.*
- *Steps.*
  - *1. Use "gen-exp" command in Maple to find generalized exponents of L_1 and L_2 at $x = p$. Denote the outputs as $[e_1, T = t_p], [e_2, c_1T^2 = t_p]$ and $[\widetilde{e_1}, T = t_p], [\widetilde{e_2}, c_2T^2 = t_p]$ for some constants c_1 and c_2.*
  - *2. Let $d_p = \widetilde{e_1} - e_1$. Compute the minpolys of e_2 and $\widetilde{e_2}$ over $\mathbb{C}(t_p)$ and denote them as m_2, $\widetilde{m_2} \in \mathbb{C}(t_p)[X]$. Say $m_2 = X^2 + a_1X + a_0$ and $\widetilde{m_2} = X^2 + b_1X + b_0$, let $d'_p = \frac{a_1 - b_1}{2}$, then check if $d_p \equiv d'_p \bmod \frac{1}{2}\mathbb{Z}$ and $\widetilde{m_2}(X + d'_p) = m_2$. If so, return d_p, otherwise return "not projectively equivalent".*

3.3.2 Case with Ramification 3

In this case, three generalized exponents of L_1 at $x = p$ have ramification 3. Denote this case as of type (3). The output of "gen-exp" on L_1 at $x = p$ is $[e, c_1T^3 = t_p]$ for some constant c_1. If $L_1 \xrightarrow{(ii),(iii)} L_2$, then generalized exponents of L_2 at $x = p$ has to be of type (3) and the output of "gen-exp" on L_2 at $x = p$ must be $[\widetilde{e}, c_2T^3 = t_p]$ with $c_2 \in \mathbb{C}$.

Remark 3.3.6 *The algorithm of computing the difference at $x = p$ of type (3) is similar to Algorithm 3.3.5 except now the minpolys have degree 3. Let m and $\widetilde{m}$ be minpolys of e and $\widetilde{e}$ over $\mathbb{C}(t_p)[X]$. Let $d_p = \frac{a_2 - b_2}{3}$ where a_2 and b_2 are coefficients of m and $\widetilde{m}$ with respect to X^2. We only need to check if $\widetilde{m}(X + d_p) = m$, if yes, then return 3 differences: d_p, $d_p - \frac{1}{3}$ and $d_p + \frac{1}{3}$, otherwise return "not projectively equivalent".*

Example 3.3.7 *At $x = 0$, the generalized exponents of $L_1 = x^5\partial^3 + x$ and $L_2 = x^5\partial^3 - \frac{3}{2}x(x^2 - 4)\partial^2 + (3(4x^5 + x^4 - 32x^3 - 8x^2 + 16)/4x^3)\partial - (16x^8 + 12x^7 - 319x^6 - 144x^5 - 12x^4 + 384x^3 + 48x^2 - 64)/(8x^7)$ are:*

	L_1	L_2
$-T^3 = t_0 = x$	$1/T + 4/3$	$1/T + 4/3 - 1/2T^3 + 2/T^9$

So $p = 0$, $e = \frac{1}{T} + \frac{4}{3}$ and $\widetilde{e} = \frac{1}{T} + \frac{4}{3} - \frac{1}{2T^3} + \frac{2}{T^9}$. Therefore

$$m = X^3 - 4X^2 + \frac{16X}{3} - \frac{64x - 27}{27t_0},$$

$\widetilde{m} = X^3 - (4 + \frac{3}{2t_0} - \frac{6}{t_0^3})X^2 + (64t_0^6 + 48t_0^5 + 9t_0^4 - 192t_0^3 - 72t_0^2 + 144)/(12t_0^6)X - (512t_0^9 + 360t_0^8 + 216t_0^7 - 2277t_0^6 - 1728t_0^5 - 324t_0^4 + 3456t_0^3 + 1296t_0^2 - 1728)/(216t_0^9)$.

Then $d_p = \frac{a_2 - b_2}{3} = \frac{1}{2t_0} - \frac{2}{t_0^3}$ *and* $\widetilde{m}(X + d_p) = m$. *So differences are*

$$\{\frac{1}{2t_0} - \frac{2}{t_0^3}, \frac{1}{2t_0} - \frac{2}{t_0^3} + \frac{1}{3}, \frac{1}{2t_0} - \frac{2}{t_0^3} - \frac{1}{3}\}.$$

CHAPTER 4

COMPUTE F_1-TYPE SOLUTIONS WITH PULLBACK FUNCTIONS OF DEGREE ONE

The most well known example of A-hypergeometric function that is not univariate is Appell's F_1 function. It is a generalization of Gauss hypergeometric series ${}_2F_1(a, b, c \,|\, x)$ and defined by:

$$F_1(a, b_1, b_2, c \,|\, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!} x^m y^n.$$

It satisfies a system of bivariate differential equations. If x and y are replaced by univariate functions, then the result satisfies a univariate differential equation of order 3.

Example 4.0.8 *Let*

$$\begin{aligned} L = \partial^3 &+ \frac{225x^3 - 140x^2 + 21x + 18}{15x(x-1)(x+1)(5x-3)}\partial^2 + \frac{4375x^4 - 10125x^3 - 660x^2 + 8235x - 1701}{900x^2(x+1)(x-1)^2(5x-3)}\partial \\ &+ \frac{625x^3 + 3200x^2 - 4995x + 1134}{600x^3(x+1)(x-1)^2(5x-3)}. \end{aligned} \tag{4.1}$$

One solution of $L(y) = 0$ at $x = 0$ is

$$r \cdot F_1(a, b_1, b_2, c \,|\, u, v) \tag{4.2}$$

where $a = \frac{1}{2}, b_1 = \frac{1}{3}, b_2 = \frac{1}{5}, c = 1,\ u = x, v = \frac{1}{x}$ and $r = \frac{1}{\sqrt{x}}$.

Our F_1-solver computes solutions in Example 4.0.8 by following steps.

- Compute candidates of functions u, v. (Section 4.1)
- Divide the candidate set of $[u, v]$ into orbits. (Section 4.2)
- For each orbit, pick one element and compute parameters a, b_1, b_2, c in (4.2), exp-product transformation and gauge transformation which send L_c to L. Here L_c is the minimal operator of $F_1(a, b_1, b_2, c \,|\, u, v)$ (Section 4.3).

4.1 Compute Candidates u, v in $F_1(a, b_1, b_2, c \,|\, u, v)$

If $u, v \in \mathbb{C}(x)$, then the minimal operator L_c of $F_1(a, b_1, b_2, c \,|\, u, v)$ can be computed by the implementation [26]. If L_c can be transformed to L by some exp-product transformations and gauge transformations ($L_c \xrightarrow{(ii),(iii)} L$), then they have the same *non-removable singularities* (Definition 2.1.14). Let $R(u, v)$ be the set of roots of u, $1-u$, $\frac{1}{u}$, v, $1-v$, $\frac{1}{v}$ and $u-v$, all non-removable singularities of L_c. To find candidates of u,v, we search for $u, v \in \mathbb{C}(x)$ such that $R(u, v) = A$ where A is the non-removable singularity set of L. At the moment we restrict u and v to degree 1, which is the most simple case. Later on we may generalize this to higher degrees.

First, compute all candidates of u.

Algorithm 4.1.1 *CandidateU*

- *Input: the non-removable singularity set, A, of the order 3 operator L.*
- *Output: the set of all Möbius transformations $f : \mathbb{P} \mapsto \mathbb{P}$ with $\{0, 1, \infty\} \subseteq f(A)$.*
- *Steps: list all combinations of 3 different elements $p, q, r \in A$. For each combination $[p, q, r]$, compute the Möbius transformation f such that $f(p) = 0$, $f(q) = 1$ and $f(r) = \infty$. Return the set of all such f.*

Example 4.1.2 *Continue with Example 4.0.8. The non-removable singularity set of L is $A = \{-1, 0, 1, \infty\}$. Note that $x = \frac{3}{5}$ is an apparent singularity (Definition 2.1.16) since the solutions at $x = \frac{3}{5}$ are analytic. Algorithm 4.1.1 returns the set $M = \{x, \frac{1}{x}, \frac{1}{x+1}, \ldots\}$ with 24 functions.*

Algorithm 4.1.3 below picks two functions u, v from the output of Algorithm 4.1.1, then check if $R(u, v) = A$.

Algorithm 4.1.3 *CandiUV*

- *Input: a set A.*
- *Output: the set of all pairs $[u, v]$ with $R(u, v) = A$.*
- *Steps.*
  - *Apply algorithm 4.1.1 on A, then obtain the set M.*
  - *Return the set of all pairs $u, v \in M$ with $R(u, v) = A$.*

Example 4.1.4 *Continue with Example 4.1.2. Algorithm 4.1.3 returns the set of 144 candidate pairs: $S = \{[x, \frac{1}{x}], [x, \frac{x}{x+1}], [x, -x], \ldots\}$.*

4.2 Divide the Candidate Set into Orbits

Example 4.1.4 shows that there may be many candidate pairs $[u, v] \in S$. Checking one pair (Section 4.3) is a substantial amount of work. The goal in this section is to reduce the number of candidate pairs $[u, v]$.

Proposition 4.2.1 *The following functions satisfy the same differential equations.*

- $F_1(a, b_1, b_2, c \,|\, u, v)$
- $r_1 \cdot F_1(c - a, b_1, b_2, c \,|\, \frac{u}{u-1}, \frac{v}{v-1})$
- $r_2 \cdot F_1(a, c - b_1 - b_2, b_2, c \,|\, \frac{u}{u-1}, \frac{v-u}{1-u})$
- $r_3 \cdot F_1(a, b_1, c - b_1 - b_2, c \,|\, \frac{v-u}{v-1}, \frac{v}{v-1})$
- $F_1(a, b_1, b_2, b_1 + b_2 + a + 1 - c \,|\, 1 - u, 1 - v)$

where $r_1 = (1-u)^{-b_1}(1-v)^{-b_2}$, $r_2 = (1-u)^{-a}$, $r_3 = (1-v)^{-a}$.

Pf : the first 4 relations are found in [12] and the last one is obtained by the implementation [31].

Proposition 4.2.1 shows that after testing a candidate of the form $[*, *, *, *, u, v]$, there is no need to try $[*, *, *, *, \frac{u}{u-1}, \frac{v}{v-1}]$, $[*, *, *, *, \frac{u}{u-1}, \frac{v-u}{1-u}]$, $[*, *, *, *, \frac{v-u}{v-1}, \frac{v}{v-1}]$ and $[*, *, *, *, 1-u, 1-v]$.

Definition 4.2.2 *Let* $L_6 = \mathbb{Q}(a, b_1, b_2, c, u, v)$ *with* a, b_1, b_2, c, u, v *algebraically independent. Let* $G = \langle R_1, R_2, R_3, R_4 \rangle \subseteq \mathrm{Aut}(L_6)$ *where* R_1, R_2, R_3, R_4 *act on* $f(a, b_1, b_2, c \,|\, u, v) \in L_6$ *as follows:*

- $R_1(f(a, b_1, b_2, c \,|\, u, v)) = f(c - a, b_1, b_2, c \,|\, \frac{u}{u-1}, \frac{v}{v-1})$.
- $R_2(f(a, b_1, b_2, c \,|\, u, v)) = f(a, c - b_1 - b_2, b_2, c \,|\, \frac{u}{u-1}, \frac{v-u}{1-u})$.
- $R_3(f(a, b_1, b_2, c \,|\, u, v)) = f(a, b_1, c - b_1 - b_2, c \,|\, \frac{v-u}{v-1}, \frac{v}{v-1})$.
- $R_4(f(a, b_1, b_2, c \,|\, u, v)) = f(a, b_1, b_2, b_1 + b_2 + a + 1 - c \,|\, 1 - u, 1 - v)$.

Proposition 4.2.3 $G \cong S_5$ *and acts faithfully on* $L_2 := \mathbb{Q}(u, v) \subseteq L_6$ *as well, so we can also interpret* G *as a subgroup of* $\mathrm{Aut}(L_2)$. *Let* $G_3 = \langle R_1, R_2, R_3 \rangle \subseteq G$, *a subgroup of* G *that preserves the point* $(u, v) = (0, 0)$. *Then* $G_3 \cong S_2 \times S_3$.

Pf : The orbit of $a \in L_6$ under G is $A = \{a, c-a, b_1+b_2+1-c, 1-b_1, 1-b_2\}$ and R_1 acts as the permutation (12) on A. Similarly, R_2, R_3 and R_4 acts as the permutation (34), (35) and (23). These actions are faithful and thus:

$$G \cong < (12), (34), (35), (23) > \cong S_5$$

and

$$G_3 \cong < (12), (34), (35) > \cong S_2 \times S_3.$$

Algorithm 4.2.4 below uses G to divide the candidate set of $[u, v]$ into orbits.

Algorithm 4.2.4 *GetOrb*

- *Input: the set of candidates of* $[u, v]$ *obtained from Algorithm 4.1.3.*
- *Output: the same candidates divided into orbits.*
- *Steps: for each pair* $[u, v]$*, find its orbit under the group* G*:*
  - *let* $B := \{[u, v]\}$.
  - *While* $B \neq B \cup R_1(B) \cup R_2(B) \cup R_3(B) \cup R_4(B)$ *do* $B := B \cup R_1(B) \cup R_2(B) \cup R_3(B) \cup R_4(B)$*. Here* $R_i \in G \subseteq \mathrm{Aut}(L_2)$ *with* $1 \leq i \leq 4$.

Example 4.2.5 *Continue with Example 4.1.4. Algorithm 4.2.4 divides* S *consisting of* 144 *pairs into only* 3 *orbits:* $\{\{[x, \frac{1}{x}], \ldots\}, \{[x, -x], \ldots\}, \{[\frac{x-1}{x+1}, \frac{x+1}{x-1}], \ldots\}\}$.

Rather than checking all candidates $[u, v]$, now it suffices to check one candidate $[u, v]$ in each orbit.

4.3 Compute Parameters a, b_1, b_2, c and Transformations

Let L_c be the minimal operator of $F_1(a, b_1, b_2, c \,|\, u, v)$ where $u, v \in \mathbb{C}(x)$ of degree 1. Table 4.1 and 4.2 relate the parameters a, b_1, b_2, c to the exponents of L_c. Let U be the set of roots of $u, 1-u$ and $\frac{1}{u}$ and let V be the set of roots of $v, 1-v$ and $\frac{1}{v}$. Table 4.1 describes the relation when $U \cap V = \emptyset$ and the multiplicity of the root of $u - v$ is 1. For example, if $x = 0$ is a root of u but not a root of v or $1 - v$ or $\frac{1}{v}$, then table 4.1 shows how to obtain $b_2 - c + 1$ from the exponents of L_c at $x = 0$.

If $U \cap V \neq \emptyset$, then the relation changes. Table 4.2 describes this case. For example, if $x = 0$ is a common root of u and $1 - v$, then Table 4.2 shows how to obtain $\{b_2 - c + 1, c - a - b_2\}$ from

Table 4.1: Exponents at Single Roots

At Root of	Exponents of L_c
u	$[0, 1, b_2 - c + 1]$
$1 - u$	$[0, 1, c - a - b_1]$
$\frac{1}{u}$	$[a, b_1, b_1 + 1]$
v	$[0, 1, b_1 - c + 1]$
$1 - v$	$[0, 1, c - a - b_2]$
$\frac{1}{v}$	$[a, b_2, b_2 + 1]$
$u - v$	$[0, 1, 1 - b_1 - b_2]$

Table 4.2: Exponents at Common Roots

At Root of	Exponents of L_c
$[u, v, u - v]$	$[0, 1 - c, 2 - c]$
$[u, v, u - v, u - v]$	$[0, 1 - c, 2 - b_1 - b_2 - c]$
$[u, 1 - v]$	$[0, b_2 - c + 1, c - a - b_2]$
$[u, \frac{1}{v}]$	$[a, b_2, 2b_2 - c + 1]$
$[1 - u, v]$	$[0, c - a - b_1, b_1 - c + 1]$
$[1 - u, 1 - v, u - v]$	$[0, c - a - b_1 - b_2, c - a - b_1 - b_2 + 1]$
$[1 - u, 1 - v, u - v, u - v]$	$[0, c - a - b_1 - b_2, c - a - 2b_1 - 2b_2 + 1]$
$[1 - u, \frac{1}{v}]$	$[a, b_2, c - a - b_1 + b_2]$
$[\frac{1}{u}, v]$	$[a, b_1, 2b_1 + 1 - c]$
$[\frac{1}{u}, 1 - v]$	$[a, b_1, c - a + b_1 - b_2]$
$[\frac{1}{u}, \frac{1}{v}]$	$[a, a + 1, b_1 + b_2]$
$[u - v, u - v]$	$[0, 1, 2 - 2b_1 - 2b_2]$

the exponents of L_c at $x = 0$. Likewise, if $x = 0$ is a common root of u and v and $x = 0$ is a root of $u - v$ with multiplicity 1, then $\{1 - c, 2 - c\}$ can be obtained from the exponents of L_c at $x = 0$. Likewise for the case in which the common root of u and v is a root of $u - v$ with multiplicity 2.

Now compute the parameters in two cases.

- $L_c \xrightarrow{(ii), r} L$.
- $L_c \xrightarrow{(ii), r} \xrightarrow{(iii), G} L$.

4.3.1 Case with Exp-product Transformation

From Section 2.4.2, the exp-product transformation ($\xrightarrow{(ii)}$) changes the exponents at one point by the same difference. So if $L_c \xrightarrow{(ii), r} L$, then the parameters can be computed by solving equations from the exponents of L_c and L.

Example 4.3.1 *Continue with Example 4.2.5. Let* $[u, v] = [x, \frac{1}{x}]$ *from the first orbit and* L_c *be the minimal operator of* $F_1(a, b_1, b_2, c \,|\, x, \frac{1}{x})$*. Then* 0 *is the common root of* u *and* $\frac{1}{v}$*. The exponents of* L_c *at* 0 *are* $[a, b_2, 2b_2 - c + 1]$ *from Table 4.2. The exponents of* L *at* 0 *are* $[1, \frac{9}{10}, \frac{7}{10}]$*. Let* d *be the difference between them. Then:*

$$\{a + d, b_2 + d, 2b_2 - c + 1 + d\} = \{1, \frac{9}{10}, \frac{7}{10}\}.$$

This implies 3! *sets of equations since there are* 3! *ways to pair elements in* $\{a+d, b_2+d, 2b_2-c+1+d\}$ *and elements in* $\{1, \frac{9}{10}, \frac{7}{10}\}$*. For example, we can pair them as* $\{a+d = \frac{7}{10}, b_2+d = \frac{9}{10}, 2b_2-c+1+d = 1\}$*. By eliminating* d*,* $\{a - b_2 = -\frac{1}{5}, c - b_2 - 1 = -\frac{1}{10}\}$*. Likewise, we can obtain* 6 *such sets of equations, which is the output of Algorithm 4.3.2 below with the input* $1, \frac{9}{10}, \frac{7}{10}, a, b_2, 2b_2 - c + 1$.

Algorithm 4.3.2 *MatchExp1*

- *Input: the exponents of the given operator* L *at some non-removable singularity* p*,* e_1, e_2, e_3*, and the exponents of* L_c *at* p*,* f_1, f_2, f_3*, where* L_c *is the minimal operator of* $F_1(a, b_1, b_2, c \,|\, u, v)$ *with unknown parameters* a, b_1, b_2 *and* c*.*
- *Output: some equations regarding parameters* a, b_1, b_2 *and* c*.*
- *Steps: compute the set of differences among* e_1, e_2, e_3 *and the set of differences among* f_1, f_2, f_3*, set equations between these two sets and return the solvable ones.*

Algorithm 4.3.3 below shows that the parameters a, b_1, b_2, c and the parameter r in $L_c \xrightarrow{(ii),\, r} L$ can be computed at the same time. Since the relation of parameters and exponents of L_c changes as the relation of u and v changes, so there are several cases to consider. Algorithm 4.3.3 deals with the case when u and $\frac{1}{v}$ have a common root, $\frac{1}{u}$ and v have a common root, $1 - u$ and $1 - v$ have a common root and it is also a root of $u - v$ with multiplicity 1. For other cases, the algorithms are very similar to Algorithm 4.3.3.

Algorithm 4.3.3 *Case1*

- *Input: the operator* L *and one candidate pair* u, v*.*
- *Output: the set of all combinations* $[a, b_1, b_2, c, u, v, r]$ *where* r *is the parameter in* $L_c \xrightarrow{(ii),\, r} L$*. Recall* L_c *is the minimal operator of* $F_1(a, b_1, b_2, c \,|\, u, v)$*.*
- *Steps.*

– *Let $x = p$ be the common root of u and $\frac{1}{v}$. According to Table 4.2, the exponents of L_c at p are $[a, b_2, 2b_2 - c + 1]$. Use Algorithm 4.3.2 to relate exponents of L at $x = p$ and $[a, b_2, 2b_2 - c + 1]$. Solve for a and b_2.*

– *Let L_c be the minimal operator of $F_1(a, b_1, b_2, c \,|\, u, v)$ where a and b_2 are written in terms of c. So L_c has only two unknown parameters b_1 and c. Take r as one third of the difference between the coefficients of L and L_c with respect to ∂^2. If L_c can be transformed to L by some exp-product transformation, then the parameter of the transformation should be r. So set an equation between L and $L_c \circledS (\partial - r)$ and then solve for b_1 and c. If there exists a solution, then the combination $[a, b_1, b_2, c, u, v, r]$ in the output is obtained.*

- *Comment. Algorithm 4.3.2 may return more than* 1 *equations, which means there may be more than* 1 *solutions for parameters in F_1. All possibilities are checked in Algorithm 4.3.3.*

Example 4.3.4 *Continue with Example 4.2.5. Let $[u, v] = [x, \frac{1}{x}]$ from one orbit.* $\text{Case1}(L, x, \frac{1}{x})$ *returns* $\{[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, 1, x, \frac{1}{x}, -\frac{1}{2x}], [\frac{8}{15}, \frac{1}{5}, \frac{1}{3}, \frac{31}{30}, x, \frac{1}{x}, -\frac{11}{30x}]\}$.

4.3.2 Case with Exp-product Transformation and Gauge Transformation

If $L_c \xrightarrow{(ii),\, r} \xrightarrow{(iii),\, G} L$, then from Section 2.4.2 and 2.4.3, there exists some $d_p \in \mathbb{C}[t_p^{-1}]$ such that {generalized exponents of L_c at $x = p$} $+d_p \sim$ {generalized exponents of L at $x = p$} for any $p \in \mathbb{P}^1$. Like what we did in Subsection 4.3.1, we can set some equations between exponents of L_c and L. The only difference is that shifting by integers is allowed in this case.

Example 4.3.5 *Continue with Example 4.3.1. The relation*

$$\{a + d, b_2 + d, 2b_2 - c + 1 + d\} \sim \{1, \frac{9}{10}, \frac{7}{10}\}.$$

also implies 3! *sets of equations. One of them is*

$$\{a + d = \frac{7}{10} \text{ mod } \mathbb{Z},\ b_2 + d = \frac{9}{10} \text{ mod } \mathbb{Z},\ 2b_2 - c + 1 + d = 1 \text{ mod } \mathbb{Z}\}.$$

By eliminating d, $\{a - b_2 = -\frac{1}{5} + n_2, c - b_2 - 1 = -\frac{1}{10} + n_1\}$ for some integers n_1 and n_2. Likewise, we can obtain 6 *such sets of equations, which is the output of Algorithm 4.3.6 below with input* $1, \frac{9}{10}, \frac{7}{10}, a, b_2, 2b_2 - c + 1, n_1, n_2$.

Algorithm 4.3.6 *MatchExp2*

- *Input: the input of Algorithm 4.3.2 and two indices which indicate two integers.*

- *Output: some equations regarding parameters* a, b_1, b_2 *and* c*.*

- *Steps: the only difference from Algorithm 4.3.2 is when setting equations between the differences, allow them shifting by some integer.*

Now at each non-removable singularity of L, we have a bunch of equation sets about parameters a, b_1, b_2 and c. Algorithm 4.3.7 below combines all of them and compute the parameters.

Algorithm 4.3.7 *Comb*

- *Input: the outputs of Algorithm 4.3.6 at all non-removable singularities of* L*, say* $S_1, \ldots, S_n$*.*

- *Output: the set of candidates for* $[a, b_1, b_2, c]$*.*

- *Steps: for each* $(s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$*, solve for parameters* a, b_1, b_2, c *and return all solutions.*

Now L_c is known. Then use Algorithm 3.1.2 to compute r and "Homomorphisms" to compute G in $L_c \xrightarrow{(ii), r} \xrightarrow{(iii), G} L$.

CHAPTER 5

COMPUTE HOMOMORPHISM(S) BETWEEN TWO D-MODULES

We have many univariate tools for hypergeometric functions, such as compute r in $\xrightarrow{(ii),\, r}$, compute G in $\xrightarrow{(iii),\, G}$ for any order operators and compute transformations $\xrightarrow{(ii),\,(iii)}$ for operators of order 2 [27] and order 3 (Section 3.3). But these tools can not deal with multivariate A-hypergeometric functions. So now we need to generalize them to multivariate. By Remark 2.2.3, to generalize transformation (iii), we need to compute homomorphisms between two D-modules. Here D can be, for example, the differential ring $\mathbb{C}(x, y)[\partial_x, \partial_y]$.[1] We will illustrate this tool via computing homomorphisms between $M := F_1^D(a, b_1, b_2, c \,|\, x, y)$ and $M' := F_1^D(a+1, b_1, b_2, c \,|\, x, y)$ (D-modules of $F_1(a, b_1, b_2, c \,|\, x, y)$ and $F_1(a+1, b_1, b_2, c \,|\, x, y)$ respectively with $D = \mathbb{C}(x, y)[\partial_x, \partial_y]$) by following steps:

- for each variable and each D-module, find a cyclic vector (Definition 5.2.1) and compute its minimal operator with respect to that variable (Section 5.2). Let L_x and L'_x be the minimal operator of M and M' with respect to x, L_y and L'_y be the minimal operator of them with respect to y.
- For each variable, use the univariate tool (DEtools[Homomorphisms] in Maple) to find homomorphisms between two minimal operators with respect to that variable. Let h_x be the homomorphism(s) between L_x and L'_x, and h_y be the homomorphism(s) between L_y and L'_y. From h_x and h_y, we can obtain H_x, the homomorphism of M and M' as $\mathbb{C}(x, y)[\partial_x]$-modules, and H_y, the homomorphism of them as $\mathbb{C}(x, y)[\partial_y]$-modules. Then compute $H_x \cap H_y$, which is the homomorphism between M and M' as D-modules. (Section 5.3)

By Definition 2.2.4, to generalize transformation (ii)+(iii), we use the tensor product of two D-modules since tensor product with a 1-dimensional D-module ($M \otimes I$) corresponds to applying the transformation (ii) on its minimal operator ($L \circledS (\partial - r)$) where I is the 1-dimensional D-module for r. (Section 5.4)

[1] We will focus on two variables through the approach, but it also works for more variables (Section 5.3.3).

5.1 The $\mathbb{C}(x, y)[\partial_x, \partial_y]$-Module of $F_1(a, b_1, b_2, c \,|\, x, y)$

Let $K = \mathbb{C}(x, y)$, $D = K[\partial_x, \partial_y]$. As in Section 2.2, a D-module is a finitely dimensional K-vector space on which D acts. To turn K^n into a D-module, take two $n \times n$ matrices M_x and M_y over K and define

$$\partial_x \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = M_x \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \partial_x(a_1) \\ \vdots \\ \partial_x(a_n) \end{bmatrix}$$

and

$$\partial_y \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = M_y \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \partial_y(a_1) \\ \vdots \\ \partial_y(a_n) \end{bmatrix}.$$

In the ring D, the elements ∂_x and ∂_y commute, so $\partial_x \partial_y$ and $\partial_y \partial_x$ must have the same action on K^n in order for K^n to be a D-module. That implies $(\frac{d}{dx} + M_x)M_y = (\frac{d}{dy} + M_y)M_x$ (integrability condition). Given $v \in K^n$ and M_x, M_y, we can compute its minimal operator in $K[\partial_x]$ or in $K[\partial_y]$.

Let $B_1 = F_1(a, b_1, b_2, c \,|\, x, y)$ for some $a, b_1, b_2, c \in \mathbb{C}$. Then its D-module $DB_1 := \{L(B_1) \,|\, L \in D\}$ generally has K-dimension 3, but can have lower dimension for specific values of the parameters (for example, if $a = 0$, then $B_1 = 1$ in which case DB_1 is just K). To avoid such drop in dimension, we define a D-module for $F_1(a, b_1, b_2, c \,|\, x, y)$, denoted as $F_1^D(a, b_1, b_2, c \,|\, x, y)$, as follows:

- as a K-vector space it is K^3,
- ∂_x and ∂_y act on K^3 as follows: if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in K^3$, then

$$\partial_x \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = M_x \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} \partial_x(a_1) \\ \partial_x(a_2) \\ \partial_x(a_3) \end{bmatrix}$$

and

$$\partial_y \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = M_y \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} \partial_y(a_1) \\ \partial_y(a_2) \\ \partial_y(a_3) \end{bmatrix}$$

where

$$M_x = \begin{bmatrix} \frac{-b_1}{x} & \frac{b_1+1-c+b_2}{x(x-1)} & 0 \\ \frac{b_1}{x} & \frac{ax^2+b_1x-cx+x-axy+b_2xy+cy-b_1y-y-b_2y}{x(y-x)(x-1)} & \frac{b_1}{x-y} \\ 0 & \frac{b_2(y-1)}{(x-1)(x-y)} & \frac{b_1}{y-x} \end{bmatrix} \tag{5.1}$$

and

$$M_y = \begin{bmatrix} \frac{-b_2}{y} & 0 & \frac{b_1+1-c+b_2}{y(y-1)} \\ 0 & \frac{b_2}{x-y} & \frac{b_1(1-x)}{(x-y)(y-1)} \\ \frac{b_2}{y} & \frac{b_2}{y-x} & \frac{b_1x-cx+x+b_2x+axy-b_1xy+cy-y-b_2y-ay^2}{y(y-1)(y-x)} \end{bmatrix}. \tag{5.2}$$

Proposition 5.1.1 *Let $B_1 = F_1(a, b_1, b_2, c \,|\, x, y)$, $B_2 = F_1(a, b_1+1, b_2, c \,|\, x, y)$, $B_3 = F_1(a, b_1, b_2+1, c \,|\, x, y)$. Then $KB_1+KB_2+KB_3$ is a D-module and DB_1 is a submodule of it. If $\dim_K(DB_1) = 3$ then $DB_1 \cong KB_1 + KB_2 + KB_3 \cong F_1^D(a, b_1, b_2, c \,|\, x, y)$ as D-modules. If $\dim_K(KB_1 + KB_2 + KB_3) < 3$ then it is a quotient module of $F_1^D(a, b_1, b_2, c \,|\, x, y)$.*

Pf : ∂_x and ∂_y act on B_1, B_2, B_3 as (implementation [31] and [24]):

- (1) $\partial_x B_1 = -\frac{b_1}{x} B_1 + \frac{b_1}{x} B_2$, $\partial_y B_1 = -\frac{b_2}{y} B_1 + \frac{b_2}{y} B_3$.
- (2) $\partial_x B_2 = \frac{b_1+b_2-c+1}{x(x-1)} B_1 - \frac{ax^2+(b_1-c+1-ay+b_2y)x+(c-b_1-b_2-1)y}{x(x-1)(x-y)} B_2 + \frac{b_2(y-1)}{(x-1)(x-y)} B_3$,
$\partial_y B_2 = \frac{b_2}{x-y} B_2 - \frac{b_2}{x-y} B_3$.
- (3) $\partial_y B_3 = \frac{b_1+b_2-c+1}{y(y-1)} B_1 + \frac{ay^2+(b_2-c+1-ax+b_1x)y+(c-b_1-b_2-1)x}{y(y-1)(x-y)} B_3 - \frac{b_1(x-1)}{(y-1)(x-y)} B_2$,
$\partial_x B_3 = \frac{b_1}{x-y} B_2 - \frac{b_1}{x-y} B_3$.

So $KB_1 + KB_2 + KB_3$ is a D-module and $DB_1 \subseteq KB_1 + KB_2 + KB_3$ is a submodule. Define the K-linear map $\psi : F_1^D(a, b_1, b_2, c \,|\, x, y) \to KB_1+KB_2+KB_3$ with $\psi(\begin{bmatrix}1\\0\\0\end{bmatrix}) = B_1$, $\psi(\begin{bmatrix}0\\1\\0\end{bmatrix}) = B_2$, $\psi(\begin{bmatrix}0\\0\\1\end{bmatrix}) = B_3$. Then ψ is a D-module homomorphism between $F_1^D(a, b_1, b_2, c \,|\, x, y)$ and $KB_1 + KB_2 + KB_3$ since the matrices M_x (5.1) and M_y (5.2) in $F_1^D(a, b_1, b_2, c \,|\, x, y)$ match precisely with (1)(2)(3) under ψ.

- If $\dim_K(KB_1 + KB_2 + KB_3) = 3$, then ψ is one-to-one, so $F_1^D(a, b_1, b_2, c \,|\, x, y) \cong KB_1 + KB_2 + KB_3$ as D-modules.
- If $\dim_K(KB_1 + KB_2 + KB_3) < 3$, then the image of ψ, $KB_1 + KB_2 + KB_3$, is a quotient module of $F_1^D(a, b_1, b_2, c \,|\, x, y)$.
- If $\dim_K(DB_1) = 3$, then $\dim_K(KB_1+KB_2+KB_3) = 3$ and therefore $DB_1 \cong KB_1+KB_2+KB_3 \cong F_1^D(a, b_1, b_2, c \,|\, x, y)$ as D-modules.

Remark 5.1.2 *From now on, use the form* $[[variables],[derivatives],[matrices]]$ *to denote modules. The first entry encodes the field* $K = \mathbb{C}(x_1, \cdots, x_p)$ *and the first two entries encode the differential ring* $K[\partial_{x_1}, \cdots, \partial_{x_p}]$*. So* $[[x,y],[\partial_x,\partial_y],[M_x,M_y]]$ *with* M_x*,* M_y *in (5.1) (5.2) denotes the module* $F_1^D(a, b_1, b_2, c \,|\, x, y)$*.*

5.2 Find a Cyclic Vector and its Minimal Operator with respect to One Variable

Definition 5.2.1 *Let* $[[x_1, x_2, \ldots, x_p], [\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_p}], [M_{x_1}, M_{x_2}, \ldots, M_{x_p}]]$ *denote a module* M*. Let* $K = \mathbb{C}(x_1, \ldots, x_p)$*. An element* $B \in M$ *is a* cyclic vector *with respect to* x_i $(1 \le i \le p)$ *if* $K[\partial_{x_i}]B = M$*. The cyclic vector theorem in [25] ensures the existence of cyclic vectors. Moreover, for irreducible modules, every nonzero vector is cyclic.*

Algorithm 5.2.2 *CycVec*

- *Input: a module* M *and a variable* x_i*.*
- *Output:* m*, a cyclic vector of* M *with respect to* x_i*, and its minimal operator.*
- *Steps: let* m *be the following element in* M*. Check if it is cyclic with respect to* x_i*. If yes, then stop and return* m *and its minimal operator with respect to* x_i*.*
  - *Elements in the standard basis of* M*, which are* n *by one vectors with only one entry equal* 1 *and others equal* 0*. Here* n *is the dimension of* M *as a* K*-vector space.*
  - *Combinations of the basis elements with random small number coefficients.*
  - *Combinations of the basis elements with random large number coefficients.*
  - *Combinations of the basis elements with degree* 1 *rational function coefficients.*

Example 5.2.3 *By Algorithm 5.2.2,* $\mathrm{CycVec}(F_1^D(a, b_1, b_2, c \,|\, x, y), x)$ *finds a cyclic vector of the D-module* $F_1^D(a, b_1, b_2, c \,|\, x, y)$ *with respect to* x*,* $\begin{bmatrix}1\\0\\0\end{bmatrix}$*, and its minimal operator* L_x *with respect to* x*.*

$$\begin{aligned} L_x = 1 &+ \frac{b_1 x + 2x + 2ax + b_2 y - ay - y - c}{ab_1}\partial_x \\ &+ \frac{4x^2 + ax^2 + 2b_1x^2 - b_1x - cx - 2x - b_1xy - axy + b_2xy - 3xy - b_2y + y + cy}{ab_1(1+b_1)}\partial_x^2 \\ &+ \frac{x^2 - x - xy + y}{ab_1(1+b_1)}\partial_x^3. \end{aligned} \tag{5.3}$$

5.3 Compute Homomorphisms between Two Modules

Let $K = \mathbb{C}(x, y)$, $D = K[\partial_x, \partial_y]$, $D_x = K[\partial_x]$ and $D_y = K[\partial_y]$. Let M, M' be D-modules. As K-vector spaces, M is K^n and M' is $K^{n'}$. Our goal is to compute the homomorphisms between M and M' as D-modules.

Theorem 5.3.1

$$\mathrm{Hom}_D(M, M') = \mathrm{Hom}_{D_x}(M, M') \cap \mathrm{Hom}_{D_y}(M, M').$$

Pf : $\mathrm{Hom}_D(M, M') \subseteq \mathrm{Hom}_{D_x}(M, M') \cap \mathrm{Hom}_{D_y}(M, M')$ since $D_x \subseteq D$ and $D_y \subseteq D$. Conversely, if $\phi \in \mathrm{Hom}_{D_x}(M, M') \cap \mathrm{Hom}_{D_y}(M, M')$, then $\forall m \in M$, $\phi(rm) = r\phi(m)$ for every $r \in D_x$ and every $r \in D_y$, in particular, for $r = \partial_x$ and $r = \partial_y$. By repeating this and using K-linearity, we have $\phi(rm) = r\phi(m)$ for all $r \in K[\partial_x, \partial_y]$. So $\phi \in \mathrm{Hom}_D(M, M')$.

Theorem 5.3.1 reduces our goal to two tasks.

- Compute $\mathrm{Hom}_{D_x}(M, M')$ and $\mathrm{Hom}_{D_y}(M, M')$.
- Given two vector spaces $V_1, V_2 \subseteq \mathrm{Mat}_{n',n}(K)$, where V_1 is a $\mathbb{C}(y)$-vector space and V_2 is a $\mathbb{C}(x)$-vector space, compute their intersection $V_1 \cap V_2$ (a $\mathbb{C}$-vector space).

5.3.1 Compute $\mathrm{Hom}_{D_x}(M, M')$

Let $(m, L_x) = \mathrm{CycVec}(M, x)$ and $(m', L'_x) = \mathrm{CycVec}(M', x)$ (Algorithm 5.2.2). Then (Remark 2.2.3):

$$M \cong D_x/D_xL_x \quad \text{and} \quad M' \cong D_x/D_xL'_x \quad \text{as} \quad D_x\text{−modules.} \tag{5.4}$$

Remark 5.3.2 *If $\phi : M \to M'$ is a homomorphism as D_x-modules and $m \in M$ is a cyclic vector with respect to x, then ϕ is completely determined by $\phi(m)$.*

Remark 5.3.3 DEtools[Homomorphisms] in Maple. *Let $L_1, L_2 \in D_x$. The "*Homomorphisms(L_2, L_1)*" command in Maple computes a basis of all $G \in D_x/D_xL_2$, for which $1 \longmapsto G$ is a D_x-homomorphism from D_x/D_xL_1 to D_x/D_xL_2.*

By (5.4) and Remark 5.3.3, if G is in the output of "Homomorphisms(L'_x, L_x)", then the corresponding element $\phi \in \mathrm{Hom}_{D_x}(M, M') \subseteq \mathrm{Hom}_K(M, M') = \mathrm{Mat}_{n',n}(K)$ (recall $\mathrm{Hom}_K(M, M') = \mathrm{Hom}_K(K^n, K^{n'})$) is $\mathrm{Mat}_1 \cdot \mathrm{Mat}_2^{-1}$ where

$$\mathrm{Mat}_1 = (G(m'), \partial_x G(m'), \ldots, \partial_x^{n-1} G(m')) \in \mathrm{Mat}_{n',n}(K)$$

and

$$\text{Mat}_2 = (m, \partial_x m, \ldots, \partial_x^{n-1} m) \in \text{GL}_n(K).$$

Mat_1 gives ϕ on the basis $m, \partial_x m, \ldots, \partial_x^{n-1} m$ and Mat_2^{-1} is the change of basis matrix from $\{m, \partial_x m, \ldots, \partial_x^{n-1} m\}$ to the standard basis. Likewise, one can compute $\text{Hom}_{D_y}(M, M')$.

5.3.2 Compute $\text{Hom}_{D_x}(M, M') \cap \text{Hom}_{D_y}(M, M')$

Lemma 5.3.4 *If $h_1, \ldots, h_d \in \text{Hom}_{D_x}(M, M')$ are $\mathbb{C}(y)$-linear independent, then $h_1, \ldots, h_d$ are K-linear independent.*

Pf : suppose $h_1, \ldots, h_d$ are K-linear dependent. Choose a minimal linear relation $\sum_{i=1}^{d} c_i h_i = 0$ with the fewest nonzero $c_i \in K$. We may assume $c_1 = 1$ (otherwise reorder to make $c_1 \neq 0$ then divide by c_1). So

$$\forall m \in M, \quad \sum_{i=1}^{d} c_i h_i(m) = 0. \tag{5.5}$$

Since $h_i \in \text{Hom}_{D_x}(M, M')$ and $\partial_x \in D_x$, applying ∂_x gives

$$\forall m \in M, \quad \sum_{i=1}^{d} [\partial_x(c_i) \cdot h_i(m) + c_i h_i(\partial_x m)] = 0. \tag{5.6}$$

Since $\sum_{i=1}^{d} c_i h_i(\partial_x m) = 0$ (by (5.5) and $\partial_x m \in M$), so $\sum_{i=1}^{d} \partial_x(c_i) \cdot h_i = 0$ with $\partial_x(c_1) = \partial_x(1) = 0$. Then $\sum_{i=1}^{d} \partial_x(c_i) \cdot h_i = 0$ has fewer nonzero terms than $\sum_{i=1}^{d} c_i h_i = 0$. By minimality, all $\partial_x(c_i)$ must be 0 and hence $c_i \in \mathbb{C}(y)$. So $h_1, \ldots, h_d$ are $\mathbb{C}(y)$-linear dependent, which means that K-linear dependency of elements in $\text{Hom}_{D_x}(M, M')$ implies $\mathbb{C}(y)$-linear dependency. Then its contrapositive is also true, which completes the proof.

Let $h_1, \ldots, h_{d_1}$ be a basis of $\text{Hom}_{D_x}(M, M')$ as a $\mathbb{C}(y)$-vector space and $H_1, \ldots, H_{d_2}$ be a basis of $\text{Hom}_{D_y}(M, M')$ as a $\mathbb{C}(x)$-vector space. All of them are $n' \times n$ matrices over K.

Algorithm 5.3.5 *Case of $d_1 = d_2 = 1$*

- *Input:* $h_1 \in \text{Hom}_{D_x}(M, M')$ *and* $H_1 \in \text{Hom}_{D_y}(M, M')$.
- *Output:* $\text{Hom}_D(M, M')$.
- *Steps.*
  - *1. Check if $h_1 = Q \cdot H_1$ for some $Q \in K$.*

– *2. If so, check if there exist* $D_1 \in \mathbb{C}(y)$ *and* $D_2 \in \mathbb{C}(x)$ *such that* $Q = D_1 \cdot D_2$*.*

– *3. If so, return* $\frac{h_1}{D_1}$ *(equals* $\frac{h_2}{D_2}$*).*

- *Comment. If the check in step* 1 *or step* 2 *fails, then return the empty set.*

Example 5.3.6 *By computations in Section 5.3.1, a* D_x*-module homomorphism* h_1 *from* $F_1(a+1, b_1, b_2, c\,|\,x, y)$ *to* $F_1(a, b_1, b_2, c\,|\,x, y)$ *and a* D_y*-module homomorphism* H_1 *are:*

$$h_1 = \begin{bmatrix} b_1(a-b_1-b_2)(y-1) & \frac{b_1(b_1+b_2-c+1)(y-1)}{x-1} & b_1(b_1+b_2-c+1) \\ b_1^2(y-1) & -\frac{b_1(a+b_1-c+1)(y-1)}{x-1} & -b_1^2 \\ b_1b_2(y-1) & \frac{-b_1b_2(y-1)}{x-1} & -b_1(a+b_2-c+1) \end{bmatrix}$$

and

$$H_1 = \begin{bmatrix} b_2(a-b_1-b_2)(x-1) & b_2(b_1+b_2-c+1) & \frac{b_2(b_1+b_2-c+1)(x-1)}{y-1} \\ b_1b_2(x-1) & -b_2(a+b_1-c+1) & -\frac{b_1b_2(x-1)}{y-1} \\ b_2^2(x-1) & -b_2^2 & -\frac{b_2(a+b_2-c+1)(x-1)}{y-1} \end{bmatrix}.$$

Now $h_1 = \frac{b_1(y-1)}{b_2(x-1)} \cdot H_1$*, so* $\frac{h_1}{b_1(y-1)} \in \mathrm{Hom}_D(F_1^D(a+1, b_1, b_2, c\,|\,x, y), F_1^D(a, b_1, b_2, c\,|\,x, y))$*.*

Algorithm 5.3.7 *Case of* $\mathrm{SPAN}_K(h_1, \ldots, h_{d_1}) = \mathrm{SPAN}_K(H_1, \ldots, H_{d_2})$ *and* $\dim(\mathrm{Hom}_D(M, M')) = d_1$*.*

- *Input: a basis of* $\mathrm{Hom}_{D_x}(M, M')$ *as* $\mathbb{C}(y)$*-vector space,* $h_1, \ldots, h_{d_1}$*, and a basis of* $\mathrm{Hom}_{D_y}(M, M')$ *as* $\mathbb{C}(x)$*-vector space,* $H_1, \ldots, H_{d_2}$*.*
- *Output: a basis of* $\mathrm{Hom}_D(M, M')$*.*
- *Steps.*
  – *1. By Lemma 5.3.4,* $d_1 = d_2$*. Let* $d = d_1$*. Write* $h_1, \ldots, h_d$ *into one* $n' \cdot n \times d$ *matrix* A*,* $(i,j)^{th}$ *entry in* h_k *being* $((i-1)n+j, k)^{th}$ *entry in* A*,* $i = 1, \ldots, n'$*,* $j = 1, \ldots, n$ *and* $k = 1, \ldots, d$*. Likewise,* $H_1, \ldots, H_d$ *give another* $n' \cdot n \times d$ *matrix* B*.*

  – *2. Solve for the change of basis matrix* $C = (C_{ij})_{d \times d}$ *such that* $h_i = \sum_{j=1}^{d} C_{ij} \cdot H_j$ *for every* $i = 1, \ldots, d$*, that is,* $A = B \cdot C$*.*

  – *3. Let* C_1 *be the invertible matrix from substituting* y *by some random value in the matrix* C*. Then* $C_1 \in \mathrm{Mat}_{d,d}(\mathbb{C}(x))$*. Let* $C_2 = C_1^{-1} \cdot C$*, then* $C_2 \in \mathrm{Mat}_{d,d}(\mathbb{C}(y))$ *and* $C = C_1 \cdot C_2$*. (The existence of such* C_1 *and* C_2 *is ensured by* $\dim(\mathrm{Hom}_D(M, M')) = d_1$*.)*

  – *4. Rewrite* $A \cdot C_2^{-1}$ *as* d $n' \times n$ *matrices, which is the inverse process of step 1. Return the set of these matrices.*

- *Comment on step 3. Let* $\{B_1, \ldots, B_d\}$ *be a basis of* $\mathrm{Hom}_D(M, M')$ *as a* $\mathbb{C}$*-vector space. Then* $\mathrm{SPAN}_{\mathbb{C}(y)}(B_1, \ldots, B_d) \subseteq \mathrm{SPAN}_{\mathbb{C}(y)}(h_1, \ldots, h_d)$*. From Lemma 5.3.4,*

$$\dim(\mathrm{SPAN}_{\mathbb{C}(y)}(B_1, \ldots, B_d)) = \dim(\mathrm{SPAN}_{\mathbb{C}}(B_1, \ldots, B_d)) = d.$$

  And $\dim(\mathrm{SPAN}_{\mathbb{C}(y)}(h_1, \ldots, h_d)) = d$*, so* $\mathrm{SPAN}_{\mathbb{C}(y)}(h_1, \ldots, h_d) = \mathrm{SPAN}_{\mathbb{C}(y)}(B_1, \ldots, B_d)$.

Case 3. $d_1 \neq d_2$ **or** $\dim(\mathrm{Hom}_D(M, M')) \neq d_1$**.** This would be detected in Step 3 in Algorithm 5.3.7. For completeness, we should implement this case too. We did not implement this case because it has not yet occurred in our computation.

5.3.3 General Case: M and M' are $\mathbb{C}(x_1, x_2, \ldots, x_p)[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_p}]$-Modules with $p > 2$

Let $K = \mathbb{C}(x_1, x_2, \ldots, x_p)$, $D = K[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_p}]$ and $D_{x_i} = K[\partial_{x_i}]$ for $i = 1, 2 \ldots, p$. Algorithm 5.3.8 below deals with the case when $d_1 = d_2 = \ldots = d_p = 1$ where d_i is the dimension of $\mathrm{Hom}_{D_{x_i}}(M, M')$ with $i = 1, 2 \ldots, p$.

Algorithm 5.3.8 *combHom*

- *Input: two matrices* $A, B \in \mathrm{Mat}_{n',n}(K)$ *and three variable sets* l_1*,* l_2 *and* l*, where* A *is unique up to* l_1*,* B *is unique up to* l_2 *and* $l = \{x_1, x_2, \ldots, x_p\}$*.*

- *Output: a list of two entries. The first entry is the matrix* $C \in \mathrm{Mat}_{n',n}(K)$*, which is equal to both* A *and* B *up to their own constants. The second entry is the variable set* ll *where* C *is unique up to it.*

- *Steps are similar as in Algorithm 5.3.5. Just replace* x *by variables in* l_2 *and replace* y *by variables in* l_1*. Then let* $ll = l_1 \cap l_2$*.*

- *Comment. If* $A \in \mathrm{Hom}_{D_{x_1}}(M, M')$*, then* $l_1 = \{x_2, \ldots, x_p\}$ *since* A *is unique up to constants in* $\mathbb{C}(x_2, \ldots, x_p)$*.*

Example 5.3.9 *Let* $K = \mathbb{C}(x, y, z)$*,* $D = K[\partial_x, \partial_y, \partial_z]$*,* $D_x = K[\partial_x]$*,* $D_y = K[\partial_y]$ *and* $D_z = K[\partial_z]$*. Let* M *be the module of* $F_1(a, b_1, b_2, c \,|\, x^2, yz)$ *and* M' *be the module of* $F_1(a{+}1, b_1, b_2, c \,|\, x^2, yz)$*, then by the computation in Section 5.3.1, homomorphisms* $h_x \in \mathrm{Hom}_{D_x}(M, M')$*,* $h_y \in \mathrm{Hom}_{D_y}(M, M')$ *and* $h_z \in \mathrm{Hom}_{D_z}(M, M')$ *are:*

$$h_x = \begin{bmatrix} b_1(a-b_1-b_2)(yz-1) & \frac{b_1(b_1+b_2-c+1)(yz-1)}{x^2-1} & b_1(b_1+b_2-c+1) \\ b_1^2(yz-1) & -\frac{b_1(a+b_1-c+1)(yz-1)}{x^2-1} & -b_1^2 \\ b_1b_2(yz-1) & \frac{-b_1b_2(yz-1)}{x^2-1} & -b_1(a+b_2-c+1) \end{bmatrix}$$

and

$$h_y = \begin{bmatrix} -b_2(a-b_1-b_2)(x^2-1)z & -b_2(b_1+b_2-c+1)z & -\frac{b_2(b_1+b_2-c+1)(x^2-1)z}{yz-1} \\ -b_1b_2(x^2-1)z & b_2(a+b_1-c+1)z & \frac{b_1b_2(x^2-1)z}{yz-1} \\ -b_2^2(x^2-1)z & b_2^2z & \frac{b_2(a+b_2-c+1)(x^2-1)z}{yz-1} \end{bmatrix}$$

and

$$h_z = \begin{bmatrix} -b_2(a-b_1-b_2)(x^2-1)y & -b_2(b_1+b_2-c+1)y & -\frac{b_2(b_1+b_2-c+1)(x^2-1)y}{yz-1} \\ -b_1b_2(x^2-1)y & b_2(a+b_1-c+1)y & \frac{b_1b_2(x^2-1)y}{yz-1} \\ -b_2^2(x^2-1)y & b_2^2y & \frac{b_2(a+b_2-c+1)(x^2-1)y}{yz-1} \end{bmatrix}$$

Since $h_x = -\frac{b_1(yz-1)}{b_2(x^2-1)z} \cdot h_y = \frac{b_1(yz-1)}{b_2z} \cdot (-\frac{1}{x^2-1}) \cdot h_y$, *so* $\text{combHom}(h_x, h_y, \{y,z\}, \{x,z\}, \{x,y,z\})$ *returns* $[h_{x,y}, z]$ *with* $h_{x,y} = \frac{b_2z}{b_1(yz-1)} \cdot h_x$ *as follows:*

$$h_{x,y} = \begin{bmatrix} b_2(a-b_1-b_2)z & \frac{b_2(b_1+b_2-c+1)z}{x^2-1} & \frac{b_2(b_1+b_2-c+1)z}{yz-1} \\ b_1b_2z & -\frac{b_2(a+b_1-c+1)z}{x^2-1} & -\frac{b_1b_2z}{yz-1} \\ b_2^2z & -\frac{b_2^2z}{x^2-1} & -\frac{b_2(a+b_2-c+1)z}{yz-1} \end{bmatrix}$$

Now $h_{x,y} = -\frac{z}{(x^2-1)y} \cdot h_z$, *so* $\text{combHom}(h_{x,y}, h_z, \{z\}, \{x,y\}, \{x,y,z\})$ *gives* $\frac{h_{x,y}}{z} \in \text{Hom}_D(M, M')$.

Likewise, when $\text{SPAN}_K(\text{Hom}_{D_{x_1}}(M,M')) = \ldots = \text{SPAN}_K(\text{Hom}_{D_{x_p}}(M,M'))$ and the dimension of $\text{Hom}_D(M,M')$ equals the dimension of $\text{Hom}_{D_{x_i}}(M,M')$ which is greater than 1, making the same changes as what we did in Algorithm 5.3.8 on Algorithm 5.3.7 gives the Algorithm "combHom2".

Algorithm 5.3.10 *hom*

- *Input: two modules* M, M' *where* M *is* K^n *and* M' *is* $K^{n'}$ *as* K*-vector spaces,* $K = \mathbb{C}(x_1, \ldots, x_p)$.
- *Output:* $\text{Hom}_D(M, M')$ *with* $D = K[\partial_{x_1}, \ldots, \partial_{x_p}]$.
- *Steps.*
  - *Compute* $\text{Hom}_{D_{x_i}}(M, M')$ *for* $i = 1 \ldots p$ *(Section 5.3.1). Let* d_i *be the dimension of* $\text{Hom}_{D_{x_i}}(M, M')$ *and* $\{A_{i,1}, \ldots, A_{i,d_i}\}$ *be a basis of* $\text{Hom}_{D_{x_i}}(M, M')$.

– *If* $d_1 = d_2 = \ldots = d_p = 1$*, then apply Algorithm 5.3.8 on* $A_{1,1}$*,* $A_{2,1}$ *to get a new matrix* A_{new}*. Then for each* i *with* $2 < i \leq p$ *(if exists), apply Algorithm 5.3.8 on* A_{new} *and* $A_{i,1}$ *until all* $A_{i,1}$ $(i = 3, \ldots, p)$ *are used once. If it returns a matrix* A_{new}*, then* $A_{new} \in \mathrm{Hom}_D(M, M')$*. If* $d_1 = d_2 = \ldots = d_p > 2$ *and the dimension of* $\mathrm{Hom}_D(M, M')$ *equals* d_1*, then use the Algorithm "combHom2" to do the same procedure.*

5.4 Compute Projective Homomorphisms between Two Modules

In this section, let $K = \mathbb{C}(x, y)$ and $D = K[\partial_x, \partial_y]$. Algorithm 5.3.10 computes homomorphisms between two D-modules of multivariate case, which corresponds to finding G in $\xrightarrow{(iii),\,G}$ for the univariate case. Now we need an algorithm to check if two modules M_1 and M_2 are projectively equivalent. By Definition 2.2.4, this is equivalent to computing the 1-dimensional module I such that the tensor product of M_1 and I $(M_1 \otimes I)$ is homomorphic to M_2. The 1-dimensional module I corresponds to r in $\xrightarrow{(ii),\,r}$ which could be computed by:

Algorithm 5.4.1 *OneDiModule*

- *Input: an algebraic function* $r \in \overline{K}$ *and a variable list* $[x, y]$*.*
- *Output: the* 1*-dimensional module corresponding to* r*:* $[[x, y], [\partial_x, \partial_y], [[\frac{\partial_x(r)}{r}], [\frac{\partial_y(r)}{r}]]]$.
- *Comment: the* 1*-dimensional* D*-module one to one corresponds to* $r \in \overline{K}$*.*

Algorithm 5.4.2 *TProModule*

- *Input: any two modules* M_1 *and* M_2 *over the same ring.*
- *Output: the tensor product module of* M_1 *and* M_2*:* $M_1 \otimes M_2$*.*
- *Comments.*
  – *The basis of* $M_1 \otimes M_2$ *is* $\{b \otimes B\}$ *for any basis element* $b \in M_1$ *and any basis element* $B \in M_2$*.*
  – $\partial_x(b \otimes B) = (\partial_x b) \otimes B + b \otimes (\partial_x B)$ *and* $\partial_y(b \otimes B) = (\partial_y b) \otimes B + b \otimes (\partial_y B)$ *for any* $b \in M_1$ *and any* $B \in M_2$*.*

Example 5.4.3 *Let* $M_1 = [[x, y], [\partial_x, \partial_y], [\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}]]$ *and* $M_2 = [[x, y], [\partial_x, \partial_y], [[c], [d]]]$*. Let* $\{b_1, b_2\}$ *be the basis of* M_1 *and* $\{B\}$ *be a basis of* M_2*. Then* $\{b_1 \otimes B, b_2 \otimes B\}$ *is a basis of* $M_1 \otimes M_2$*. And the relations:*

$$\partial_x b_1 = a_{11} b_1 + a_{21} b_2,$$

$$\partial_x b_2 = a_{12} b_1 + a_{22} b_2,$$

$$\partial_x B = cB.$$

Then:

$$\partial_x (b_1 \otimes B) = (\partial_x b_1) \otimes B + b_1 \otimes (\partial_x B) = (a_{11} + c)(b_1 \otimes B) + a_{21}(b_2 \otimes B),$$

$$\partial_x (b_2 \otimes B) = (\partial_x b_2) \otimes B + b_2 \otimes (\partial_x B) = a_{12}(b_1 \otimes B) + (a_{22} + c)(b_2 \otimes B).$$

So the matrix of $M_1 \otimes M_2$ *with respect to* x *is* $\begin{bmatrix} a_{11}+c & a_{12} \\ a_{21} & a_{22}+c \end{bmatrix}$, *likewise, one can obtain the matrix with respect to* y: $\begin{bmatrix} b_{11}+d & b_{12} \\ b_{21} & b_{22}+d \end{bmatrix}$. *So* $M_1 \otimes M_2$ *is* $[[x, y], [\partial_x, \partial_y], [\begin{bmatrix} a_{11}+c & a_{12} \\ a_{21} & a_{22}+c \end{bmatrix}, \begin{bmatrix} b_{11}+d & b_{12} \\ b_{21} & b_{22}+d \end{bmatrix}]]$.

Remark 5.4.4 *Let* I *be the* 1*-dimensional module of function* r, *then* $M_1 \otimes I$ *is equivalent to applying* $\xrightarrow{(ii), r}$ *on the minimal operator of* M_1.

Algorithm 5.4.5 *projHom*

- *Input: two D-modules* M_1, M_2 *and a number* N *indicating options (comment).*
- *Output: the set of lists* $[r, h]$ *which gives* $M_1 \xrightarrow{(ii), r} \xrightarrow{(iii), h} M_2$, *i.e.,* M_2 *is homomorphic (under* h*) to* $M_1 \otimes I$ *where* I *is the* 1*-dimensional module for* r.
- *Steps.*
  - *1. For each variable* (x, y), *use Algorithm 5.2.2 to find cyclic vector of* M_1 *and its minimal operators* L_{1x} *and* L_{1y}. *Likewise, find minimal operators* L_{2x} *and* L_{2y} *of* M_2. *Let* S_1 *be the set of singularities of* L_{1x} *and* L_{2x} *with form* $x = p \in \mathbb{P}^1$. *Let* S_2 *be the set of singularities of* L_{1y} *and* L_{2y} *with form* $y = q \in \mathbb{P}^1$. *Let* S_3 *be the set of other singularities (e.g.* $x = y$*).*
  - *2. For singularities in* S_1, *compute exponents of* L_{1x}, *say* $\{e_1, \ldots, e_m\}$, *and exponents of* L_{2x}, *say* $\{f_1, \ldots, f_n\}$. *Let* D_{x-p} *be the exponent difference set of all possible* $f_i - e_j$ *with* $i = 1, \ldots, n$, $j = 1, \ldots, m$. *Let*

$$f_p = \begin{cases} x - p, & p \in \mathbb{C} \\ \frac{1}{x}, & p = \infty. \end{cases}$$

Then $(f_p)^d$ *with* $d \in D_{x-p}$ *is a candidate factor of* r*. For singularities in* S_2*, let* D_{y-q} *be the exponent difference set of* L_{1y} *and* L_{2y}*. Let*

$$f_q = \begin{cases} y - p, & p \in \mathbb{C} \\ \frac{1}{y}, & p = \infty. \end{cases}$$

Then $(f_q)^d$*,* $d \in D_{y-q}$*, is a candidate factor of* r*.*

– *3. For singularities in* S_3*, say* $F(x, y)$*, find the exponent difference set* D_{Fx} *from* L_{1x}*,* L_{2x} *and* D_{Fy} *from* L_{1y}*,* L_{2y}*. For each* $d \in D_{Fx}$*, if there exists* $\widetilde{d} \in D_{Fy}$ *such that* $d - \widetilde{d} \in \mathbb{Z}$*, then* $F(x, y)^d$ *is a candidate factor of* r*.*

– *4. For each possible function* $r = \prod_{p \in S_1,\, q \in S_2,\, F \in S_3} (f_p)^{d_1} (f_q)^{d_2} F(x, y)^{d_3}$ *where* d_1*,* d_2 *and* d_3 *are exponent differences at singularities, use Algorithm 5.4.1 to obtain the module for* r*, say* I*, then use Algorithm 5.4.2 to obtain* $M_1 \otimes I$*, then apply Algorithm 5.3.10 on* $M_1 \otimes I$ *and* M_2*, if there exists a homomorphism* h*, then* $[r, h]$ *is a list in the output.*

- *Comment: there may be hundreds of candidates* r*. Adding options to the projective homomorphism, such as surjective or injective, may rule out lots of candidates (Example 5.4.7 below). The number in the input controls this:* $N = 1$ *for surjective,* $N = 2$ *for injective and* $N = 0$ *for the general case.*

Remark 5.4.6 *Suppose there exists a projective homomorphism* h *from D-module* M_1 *to* M_2*, then*

- *If* M_1 *is irreducible, then* h *is injective.*
- *If* M_2 *is irreducible, then* h *is surjective.*

Example 5.4.7 *Let* $M_1 = F_1^D(1, b_1, b_2, c \,|\, x, y)$ *(reducible) and* M_2 *be the D-module of* ${}_2F_1(1 - b_1, c - b_1 - b_2, c - b_1 \,|\, \frac{y(x-1)}{x(y-1)})$*. Algorithm 5.4.5 gives projective homomorphism from* M_1 *to* M_2*:* $r = x^{c-b_1-b_2-1}(x-1)^{b_1-c}(x-y)^{b_1-2+b_2}(y-1)^{1-b_1}$ *and the homomorphism:*

$$h = \begin{bmatrix} \frac{1}{x(x^2-x-xy+y)} & \frac{cx-x-b_1x+xy-b_2xy+b_2y+2b_1y-cy}{b_1x(x-1)(x-y)(x^2-x-xy+y)} & \frac{b_2xy-b_2x-xy+b_1xy+y-b_1y}{b_2x(x-1)(x-y)^2(y-1)} \\ 0 & \frac{(b_1-1)(b_1+b_2-c)y}{b_1(b_1-c)x^2(y-1)(x^2-x-xy+y)} & \frac{(b_1-1)(b_1+b_2-c)y}{b_2(b_1-c)x^2(x-y)(y-1)^2} \end{bmatrix}$$

There are 243 *candidates for* r *when using Algorithm 5.4.5 by projHom(*M_1*,*M_2*,0). But* M_2 *is irreducible, so by Remark 5.4.6, the projective homomorphism is surjective. By adding this option, the number of candidates of* r *drops to* 1 *by projHom(*M_1*,*M_2*,1), which saves a lot of computation.*

Remark 5.4.8 *[24] gives a similar relation between* $F_1(a, b_1, b_2, b_1 + b_2 \,|\, x, y)$ *(reducible) and some* ${}_2F_1$ *function as in Example 5.4.7.*

CHAPTER 6

APPLICATIONS

Using the tools in Chapter 5, this chapter mainly discusses if the order 2 factors of reducible A-hypergeometric systems of order 3 come from the ${}_2F_1$ function, the only globally bounded (Definition 1.1.4) A-hypergeometric function of order 2. This partly answers the Question 3 in Section 1.1.

6.1 Reducible Appell F_1

Let $K = \mathbb{C}(x, y)$ and $D = K[\partial_x, \partial_y]$. Example 5.4.7 gives one reducible case of $F_1^D(a, b_1, b_2, c \,|\, x, y)$. This section discusses all reducible cases: $F_1(a, b_1, b_2, c \,|\, x, y)$ is reducible if and only if [4]

$$a \in \mathbb{Z} \text{ or } b_1 \in \mathbb{Z} \text{ or } b_2 \in \mathbb{Z} \text{ or } c - a \in \mathbb{Z} \text{ or } c - b_1 - b_2 \in \mathbb{Z}. \tag{6.1}$$

Theorem 6.1.1 *Let $M = F_1^D(a, b_1, b_2, c \,|\, x, y)$ as defined in Section 5.1. In each reducible case (6.1), M is projectively equivalent to $F_1^D(a' \in \mathbb{Z}, b_1', b_2', c' \,|\, x', y')$ where $\mathbb{C}(x', y') = \mathbb{C}(x, y)$.*

Pf : Proposition 4.2.1 gives (up to automorphisms of $\mathbb{C}(u, v)$, take $u = x$ and $v = y$) projective equivalences among the modules of:

- (i) $F_1(a, b_1, b_2, c)$
- (ii) $F_1(c - a, b_1, b_2, c)$
- (iii) $F_1(a, c - b_1 - b_2, b_2, c)$
- (iv) $F_1(a, b_1, c - b_1 - b_2, c)$
- (v) $F_1(a, b_1, b_2, b_1 + b_2 + a + 1 - c)$.

Observe that[1]: $a \in \mathbb{Z} \overset{(ii)}{\longleftrightarrow} c - a \in \mathbb{Z} \overset{(i)}{\longleftrightarrow} 4^{th} - 1^{st}\,\text{entry in}\,\mathbb{Z} \overset{(v)}{\longleftrightarrow} b_1 + b_2 + 1 - c \in \mathbb{Z} \Leftrightarrow c - b_1 - b_2 \in \mathbb{Z} \overset{(iii)}{\longleftrightarrow} 2^{nd}\,\text{entry in}\,\mathbb{Z} \overset{(i)}{\longleftrightarrow} b_1 \in \mathbb{Z}$. Likewise, $b_2 \in \mathbb{Z}$ reduces to $a \in \mathbb{Z}$ using (iv) instead of (iii). So all reducible cases in (6.1) reduce to $a \in \mathbb{Z}$ under the transformations from Proposition 4.2.1.

[1]note $\leftrightarrow$ refers to a transformation from proposition 4.2.1 and $\Leftrightarrow$ refers to equivalent statements.

Theorem 6.1.2 *If $c \notin \mathbb{Z}$, then modules $F_1^D(a \in \mathbb{Z}, b_1, b_2, c \,|\, x, y)$ reduce to $F_1^D(0, b_1, b_2, c \,|\, x, y)$ or $F_1^D(1, b_1, b_2, c \,|\, x, y)$ under isomorphisms, otherwise they reduce to $F_1^D(0, b_1, b_2, c \,|\, x, y)$ or $F_1^D(1, b_1, b_2, c \,|\, x, y)$ or $F_1^D(c, b_1, b_2, c \,|\, x, y)$ when $c \in \mathbb{Z}$ and $c \geq 1$ or $F_1^D(c-1, b_1, b_2, c \,|\, x, y)$ when $c \in \mathbb{Z}$ and $c \leq 0$.*

Pf : Example 5.3.6 gives a homomorphism between $F_1^D(a, b_1, b_2, c \,|\, x, y)$ and $F_1^D(a{+}1, b_1, b_2, c \,|\, x, y)$:

$$H = \begin{bmatrix} a - b_1 - b_2 & \frac{b_1+b_2-c+1}{x-1} & \frac{b_1+b_2-c+1}{y-1} \\ b_1 & -\frac{a+b_1-c+1}{x-1} & -\frac{b_1}{y-1} \\ b_2 & \frac{-b_2}{x-1} & -\frac{a+b_2-c+1}{y-1} \end{bmatrix}.$$

This is an isomorphism if $a \neq 0$ and $a \neq c-1$ since then $\det(H) = \frac{a(a+1-c)^2}{(x-1)(y-1)} \neq 0$. Let $M_a = F_1^D(a, b_1, b_2, c \,|\, x, y)$.

- (i) If $c \notin \mathbb{Z}$, then $M_1 \cong M_2 \cong M_3 \cong \ldots$ and $M_0 \cong M_{-1} \cong M_{-2} \cong \ldots$, so for any $a \in \mathbb{Z}$, $M_a \cong M_0$ or $M_a \cong M_1$.
- (ii) If $c \in \mathbb{Z}$ and $c \geq 1$, then $M_0 \cong M_{-1} \cong M_{-2} \cong \ldots$, $M_1 \cong \ldots \cong M_{c-1}$ and $M_c \cong M_{c+1} \cong M_{c+2} \cong \ldots$, so for any $a \in \mathbb{Z}$, M_a is isomorphic to M_0, M_1 or M_c.
- (iii) If $c \in \mathbb{Z}$ and $c \leq 0$, similar to (ii), M_a is isomorphic to M_0, M_1 or M_{c-1}.

Definition 6.1.3 *Given a D-module $M = [[x, y], [\partial_x, \partial_y], [M_x, M_y]]$ with a basis $\{B_1, \ldots, B_n\}$, recall that M_x and M_y are the derivative matrices of the basis with respect to x and y respectively. The dual module of M is the D-module with basis $\{B_1^*, \ldots, B_n^*\}$ and:*

$$B_i^*(B_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j, \end{cases}$$

$i, j = 1, \ldots, n.$

Algorithm 6.1.4 *dualmodule*

- *Input: any D-module $[[x, y], [\partial_x, \partial_y], [M_x, M_y]]$.*
- *Output: the dual module of the input $[[x, y], [\partial_x, \partial_y], [-M_x^T, -M_y^T]]$.*
- *Comment. Let $X_{j,i} = (M_x)_{j,i}$ for $i, j = 1, \ldots, n$. Since $0 = \partial_x(B_i^* \cdot B_j) = \partial_x(B_i^*)(B_j) + B_i^*(\partial_x(B_j))$, then*

  $$\partial_x(B_i^*)(B_j) = -B_i^*(\partial_x(B_j)) = -B_i^*(X_{j,1}B_1 + \ldots + X_{j,i}B_i + \ldots) = -X_{j,i}.$$

  So the derivative matrix with respect to x is $-M_x^T$. Likewise, the derivative matrix with respect to y is $-M_y^T$.

Theorem 6.1.5 *The modules* $F_1^D(0, b_1, b_2, c\,|\,x, y)$ *and* $F_1^D(1, b_1, b_2, c\,|\,x, y)$ *reduce to each other under the dual.*

Pf : there is a homomorphism between the dual module of $F_1(1-a, 1-b_1, 1-b_2, 3-c\,|\,x, y)$ and the module of $F_1(a, b_1, b_2, c\,|\,x, y)$, which is computed by Algorithms 6.1.4 and 5.3.10.

Definition 6.1.6 *A D-module M comes from* ${}_2F_1$ *if there exists a* 1*-dimensional module I such that M is homomorphic to the tensor product of I and the module of* ${}_2F_1(a, b, c\,|\,g(x, y))$ *for some* $a, b, c \in \mathbb{Q}$ *and* $g(x, y) \in \mathbb{C}(x, y)$.

Theorem 6.1.7 *Any irreducible* 2^{nd} *order submodule or quotient module of* $F_1^D(a, b_1, b_2, c\,|\,x, y)$ *comes from* ${}_2F_1$.

Pf : Example 5.4.7 shows that the reducible case of $a = 1$ comes from ${}_2F_1$. Now Theorem 6.1.7 is a corollary of Theorems 6.1.1, 6.1.2 and 6.1.5 combined.

6.2 Horn G_2 and Appell F_1

Like Appell's F_1 function, Horn G_2 function is a bivariate order 3 A-hypergeometric function with 4 parameters as well. It is defined by

$$G_2(a_1, a_2, b_1, b_2\,|\,x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_{n-m} (b_2)_{m-n}}{m!n!} x^m y^n.$$

One can compute its D-module ($D = \mathbb{C}(x, y)[\partial_x, \partial_y]$) using the same method as in the computation of $F_1^D(a, b_1, b_2, c\,|\,x, y)$. From now on, denote its D-module as $G_2^D(a_1, a_2, b_1, b_2\,|\,x, y)$. All non-removable singularities of the minimal operator of $G_2(a_1, a_2, b_1, b_2\,|\,x, y)$ are

$$\{x = -1, x = 0, x = \infty, y = -1, y = 0, y = \infty, xy = 1\}.$$

Proposition 6.2.1 *The group of automorphisms on* $[x, y]$ *which preserves the non-removable singularities of* $G_2(a_1, a_2, b_1, b_2\,|\,x, y)$ *is isomorphic to* S_5 *and it is generated by the maps which send* $[x, y]$ *to* $[y, x]$, $[\frac{1}{x}, \frac{1}{y}]$, $[x, -\frac{1}{xy}]$ *and* $[x, \frac{y+1}{xy-1}]$. *Furthermore, the D-modules generated by the following functions are projectively equivalent.*

- $G_2(a_1, a_2, b_1, b_2\,|\,x, y)$.

- $G_2(a_2, a_1, b_2, b_1 \,|\, y, x)$.
- $G_2(a_1, a_2, a_2 + b_2 - a_1, a_1 + b_1 - a_2 \,|\, \frac{1}{x}, \frac{1}{y})$.
- $G_2(b_1 + b_2, a_2, a_2 - b_1, a_1 - a_2 + b_1 \,|\, x, -\frac{1}{xy})$.
- $G_2(1 - a_1 - b_1, a_2, b_1, 1 - a_2 - b_1 - b_2 \,|\, x, \frac{y+1}{xy-1})$.

Pf : count the elements in the group generated by the maps which send $[x, y]$ to $[y, x]$, $[\frac{1}{x}, \frac{1}{y}]$, $[x, -\frac{1}{xy}]$, $[x, \frac{y+1}{xy-1}]$, then one can verify that the transformation group is isomorphic to S_5. Next, for each generator, take $[\frac{1}{x}, \frac{1}{y}]$ for instance, do

- Use Algorithm 5.2.2 to compute the minimal operator (of order 3) of $G_2(a_1, a_2, b_1, b_2 \,|\, x, y)$ and $G_2(A_1, A_2, B_1, B_2 \,|\, \frac{1}{x}, \frac{1}{y})$ with respect to x. Denote them as L_1 and L_2.
- Let $r = \frac{1}{3} \cdot (c_1 - c_2)$ where c_1 and c_2 are coefficients of L_1 and L_2 with respect to ∂_x^2.
- Equate L_2 and $L_1 Ⓢ (\partial_x - r)$ to solve for A_1, A_2, B_1 and B_2. (Now $A_1 = a_1$, $A_2 = a_2$, $B_1 = a_2 + b_2 - a_1$ and $B_2 = a_1 + b_1 - a_2$)
- Repeat the above steps with respect to y and make sure it gives the same solution for A_1, A_2, B_1 and B_2.
- Use Algorithm 5.4.5 to compute the projective homomorphism between $G_2^D(a_1, a_2, a_2 + b_2 - a_1, a_1 + b_1 - a_2 \,|\, \frac{1}{x}, \frac{1}{y})$ and $G_2^D(a_1, a_2, b_1, b_2 \,|\, x, y)$.
- The result of last step is $\{[x^{-1-a_1} y^{-1-a_2}, H]\}$. So the tensor product of the two modules, $G_2^D(a_1, a_2, a_2 + b_2 - a_1, a_1 + b_1 - a_2 \,|\, \frac{1}{x}, \frac{1}{y})$ and the module for $x^{-1-a_1} y^{-1-a_2}$, is isomorphic to $G_2^D(a_1, a_2, b_1, b_2 \,|\, x, y)$. Here H is the homomorphism:

  $$\begin{bmatrix} \frac{1}{xy} & -\frac{a_1}{y} & -\frac{a_2}{x} \\ 0 & -\frac{x}{y} & 0 \\ 0 & 0 & -\frac{y}{x} \end{bmatrix}$$

  with non-zero determinant $\frac{1}{x^2 y}$. So $G_2^D(a_1, a_2, a_2 + b_2 - a_1, a_1 + b_1 - a_2 \,|\, \frac{1}{x}, \frac{1}{y})$ is projectively equivalent to $G_2^D(a_1, a_2, b_1, b_2 \,|\, x, y)$.

Recall that the group of automorphisms which preserves the non-removable singularities of $F_1(a, b_1, b_2, c \,|\, x, y)$ is also S_5 (Propsition 4.2.3), so we want to know if G_2 and F_1 define the equivalent system. To uncover this, we use F_1-solver to check if G_2 is solvable in terms of F_1 and it turns out to be true.

Table 6.1: Singularities of F_1 and G_2

A-hypergeometric Functions	$F_1(a, b_1, b_2, c \,\vert\, x, y)$	$G_2(a_1, a_2, b_1, b_2 \,\vert\, x, y)$
Non-removable Singularities	$x = 0,\ x = 1,\ x = \infty,$ $y = 0,\ y = 1,\ y = \infty,$ $x = y$	$x = -1,\ x = 0,\ x = \infty,$ $y = -1,\ y = 0,\ y = \infty,$ $xy = 1$

Example 6.2.2 *Let L be the minimal operator of $G_2(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7} \,|\, x, 2x-1)$. The F_1-solver in Chapter 4 finds the F_1-type solution*

$$\frac{5x+2}{6x(x+1)} \cdot F_1(\frac{23}{35}, \frac{1}{2}, \frac{1}{3}, \frac{19}{14} \,|\, \frac{1}{x+1}, \frac{2x-1}{2x})$$

in the following steps.

- *Algorithm 4.1.3 finds* 408 *candidate pairs $[u, v]$ in $F_1(a, b_1, b_2, c \,|\, u, v)$.*
- *Algorithm 4.2.4 divides these candidates $[u, v]$ into* 14 *orbits.*
- *For each orbit, take the first pair $[u, v]$ and use Algorithm 4.3.3 to compute the parameters in F_1 and the exp-product parameter r.*

Next we try to find the exact relation between G_2 and F_1. To compute transformations on $[x, y]$ which send the non-removable singularities of F_1 to those of G_2, compare their singularities (Table 6.1). By observation, one of such transformations is: $[x, y] \mapsto [-x, -\frac{1}{y}]$. Using the same method in Proposition 6.2.1, we found the D-modules of the following functions are projectively equivalent

- $F_1(a, b_1, b_2, c \,|\, x, y)$
- $G_2(b_1, b_2, 1 + b_2 - c, a - b_2 \,|\, -x, -\frac{1}{y})$

and Algorithm 5.4.5 gives the projective homomorphism: $[y^{-b_2-1}, H]$. Here H is the homomorphism

$$\begin{bmatrix} \frac{1}{y} & -\frac{b_1}{xy} & 0 \\ 0 & -\frac{b_1}{xy} & 0 \\ 0 & 0 & b_2 \end{bmatrix}$$

with the determinant $-\frac{b_1 b_2}{xy^2}$.

This implies the systems of Appell F_1 and Horn G_2 are equivalent. So Theorem 6.1.7 also hold for G_2: any irreducible 2^{nd} order submodule or quotient module of $G_2^D(a_1, a_2, b_1, b_2 \,|\, x, y)$ comes from ${}_2F_1$.

Remark 6.2.3 *In [16], it is stated that* $y^{-b_2} \cdot G_2(b_1, b_2, 1+b_2-c, a-b_2 \,|\, -x, -\frac{1}{y})$ *and* $F_1(a, b_1, b_2, c \,|\, x, y)$ *satisfy the same differential equations.*

6.3 Reducible Horn G_3

Horn G_3 function is another A-hypergeometric function of order 3 which is defined by:

$$G_3(a, b \,|\, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m-n}(b)_{2n-m}}{m!n!} x^m y^n.$$

One can compute its D-module ($D = \mathbb{C}(x, y)[\partial_x, \partial_y]$) and we denote it as $G_3^D(a, b \,|\, x, y)$. From [4], $G_3(a, b \,|\, x, y)$ is reducible if and only if

$$a + 2b \in \mathbb{Z} \;\text{ or }\; 2a + b \in \mathbb{Z}.$$

Theorem 6.3.1 *All reducible cases of* $G_3(a, b \,|\, x, y)$ *reduce to* $a+2b = 0$ *or* $a+2b = 1$ *or* $2a+b = 0$ *or* $2a + b = 1$ *under isomorphisms.*

Pf : Algorithm "hom" gives a homomorphism between $G_3^D(a, b \,|\, x, y)$ and $G_3^D(a+1, b \,|\, x, y)$ and the determinant is nonzero if and only $a + 2b \neq 0$. So the reducible cases $a + 2b \in \mathbb{Z}$ reduce to $a + 2b = 0$ and $a + 2b = 1$. Likewise the reducible cases $2a + b \in \mathbb{Z}$ reduce to $2a + b = 0$ and $2a + b = 1$.

Theorem 6.3.2 *The reducible* $G_3(1 - 2b, b \,|\, x, y)$ *satisfies the same differential equations as*

$$(3y + 1)^{\frac{3b}{2}-1} y^{1-2b} \cdot {}_2F_1(\frac{1}{3} - \frac{1}{2}b, \frac{2}{3} - \frac{1}{2}b, \frac{1}{2} \,|\, \frac{(27xy^2 - 9y - 2)^2}{4(3y + 1)^3}).$$

One can test this relation using Algorithm 5.4.5. Steps to find this relation:

- (i) Obtain the pullback function $\frac{(27xy^2-9y-2)^2}{4(3y+1)^3}$ in ${}_2F_1$.
  - Fix b in $G_3(1 - 2b, b \,|\, x, y)$ with some value. Substitute y by some function $f \in \mathbb{C}(x)$. Let L be the minimal operator of the cyclic vector in $G_3^D(1 - 2b, b \,|\, x, f)$ with respect to x, then L has a right factor L_2, an order 2 operator.
  - For each L_2, use Algorithm 5.2.1 in [19] to obtain its ${}_2F_1$-type solution containing a pullback function $p \in \mathbb{C}(x)$.
  - Collect a bunch of pairs $[f, p]$ and interpolate p in terms of f and x.

- (ii) Obtain parameters in ${}_2F_1$: fix $y \in \mathbb{C}(x)$. Assign b to different values. Each b corresponds to a list of parameters in ${}_2F_1$ solution as above. Then using these data, interpolate parameters in terms of b.

- (iii) Now use Algorithm 5.4.5 to find the projective homomorphism between $G_3^D(1-2b, b \,|\, x, y)$ and ${}_2F_1^D(\frac{1}{3} - \frac{1}{2}b, \frac{2}{3} - \frac{1}{2}b, \frac{1}{2} \,|\, \frac{(27xy^2-9y-2)^2}{4(3y+1)^3})$.

Theorem 6.3.3 *Any irreducible* 2^{nd} *order submodule or quotient module of* $G_3^D(a, b \,|\, x, y)$ *comes from* ${}_2F_1$.

6.4 Reducible Horn G_1

Horn G_1 function is another A-hypergeometric function of order 3 and defined by

$$G_1(a, b_1, b_2 \,|\, x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}}{m!n!} x^m y^n$$

From [4], $G_1(a, b_1, b_2 \,|\, x, y)$ is reducible if and only if

$$\{a, a+b_1, a+b_2, b_1+b_2\} \cap \mathbb{Z} \neq \emptyset.$$

Reference [16] gives the relation between G_1 and F_1:

$$(1+x+y)^a \cdot G_1(a, b_1, b_2 \,|\, x, y) = F_1(1-b_1-b_2, a, a, a-b_1+1 \,|\, \frac{1+2x+\sqrt{1-4xy}}{2(1+x+y)}, \frac{1+2x-\sqrt{1-4xy}}{2(1+x+y)}). \tag{6.2}$$

As what we did on D-module of F_1, one can compute the D-module of G_1 and then apply the algorithms on it. After applying the same trick as in the proof of Theorem 6.1.1 on the parameters in the Relation (6.2), Theorem 6.1.7 implies the same result for G_1:

Theorem 6.4.1 *Any irreducible* 2^{nd} *order submodule or quotient module of* $G_1^D(a, b_1, b_2 \,|\, x, y)$ *comes from* ${}_2F_1$.

CHAPTER 7

ORDER 3 A-HYPERGEOMETRIC FUNCTIONS AND THEIR POLYTOPES

7.1 Structure of A-hypergeoemtric Functions[1]

The definition of A-hypergeometric functions begins with a finite subset $A \subseteq \mathbb{Z}^r$ (hence the name) consisting of N vectors $a_1, \ldots, a_N$ such that:

- $\mathrm{SPAN}_{\mathbb{Z}}(a_1, \ldots, a_N) = \mathbb{Z}^r$ and
- there exists a linear form h on $\mathbb{R}^r$ such that $h(a_i) = 1$ for all i. [2]

Let A denote the matrix $(a_1, \ldots, a_N)_{r \times N}$. A vector of parameters $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$ is also given. The lattice $L := \{(l_1, \ldots, l_N) \in \mathbb{Z}^N \mid \sum_{i=1}^{N} l_i a_i = 0\}$.

The A-hypergeometric equations are a set of partial differential equations with independent variables $v_1, \ldots, v_N$. This set includes two groups. The first group consists of the structure equations:

$$\Box_1 \Phi := \prod_{l_i > 0} \partial_i^{l_i} \Phi - \prod_{l_i < 0} \partial_i^{|l_i|} \Phi = 0 \tag{7.1}$$

for all $l = (l_1, \ldots, l_N) \in L$. The second group consists of the homogeneity or Euler equations.

$$Z_i \Phi := (a_{1,i} v_1 \partial_1 + \ldots + a_{N,i} v_N \partial_N - \alpha_i) \Phi = 0, \; i = 1, \ldots, r \tag{7.2}$$

where $a_{k,i}$ denotes the i-th coordinate of a_k.

Now a formal solution of the A-hypergeometric system (7.1), (7.2) can be given by

$$\Phi_{L,\gamma}(v_1, \ldots, v_N) = \sum_{l \in L} \frac{v_1^{l_1+\gamma_1} \cdot \ldots \cdot v_N^{l_N+\gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdot \ldots \cdot \Gamma(l_N + \gamma_N + 1)}$$

[1] This whole section is from [4].
[2] This condition ensures the function is regular singular.

where $l = (l_1, \ldots, l_N)$ and $(\gamma_1, \ldots, \gamma_N)$ satisfies $\alpha = \gamma_1 a_1 + \ldots + \gamma_N a_N$.

Example 7.1.1 *Let the matrix* $A = (a_1, a_2, a_3, a_4) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. *Then* $L = \mathrm{SPAN}_{\mathbb{Z}}([-1, -1, 1, 1])$. $\alpha = (-a, -b, c-1)$, *choose* $\gamma = (-a, -b, c-1, 0)$. *Then the formal solution is*

$$\Phi_{L,\gamma} = v_1^{-a} \cdot v_2^{-b} \cdot v_3^{c-1} \sum_{k \in \mathbb{Z}} \frac{v_1^{-k} v_2^{-k} v_3^k v_4^k}{\Gamma(-k-a+1)\Gamma(-k-b+1)\Gamma(c+k)\Gamma(k+1)} \tag{7.3}$$

$$= v_1^{-a} \cdot v_2^{-b} \cdot v_3^{c-1} \cdot \frac{sin(\pi a) sin(\pi b)}{\pi^2} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} (\frac{v_3 v_4}{v_1 v_2})^k. \tag{7.4}$$

Let $v_1 = v_2 = v_3 = 1$ *and* $v_4 = z$ *then it becomes* ${}_2F_1(a, b, c \,|\, z)$.

Definition 7.1.2 *The polytope corresponding to the matrix* $A = (a_1, \ldots, a_N)_{(r \times N)}$ *is the convex hull of the endpoints of* a_i.

Remark 7.1.3 *Example 7.1.1 gives a way to connect matrix A and A-hypergeometric functions. In this way, given an A-hypergeometric function, one can construct its matrix A and therefore find its corresponding polytope.*

7.2 Relations among A-hypergeometric Functions

Theorem 7.2.1 *Suppose the matrices of A-hypergeometric functions* f_1 *and* f_2 *are* $A = (a_1, \ldots, a_N)_{r \times N}$ *and* $B = (b_1, \ldots, b_N)_{r \times N}$. *Then* f_1 *and* f_2 *are equivalent if*

- $\{a_1, \ldots, a_N\} = \{b_1, \ldots, b_N\}$ *or*
- *there exists an* $r \times r$ *matrix* C *such that* $\{B_1, \ldots, B_N\} = \{b_1, \ldots, b_N\}$ *where* $(B_1, \ldots, B_N) := C \cdot (a_1, \ldots, a_N)$.

Algorithm 7.2.2 *Check Equivalence*

- *Input: two matrices* $(A)_{r \times N}$ *and* $(B)_{r \times N}$ *with* $r < N$ *corresponding to two A-hypergeometric functions.*
- *Output: one* $r \times r$ *matrix or "No equivalence".*
- *Steps.*
  - *1. Consider A and B as two sets of columns rather than matrices and check if they are equal as sets. If yes, then return the* $r \times r$ *identity matrix, otherwise, do step 2.*

– *2. Take r linearly independent columns of A, denoted as $(M_A)_{r\times r}$.*

– *3. Take r columns of B, denoted as $(M_B)_{r\times r}$. Let $M := M_B \cdot M_A^{-1}$, i.e., $M \cdot M_A = M_B$, check if $M \cdot A$ and B satisfy the condition in step 1. If yes, then return M, otherwise, repeat Step 3 until all such M_B are tested. If none of them gives a matrix M, then return "No equivalence".*

Example 7.2.3 *Appell's F_1 function can be obtained by the matrix A and Horn G_2 function by the matrix B as follows: $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.*

Algorithm 7.2.2 returns the matrix $M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, which verifies the equivalence of these two functions as stated in Section 6.2. Likewise, Algorithm 7.2.2 also finds the equivalences among Horn functions G_1, H_3 and H_6 (the equivalence of H_3 and H_6 was stated in [16]).

Definition 7.2.4 *Suppose the matrices of two A-hypergeometric functions f_1 and f_2 are $A = (a_1, \ldots, a_N)_{r\times N}$ and $B = (b_1, \ldots, b_N)_{r\times N}$. If there exists a matrix $(C)_{r\times r}$ such that*

- *$\{B_1, \ldots, B_N\} \subseteq \{b_1, \ldots, b_N\}$ with $(B_1, \ldots, B_N) = C \cdot A$ and*
- *After reordering rows and columns (if necessary), $C \cdot A$ can be written in $\begin{bmatrix} (\widetilde{A})_{(r-1)\times(N-1)} & \alpha \\ 0 & 0 \end{bmatrix}$ and there exists a column in $\widetilde{A}$ which equals α and the matrix $\widetilde{A}_{(r-1)\times(N-1)}$ also defines f_2,*

then f_1 and f_2 has weak equivalence. In fact, f_2 is a special case of f_1.

Similar to Algorithm 7.2.2, we developed an algorithm to test weak equivalence and using it, we found:

$$G_3 \subseteq G_1 \sim H_3 \sim H_6 \subseteq F_1 \sim G_2$$

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APPENDIX A

GLOBALLY BOUNDED BUT NOT ${}_3F_2$-SOLVABLE THIRD ORDER OPERATOR

The order 3 differential operator L is:

$$L = \partial^3 + \frac{5891x^3 + 9388x^2 - 11890x + 3000}{15x(x+4)(43x-20)(2x-1)}\partial^2 + \frac{235296x^3 + 30775x^2 - 191300x + 36000}{900x^2(x+4)(43x-20)(2x-1)}\partial + \frac{3096x^2 - 5005x - 1900}{900x^2(x+4)(43x-20)(2x-1)}.$$

Our goal is to show:

- L is not projectively equivalent (2.1.11) to $L_B^{x\mapsto f}$ for any pullback function $f \in \mathbb{C}(x)$ where L_B is a minimal operator of a ${}_3F_2$ function and $L_B^{x\mapsto f}$ denotes the operator after applying a change of variables (2.1.8) on L_B.
- L has a globally bounded (1.1.1) solution.

After showing these, L is a counter-example for Question 1.

A.1 Not ${}_3F_2$-solvable

Lemma A.1.1 *If $L_B^{x\mapsto f}$ is projectively equivalent to L, then its non-removable singularities (2.1.16) are $\{-4, \frac{1}{2}, \infty, 0\}$ and they have the following local properties.*

Singularities	*Logarithmic*	$\Delta(L_B^{x\mapsto f}, p)$ *up to* $\sim_{\mathbb{Z}}$
-4	*No*	$1, 3/10$
$1/2$	*No*	$1, 19/30$
∞	*No*	$1, -1/30$
0	*Yes*	$0, 1/2$

Here Δ gives the exponent differences at the singularity $x = p$ and the equivalence $\sim_{\mathbb{Z}}$ is defined in Definition A.1.2 below.

To prove Lemma A.1.1, we need some definitions and lemmas.

Definition A.1.2 *Let $\{e_1, e_2, e_3\}$ be the exponents of a regular singular (2.1.18) third order operator L at $x = p$, then define $\Delta(L, p)$ as $(e_2 - e_1, e_3 - e_1)$. Note that Δ is only well-defined up to an equivalence denoted as $\sim_2$ where the equivalence class of $(d_1, d_2) \in \mathbb{C}^2$ is $\{(d_1, d_2), (d_2, d_1), (-d_1, d_2 - d_1), (d_2 - d_1, -d_1), (-d_2, d_1 - d_2), (d_1 - d_2, -d_2)\}$ (one entry for each permutation of $[e_1, e_2, e_3]$).*

Lemma A.1.3 *If $L_2 = L_1 \circledS (\partial - r)$ then $\Delta(L_1, p) \sim_2 \Delta(L_2, p)$.*

Lemma A.1.4 *If L_1 is projectively equivalent to L_2, then $\Delta(L_1, p) \sim_{\mathbb{Z}} \Delta(L_2, p)$ where $(d_1, d_2) \sim_{\mathbb{Z}} (d_1', d_2')$ if there exist $n_1, n_2 \in \mathbb{Z}$ such that $(d_1, d_2) - (n_1, n_2) \sim_2 (d_1', d_2')$.*

Now Lemma A.1.1 follows from Lemma A.1.3, A.1.4 and the local properties of L:

Singularities	Logarithmic	Exponents	$\Delta(L, p)$ up to $\sim_2$
-4	No	$0, 1, 3/10$	$1, 3/10$
$1/2$	No	$0, 1, 19/30$	$1, 19/30$
∞	No	$1/5, 6/5, 1/6$	$1, -1/30$
0	Yes	$0, 0, 1/2$	$0, 1/2$

As shown in table 3.1, L_B has non-removable singularities $\{0, 1, \infty\}$. If f has degree d and has r_0 roots, r_∞ poles and r_1 roots of $1 - f$, then we expect that $L_B^{x \mapsto f}$ has $r_0 + r_1 + r_\infty$ non-removable singularities. But under some condition (Lemma A.1.6 below), some of them become regular or removable singular. Use D_0, D_1, D_∞ to denote the number of such points corresponding to $0, 1, \infty$, then

$$r_0 + r_1 + r_\infty - D_0 - D_1 - D_\infty = 4.$$

Definition A.1.5 *Say $f(q) = p$ with ramification index m if q is a root of $f(x) - p$ of order m.*

Lemma A.1.6 *For $L_B^{x \mapsto f}$ with $f \in \mathbb{C}(x)$, 2 is the least possible ramification index to make roots of $1 - f$ regular or removable singular and 3 is the least possible ramification index to make roots of f and poles of f regular or removable singular.*

Pf : suppose the exponents of L_B at $x = 1$ are $(0, 1, \frac{1}{2})$ and $f(5) = 1$ with ramification index 2, then the exponents of $L_B^{x \mapsto f}$ at $x = 5$ are $(0, 1, 2)$, so $L_B^{x \mapsto f}$ is regular at $x = 5$. But the situations at $x = 0$ and $x = \infty$ are a little different. Roots of f or poles of f may become regular or removable singular only if the solution of L_B at $x = 0$ and $x = \infty$ is not logarithmic, in which case the exponents cannot be $(0, \frac{1}{2}, n)$ or $(0, \frac{1}{2}, \frac{1}{2} + n)$ with $n \in \mathbb{Z}$ (table 3.1). So the ramification index to make roots of f or poles of f regular or removable singular is at least 3.

Claim A.1.7 *If L is projectively equivalent to $L_B^{x\mapsto f}$ with $f \in \mathbb{C}(x)$, then the degree of f, $d \leq 12$.*

First, we may assume the solution of L_B is logarithmic at $x = \infty$ by the following two reasons.

- By Table 3.1, the solution of L_B is logarithmic at $x = 1$ if and only if $\Delta(L_B, 1) \in \mathbb{Z}^2$ and therefore $\Delta(L_B^{x\mapsto f}, 0) \in \mathbb{Z}^2$. But $\Delta(L_B^{x\mapsto f}, 0) = (0, \frac{1}{2}) \notin \mathbb{Z}^2$, so the solution of L_B is logarithmic either at $x = 0$ or at $x = \infty$.

- Let L_1 and L_2 be the minimal operator of ${}_3F_2(a_1, a_2, a_3; b_1, b_2 \mid x)$ and ${}_3F_2(a_1, 1 - b_1 + a_1, 1 - b_2 + a_1; 1 - a_2 + a_1, 1 + a_1 - a_3 \mid \frac{1}{x})$, then L_1 and L_2 are projectively equivalent. So for any L_B, one can find L'_B which is projectively equivalent to L_B and swaps 0 and ∞.

Next, Hurwitz's equation (introduced below) relates the degree of f, d, to the ramification indices of f at singularities of $L_B^{x\mapsto f}$.

$$\sum_{p\in\mathbb{P}^1} (e_p - 1) = 2d - 2.$$

Here p is a singularity of $L_B^{x\mapsto f}$ and e_p is the ramification index of f at $x = p$. Since the solution of $L_B^{x\mapsto f}$ is logarithmic only at $x = 0$, so $f(0) = \infty$ with ramification index d, which contributes $d - 1$ in $\sum(e_p - 1)$. Therefore:

$$\sum_{f(p)\in\{0,1\}} (e_p - 1) \leq d - 1.$$

So the total number of roots of f and $1 - f$ is

$$r_0 + r_1 = 2d - \sum_{f(p)\in\{0,1\}} (e_p - 1) \geq d + 1.$$

Since $\{-4, \frac{1}{2}, \infty\}$ are non-removable singularities of $L_B^{x\mapsto f}$ from $f^{-1}(\{0, 1\})$, so

$$D_0 + D_1 = r_0 + r_1 - 3 \geq d - 2.$$

On the other hand, by Lemma A.1.6,

$$D_0 \leq \frac{d}{3},$$

$$D_1 \leq \frac{d}{2}.$$

So for any $p \in \{0,1\}$, the number of roots of $f(x) - p$ at which $L_B^{x \mapsto f}$ is regular or removable singular is

$$D_p \geq d - 2 - \max(D_0, D_1) = d - 2 - \frac{d}{2} = \frac{d}{2} - 2. \tag{A.1}$$

Combining with $D_0 \leq \frac{d}{3}$ gives

$$\frac{d}{3} \geq \frac{d}{2} - 2,$$

therefore $d \leq 12$.

Claim A.1.8 *If L is projectively equivalent to $L_B^{x \mapsto f}$ with $f \in \mathbb{C}(x)$, then the degree of f, $d \geq 12$.*

Lemma A.1.9 *Let m_1, m_2 be positive integers and $e \in \mathbb{C}$. If*

$$m_1 e \equiv \frac{3}{10} \bmod \mathbb{Z},$$
$$m_2 e \equiv \frac{19}{30} \bmod \mathbb{Z}.$$

then $m_1 + m_2 \geq 16$.

Pf : refer to the computation file [31].

Lemma A.1.10 *Let m_1, m_2 be positive integers and $d_1, d_2 \in \mathbb{C}$. If the following equivalences hold,*

$$m_1 \cdot (d_1, d_2) \sim_{\mathbb{Z}} (1, \frac{3}{10}) \tag{A.2}$$

$$m_2 \cdot (d_1, d_2) \sim_{\mathbb{Z}} (1, \frac{19}{30}) \tag{A.3}$$

then $m_1 + m_2 \geq 16$.

Pf : By Definition A.1.2 and Lemma A.1.4, Relations (A.2) and (A.3) can be interpreted into 6 cases since we may assume $m_2(d_1, d_2) \equiv (1, \frac{19}{30}) \bmod \mathbb{Z}$.

- CASE 1: (i) $m_1 d_1 \equiv 1 \bmod \mathbb{Z}$, (ii) $m_1 d_2 \equiv \frac{3}{10} \bmod \mathbb{Z}$, (iii) $m_2 d_1 \equiv 1 \bmod \mathbb{Z}$, (iv) $m_2 d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$.

$$\text{(ii)} + \text{(iv)} \xrightarrow{\text{Lemma A.1.9}} m_1 + m_2 \geq 16.$$

- CASE 2: (i) $m_1d_1 \equiv \frac{3}{10} \bmod \mathbb{Z}$, (ii) $m_1d_2 \equiv 1 \bmod \mathbb{Z}$, (iii) $m_2d_1 \equiv 1 \bmod \mathbb{Z}$, (iv) $m_2d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$.
$$\text{(iv)} \Rightarrow d_2 = \frac{n_1}{30n_2},\ n_1, n_2 \in \mathbb{Z},\ \gcd(30, n_1) = 1 \overset{\text{(ii)}}{\Longrightarrow} m_1 \geq 30.$$

- CASE 3: $m_1d_1 \equiv -1 \bmod \mathbb{Z}$, $m_1d_2 \equiv -\frac{7}{10} \bmod \mathbb{Z}$, $m_2d_1 \equiv 1 \bmod \mathbb{Z}$ and $m_2d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$. Same as CASE 1.

- CASE 4: $m_1d_1 \equiv -\frac{7}{10} \bmod \mathbb{Z}$, $m_1d_2 \equiv -1 \bmod \mathbb{Z}$, $m_2d_1 \equiv 1 \bmod \mathbb{Z}$ and $m_2d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$. Same as CASE 2.

- CASE 5: (i) $m_1d_1 \equiv -\frac{3}{10} \bmod \mathbb{Z}$, (ii) $m_1d_2 \equiv \frac{7}{10} \bmod \mathbb{Z}$, (iii) $m_2d_1 \equiv 1 \bmod \mathbb{Z}$, (iv) $m_2d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$.
$$\text{(i)} + \text{(iii)} \Rightarrow m_2 \in 10\mathbb{Z} \overset{\text{(iv)}}{\Longrightarrow} d_2 = \frac{n_1}{300n_2} \overset{\text{(ii)}}{\Longrightarrow} m_1 \geq 30.$$
Here $n_1, n_2 \in \mathbb{Z}$ and $\gcd(n_1, 300) = 1$.

- CASE 6: $m_1d_1 \equiv \frac{7}{10} \bmod \mathbb{Z}$, $m_1d_2 \equiv -\frac{3}{10} \bmod \mathbb{Z}$, $m_2d_1 \equiv 1 \bmod \mathbb{Z}$ and $m_2d_2 \equiv \frac{19}{30} \bmod \mathbb{Z}$. Same as CASE 5.

Corollary A.1.11 *Suppose L is projectively equivalent to $L_B^{x\mapsto f}$. If $f(-4) = f(\frac{1}{2}) = p \in \{0, 1\}$, then $d \geq 16$.*

Pf : all non-removable singularities of L come from the non-removable singularities of L_B, which are $\{0, 1, \infty\}$. Recall that we assume L_B has a logarithmic solution at $x = \infty$, so $\{-4, \frac{1}{2}, \infty\}$, non-logarithmic singularities of $L_B^{x\mapsto f}$, must come from $\{0, 1\}$. Suppose $f(-4) = f(\frac{1}{2}) = p \in \{0, 1\}$ and the ramification indices of $f(x) - p$ at $x = -4$, $x = \frac{1}{2}$ are m_1, m_2, then
$$\Delta(L, -4) \sim_{\mathbb{Z}} m_1\Delta(L_B, p) \quad \text{and} \quad \Delta(L, \frac{1}{2}) \sim_{\mathbb{Z}} m_2\Delta(L_B, p).$$
Denote $\Delta(L_B, p)$ as $(d_1, d_2) \in \mathbb{C}^2$, now
$$m_1(d_1, d_2) \sim_{\mathbb{Z}} \Delta(L, -4) = (1, \frac{3}{10}) \ \text{ and } \ m_2(d_1, d_2) \sim_{\mathbb{Z}} \Delta(L, \frac{1}{2}) = (1, \frac{19}{30}).$$
So the degree of f, $d \geq m_1 + m_2 \geq 16$ by Lemma A.1.10.

Remark A.1.12 *Likewise, the lower degree bound of f can be computed in the following cases.*

Cases	*Lower Degree Bound of f*
$f(-4) = f(\frac{1}{2}) \in \{0, 1\}$	16
$f(-4) = f(\infty) \in \{0, 1\}$	16
$f(\infty) = f(\frac{1}{2}) \in \{0, 1\}$	12

Now prove Claim A.1.8.

Pf : by the proof of Corollary A.1.11, singularities $\{-4, \frac{1}{2}, \infty\}$ come from $\{0, 1\}$. So at least two of them come from the same point. So the table in Remark A.1.12 gives all possible cases, which implies that $d \geq 12$.

Claim A.1.13 *L is not projectively equivalent to $L_B^{x \mapsto f}$ with the pullback function $f \in \mathbb{C}(x)$.*

Pf : if not, then $d = 12$ by Claims A.1.7 and A.1.8. And it only happens when $f(\infty) = f(\frac{1}{2}) = p \in \{0, 1\}$ from the table in Remark A.1.12. Now suppose this holds, then $\frac{1}{2}$ is a root of $f(x) - p$. But the exponent differences of L at $x = \frac{1}{2}$ are $(1, \frac{19}{30})$, so L_B has to have an exponent difference in $\frac{n}{30 \cdot \mathbb{Z}}$ at $x = p$ where $n \in \mathbb{Z}$ and $\gcd(30, n) = 1$. So to make roots of $f(x) - p$ regular or removable singular, the ramification index at that root has to be greater than or equal to 30, which can not happen since the degree of f is $12 < 30$. This fact contradicts with the inequality (A.1): $D_p \geq \frac{d}{2} - 2 = 4$.

A.2 A Globally Bounded Solution

Recall the Appell's series F_1 (Chapter 4) is defined by

$$F_1(a, b_1, b_2, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} \, (b_1)_m \, (b_2)_n}{(c)_{m+n} \, m! n!} x^m y^n,$$

where $(q)_n = q(q+1) \cdots (q+n-1)$.

Proposition A.2.1 *The following recurrence relations follow from the definition of F_1.*

- (R_1) $(a - b_1 - b_2)F_1(a, b_1, b_2, c; x, y) - aF_1(a+1, b_1, b_2, c; x, y) + b_1 F_1(a, b_1+1, b_2, c; x, y) + b_2 F_1(a, b_1, b_2+1, c; x, y) = 0$.
- (R_2) $cF_1(a, b_1, b_2, c; x, y) - (c-a)F_1(a, b_1, b_2, c+1; x, y) - aF_1(a+1, b_1, b_2, c+1; x, y) = 0$.
- (R_3) $cF_1(a, b_1, b_2, c; x, y) + c(x-1)F_1(a, b_1+1, b_2, c; x, y) - (c-a)xF_1(a, b_1+1, b_2, c+1; x, y) = 0$.
- (R_4) $cF_1(a, b_1, b_2, c; x, y) + c(y-1)F_1(a, b_1, b_2+1, c; x, y) - (c-a)yF_1(a, b_1, b_2+1, c+1; x, y) = 0$.

Algorithm A.2.2 *Simplify Appell function: F_1^{simp}.*

- *Input: nonnegative integers n_0, n_1, n_2, n_3 and $a, b_1, b_2, c \notin \mathbb{C} - \{0, -1, -2, ...\}$. Assume $a \not\equiv c \bmod \mathbb{Z}$ and $b_1 + b_2 \not\equiv c \bmod \mathbb{Z}$.*

- *Output: Rewrite* $F_1(a+n_0, b_1+n_1, b_2+n_2, c+n_3; x, y)$ *as a* $\mathbb{C}(x,y)$*-linear combination of* $\{F_1(a, b_1, b_2, c \,|\, x, y), F_1(a, b_1{+}1, b_2, c \,|\, x, y), F_1(a, b_1, b_2{+}1, c \,|\, x, y)\}$*, a basis of* $F_1^R(a, b_1, b_2, c \,|\, x, y)$ *(Section 5.1).*

- *Steps.*

  - *1. If* $(n_0, n_1, n_2, n_3) \in \{(0,0,0,0), (0,1,0,0), (0,0,1,0)\}$*, then return* $F_1(a+n_0, b_1+n_1, b_2+n_2, c+n_3; x, y)$.
  - *2. Let* $S_1 := F_1^{simp}(a+n_0-1, b_1+n_1+1, b_2+n_2, c+n_3; x, y)$,
    $S_2 := F_1^{simp}(a+n_0-1, b_1+n_1, b_2+n_2+1, c+n_3; x, y)$,
    $S_3 := F_1^{simp}(a+n_0-1, b_1+n_1, b_2+n_2, c+n_3; x, y)$.
    If $n_0 > 0$*, then return* $\frac{1}{a+n_0-1}[(b_1{+}n_1)\cdot S_1 + (b_2{+}n_2)\cdot S_2 + (a{+}n_0{-}1{-}b_1{-}b_2{-}n_1{-}n_2)\cdot S_3]$. *(This follows from* R_1*.)*
  - *3. Let* $S_4 := F_1^{simp}(a+n_0, b_1+n_1, b_2+n_2, c+n_3-1; x, y)$,
    $S_5 := F_1^{simp}(a+n_0, b_1+n_1, b_2+n_2+1, c+n_3-1; x, y)]$,
    $S_6 := F_1^{simp}(a+n_0, b_1+n_1+1, b_2+n_2, c+n_3-1; x, y)]$.
    Now $n_0 = 0$*. If* $n_3 > 0$*, then return* $\frac{c+n_3-1}{(a+n_0-c-n_3+1)(b_1+n_1-c-n_3+1+b_2+n_2)xy}[((c+n_3-1-a-n_0)xy - (b_2+n_2)x - (b_1+n_1)y)\cdot S_4 - (b_2+n_2)x(y-1)\cdot S_5 - (b_1+n_1)y(x-1)\cdot S_6]$. *(To get this, first use step 1 to reduce* R_2*, then use* R_3 *and* R_4 *to reduce it further.)*
  - *4. Let* $S_7 := F_1^{simp}(a+n_0, b_1+n_1, b_2+n_2-1, c+n_3; x, y)$,
    $S_8 := F_1^{simp}(a+n_0, b_1+n_1-1, b_2+n_2, c+n_3; x, y)$.
    Now $n_0 = n_3 = 0$*. If* $n_1 \cdot n_2 > 0$*, then return* $\frac{1}{x-y}(x\cdot S_7 - y\cdot S_8)$*. (Do substitutions* $b_1 = b_1+1$ *in* R_4 *and* $b_2 = b_2+1$ *in* R_3*. Denote the new relations as* R_6 *and* R_5*. Then* $x\cdot R_6 - y\cdot R_5$ *gives the term.)*
  - *5. Let* $S_9 := F_1^{simp}(a+n_0, b_1+n_1-2, b_2+n_2, c+n_3; x, y)$,
    $S_{10} := F_1^{simp}(a+n_0, b_1+n_1-1, b_2+n_2+1, c+n_3; x, y)$.
    Now $n_0 = n_3 = 0$ *and* $n_1 \cdot n_2 = 0$*. If* $n_1 > 1$*, then return* $\frac{1}{(b_1+n_1-1)(x-1)y}[(b_1+n_1-1-c-n_3+b_2+n_2)y\cdot S_9 + (-2y+xy+(c+n_3)y-(b_2+n_2)x-(b_2+n_2)y+(b_1+n_1-2)xy-(a+n_0)xy+(b_2+n_2)xy-2(b_1+n_1-2)y)\cdot S_8 + ((b_2+n_2)(x-xy)\cdot S_{10})]$*. (This follows from* R_3 *reduced from step 1 to step 2.)*
  - *6. Let* $S_{11} := F_1^{simp}(a+n_0, b_1+n_1, b_2+n_2-2, c+n_3; x, y)$,
    $S_{12} := F_1^{simp}(a+n_0, b_1+n_1+1, b_2+n_2-1, c+n_3; x, y)$.
    Now $n_0 = n_1 = n_3 = 0$*. If* $n_2 > 1$*, then return* $\frac{1}{(b_2+n_2-1)x(y-1)}[(b_2+n_2-1-c-n_3+b_1+n_1)x\cdot S_{11} + (-2x+xy+(c+n_3)x-(b_1+n_1)y-(b_1+n_1)x+(b_2+n_2-2)xy-(a+n_0)xy+(b_1+n_1)xy-2(b_2+n_2-2)x)\cdot S_7 + ((b_1+n_1)(y-xy)\cdot S_{12})]$*. (This follows from* R_4 *reduced from step 1 to step 2.)*

Proposition A.2.3 *Following derivatives result from definition of Appell series.*

- $\partial_x F_1(a, b_1, b_2, c; x, y) = \frac{ab_1}{c} F_1(a+1, b_1+1, b_2, c+1; x, y)$.

- $\partial_y F_1(a, b_1, b_2, c; x, y) = \frac{ab_2}{c} F_1(a+1, b_1, b_2+1, c+1; x, y)$.

Lemma A.2.4 *Let* $F(x) = \frac{x(x+1)...(x+m-1)}{m!}$. *Let* a *be a rational number with denominator* $d = p_1^{e_1} \dots p_k^{e_k}$. *Let* $D = p_1^{e_1+1} \dots p_k^{e_k+1}$. *Then* $D^m \cdot F(a)$ *is an integer.*

Pf : Let p be a prime and e be the multiplicity of p in the denominator of a.

- Case 1: $e = 0$. In this case, a is an element of the p-adic integers $\mathbb{Z}_p$. Choose an integer A for which $v_p(a - A) > v_p(m!)$ where v_p is the p-adic valuation. Then $v_p(F(A) - F(a)) > 0$. So $v_p(F(a)) \geq 0$ since $v_p(F(A)) \geq 0$.

- Case 2: $e > 0$. We now have to prove that $v_p(F(a)) \geq -m(e+1)$, which follows from $v_p(a(a+1)...(a+m-1)) = -m \cdot e$ and $v_p(m!) \leq m$.

Claim A.2.5 $F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, x, y)$ *is globally bounded (1.1.1).*

Pf : by definition,

$$F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, x, y) = \sum_{m,n=0}^{\infty} \frac{(\frac{1}{6})_{m+n} (\frac{1}{5})_m (\frac{1}{5})_n}{(m+n)!m!n!} x^m y^n.$$

Take $q_1 = 36$ and $q_2 = 25$, from Lemma A.2.4, for $\forall m \in \mathbb{N}$, $\frac{(\frac{1}{6})_m (q_1)^m}{m!} \in \mathbb{Z}$ and $\frac{(\frac{1}{5})_m (q_2)^m}{m!} \in \mathbb{Z}$. Let $q = q_1 \cdot q_2$, then

$$\begin{aligned} F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, qx, qy) &= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{6})_{m+n} (\frac{1}{5})_m (\frac{1}{5})_n}{(m+n)!m!n!} (qx)^m (qy)^n \\ &= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{6})_{m+n} (\frac{1}{5})_m (\frac{1}{5})_n}{(m+n)!m!n!} \cdot (q_1)^m \cdot (q_2)^m \cdot x^m \cdot (q_1)^n \cdot (q_2)^n \cdot y^n \\ &= \sum_{m,n=0}^{\infty} \frac{(\frac{1}{6})_{m+n} (q_1)^{m+n}}{(m+n)!} \cdot \frac{(\frac{1}{5})_m (q_2)^m}{(m)!} \cdot \frac{(\frac{1}{5})_n (q_2)^n}{(n)!} \cdot x^m y^n \end{aligned}$$

So $F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, x, y)$ is globally bounded.

Definition A.2.6 *The valuation of* $f \in \mathbb{Q}[[x]] = \{\sum_{i=0}^{\infty} a_i x^i | a_i \in \mathbb{Q}\}$, *denoted as* $v(f)$, *is defined by*

$$v(f) = \begin{cases} min\{i \,|\, a_i \neq 0\}, & f \neq 0 \\ \infty, & f = 0. \end{cases}$$

Remark A.2.7

- $\forall f, g \in \mathbb{Q}[[x]]$, $v(f+g) \geq min\{v(f), v(g)\}$.
- $\forall\ n \in \mathbb{N}$, $\forall f \in \mathbb{Q}[[x]]$, $v(f^n) = n \cdot v(f)$.

Definition A.2.8 *For any $f_1, f_2 \in \mathbb{Q}[[x]]$, their distance $d(f_1, f_2)$ is defined as*

$$d(f_1, f_2) = \begin{cases} 0, \ f_1 = f_2 \\ 2^{-v(f_1 - f_2)}, \ f_1 \neq f_2. \end{cases}$$

Remark A.2.9 $\mathbb{Q}[[x]]$ *is a metric space with the above distance, also the completion of* $\mathbb{Q}[x]$ *w.r.t this metric.*

Lemma A.2.10 *Let* $R = \{f \in \mathbb{Q}[[x]] \,|\, \exists\, m, n > 0 \text{ s.t. } nf_{(mx)} \in \mathbb{Z}[[x]]\}$.

- (a) *If $f_n \in \mathbb{Q}[x]$, $n = 0, 1, 2, ...$, then $\sum\limits_{n=0}^{\infty} f_n$ converges w.r.t metric d if and only if f_n converges to* 0.
- (b) *If $f, g \in \mathbb{Q}[[x]]$ with $v(g) > 0$, then $f \circ g$ is well defined in $\mathbb{Q}[[x]]$ by (a). Moreover, if $f, g \in R$, then $f \circ g \in R$.*

Claim A.2.11 *Let $y_1 = \frac{x+\sqrt{x^2+4x}}{2}$ and $y_2 = \frac{x-\sqrt{x^2+4x}}{2}$. Then $F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, y_1, y_2)$ is globally bounded.*

This follows from Claim A.2.5 and Lemma A.2.10.

Claim A.2.12 *$F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, y_1, y_2)$ is a solution of L.*

Apply L on $F_1(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1, y_1, y_2)$ using Algorithm A.2.2 and relations in A.2.3.

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BIOGRAPHICAL SKETCH

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