Closed Form Solutions for Linear Differential and Difference Equations

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ABSTRACT

Finding closed form solutions of differential equations has a long history in computer algebra. For example, the Risch algorithm (1969) decides if the equation $y' = f$ can be solved in terms of elementary functions. These are functions that can be written in terms of exp and log, where "in terms of" allows for field operations, composition, and algebraic extensions. More generally, functions are in closed form if they are written in terms of commonly used functions. This includes not only exp and log, but other common functions as well, such as Bessel functions or the Gauss hypergeometric function. Given a differential equation L, to find solutions written in terms of such functions, one seeks a sequence of transformations that sends the Bessel equation, or the Gauss hypergeometric equation, to L. Although random equations are unlikely to have closed form solutions, they are remarkably common in applications. For example, if $y = \sum_{n=0}^{\infty} a_n x^n$ has a positive radius of convergence, integer coefficients $a_n \in \mathbb{Z}$, and satisfies a second order homogeneous linear differential equation L with polynomial coefficients, then L is conjectured to be solvable in closed form. Such equations are common, not only in combinatorics, but in physics as well. The talk will describe recent progress in finding closed form solutions of differential and difference equations, as well as open questions.

CCS CONCEPTS

• **Mathematics of computing** → **Mathematical software**; • **Computing methodologies** → **Symbolic and algebraic manipulation**;

KEYWORDS

Symbolic Computation, Closed form solutions

A closed form solution of an equation is an exact solution presented with a finite amount of data. The expression $1.4142\ldots$ is not a closed form solution of equation $E_1 : \; x^2 - 2 = 0$ because an exact decimal expansion would be infinitely long. Likewise, the solution $1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots$ of the equation $E_2 : \; xy'' + y' - xy = 0$ is also not in closed form. Approximate solutions can be made more precise by using more digits or terms. They are very useful and widely used.

The key novelty of computer algebra systems was computing with exact expressions. A computer can represent solutions of E_1 exactly as $\pm\sqrt{2}$. However, solving E_1 by returning $\pm\sqrt{2}$ is a tautology because the expression $\sqrt{2}$ is *defined* to be a solution of $x^2 - 2 = 0$. Likewise, if we ask the computer for solutions of E_2, then "Bessel functions $I_0(x)$ and $K_0(x)$" is a tautological answer because they are by definition solutions of E_2.

Finding closed form solutions involves:

(1) "Base equations" are solved tautologically by defining "base functions" to be solutions.
(2) Solve other equations in terms of base functions using field operations $+, -, \cdot, \div$ and other operations such as composition, differentiation, and in some contexts, integration and algebraic extensions.
(3) To prevent the problem from becoming trivial:
    (a) Only allow base functions that are well known, such the square root, exp, log, Airy, Bessel, Whittaker, Gauss hypergeometric, ...
    (b) Only introduce base equations for functions that can not be expressed in terms of prior base functions.

The Ph.D theses of Yongjae Cha and Giles Levy [5, 6] follow the above approach for linear difference equations. First one defines transformations of difference equations that correspond to explicit transformations between their solutions. These transformations should be as general as possible, in order to minimize the number of base equations that are needed, and to maximize the strength of the solver. Next one implements a program to find such transformations, and a program that can decide if one of the base equations, for some values of its parameters, can be transformed to the input equation.

The same approach can be illustrated with something as easy as solving quadratic equations. Start for example with the field $\mathbb{Q}$, and field operations. Then example E_1 is not solvable. To remedy this, introduce $x^2 - d = 0$ as base equation with one parameter d, and solve it trivially by defining $\sqrt{d}$ to be a solution. The equation $ax^2 + bx + c = 0$ reduces to $x^2 - d = 0$ via a nearly-trivial transformation and should thus not be a base equation by item 3(b).

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Now consider linear differential equations. To solve example E_2, introduce a base equation $L_B: \; xy'' + y' - (x + \frac{\nu^2}{x})y = 0$ with one parameter ν and define the Bessel functions $I_\nu(x)$ and $K_\nu(x)$ as solutions of L_B. Then E_2 is solved trivially because it is L_B with $\nu = 0$.

Now consider equation $E_3: \; y'' + (2 - 10x + 4x^2 - 4x^4)y = 0$. It can be solved trivially by letting E_3 be a base equation, but that would be contrary to item 3(a). It is nearly trivial to reduce E_3 to a triconfluent Heun equation, so we could add that as a base equation. However, by item 3(b) we should first check if E_3 is also solvable in terms of the already-introduced functions. A solution is:

$$(2x^2 + x - 1)\mathrm{Ai}(x^2 - 1) + (2x + 1)\mathrm{Ai}'(x^2 - 1)$$

where Ai is the Airy Ai function, and Ai$'$ is its derivative (replacing Ai by Bi gives another solution). Airy functions can be rewritten in terms of Bessel functions with $\nu = \frac{1}{3}$. So E_3 is solvable in terms of Bessel functions. Equivalently, E_3 comes from base equation L_B with $\nu = \frac{1}{3}$ via a sequence of transformations. So no new base equation is needed for this example E_3 (of course this may change for other examples).

Item 3(b) raises the main question for closed form solutions: Given an equation, how do we know if it can be solved in terms of the current set of base functions? Suppose we want to find closed form solutions in the following context C: linear differential equations with rational functions as coefficients. One of the base equations resp. functions is L_B resp. $B_\nu(x)$ above. The only property of $B_\nu(x)$ that we will use is that it is a solution of L_B. We now search for transformations that we can apply to any solution of an equation in context C, such that the resulting function is again a solution of an equation in context C. The composition $B_0(B_0(x))$ is in closed form according to item 2. However, it does not satisfy a non-zero equation in context C. So for solving equations in this context, the function $B_0(B_0(x))$ is irrelevant.

The most general expression in terms of $B_\nu(x)$ that satisfies a second order equation with rational function coefficients is:

$$\exp(\int r) \cdot (r_1\partial + r_0)(B_\nu(\sqrt{f}))$$

where $f, r, r_0, r_1 \in \mathbb{C}(x)$ are rational functions. Here $r_1\partial + r_0$ is an operator that sends y to $r_1 y' + r_0 y$.

The Ph.D thesis of Quan Yuan [4] gives a complete algorithm to decide if an equation has a solution in this form. The same also works for Airy functions, Whittaker functions, and related functions. The algorithm works by examining the so-called irregular singularities of the input equation. These are points where solutions have essential singularities. The Bessel function has an essential singularity at ∞, in other words, the Bessel equation L_B has an irregular singularity there. These singularities can not disappear under any of the transformations: (1) composition of B_ν with $\sqrt{f}$, (2) applying $r_1\partial + r_0$, and (3) multiplying by $\exp(\int r)$. This means that all poles of f are easily found: they are irregular singularities of the input equation. Moreover, local data at each irregular singularity p provides a portion of the polar part of f at p. Combined with a small amount of additional data (coming from regular singularities) this suffices to reconstruct f and ν, after which r_0, r_1, r can be computed.

Conjecture 0.1. *Let β be a non-zero constant, let $y = \sum_{n=0}^{\infty} a_n \beta^n x^n$ with $a_n \in \mathbb{Z}$. Suppose that y has a positive radius of convergence, and satisfies a second order linear differential equation $P_2 y'' + P_1 y' + P_0 y = 0$ with polynomial coefficients $P_i = P_i(x)$, not all zero. Then this equation has a solution of the form: an algebraic function, or a ${}_2F_1$-type solution:*

$$(r_1\partial + r_0)({}_2F_1(a, b; c; f))$$

where ${}_2F_1$ denotes the Gauss hypergeometric function, a, b, c are rational numbers, $c = 1$, and r_0, r_1, f are algebraic functions.

Such differential equations are called "globally bounded". They are common in many applications, not only in combinatorics where it is natural to have $a_n \in \mathbb{Z}$, but in physics as well. So according to the conjecture, closed form solutions should be common for second order equations from many applications.

Finding ${}_2F_1$-type solutions turned out to be more complicated than finding Bessel-type solutions. The irregular singularity of L_B at $x = \infty$ introduces an irregular singularity at every pole of f under transformation (1), and these can not disappear under transformations (2),(3). However, this does not hold for the singularities of the Gauss hypergeometric equations, which are all regular singular. Some poles of f may become regular points of the input equation. This makes it much harder to reconstruct f.

Partial algorithms for finding ${}_2F_1$-type solutions for a number of different cases were developed in the Ph.D theses of Tingting Fang and Vijay Kunwar [2, 3]. The recent PhD thesis of Erdal Imamoglu [1] gave a method that, though not proven complete, appears to cover essentially all cases for which $P_0, P_1, P_2 \in \mathbb{Q}[x]$. One of the key ideas in this work is as follows. The POLRED algorithm (Cohen and Diaz Y Diaz, 1991) tries to reduce a polynomial P, which means find a polynomial Q that defines the same number field, and that is close to optimal in size. It turns out that the same strategy can also be used to reduce regular singular differential equations. This reduction preserves the order, but tends to lead to equations that easier to solve, and are close to optimal in size.

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