Closed Form Solutions

Mark van Hoeij¹

Florida State University

Slides of this talk:
www.math.fsu.edu/~hoeij/issac2017.pdf

¹Supported by NSF 1618657
What is a closed form solution?

**Example:** Solve this equation for $y = y(x)$.

$$y' = \frac{4 - x^3}{(1 - x)^2} e^x$$

**Definition**

A **closed form** solution is an expression for an **exact** solution given with a **finite amount of data**.

This is not a closed form solution:

$$y = 4x + 6x^2 + \frac{22}{3} x^3 + \frac{95}{12} x^4 + \cdots$$

because making it exact requires infinitely many terms.

The Risch algorithm finds a closed form solution:

$$y = \frac{2 + x^2}{1 - x} e^x$$
Previous slide: A **closed form** solution is an **expression** for an exact solution with only a finite amount of data.

Risch algorithm finds (if it exists) a **closed form** solution \( y \) for:

\[
y' = f
\]

To make that well-defined, specify **which expressions** are allowed:

Define \( E_{\text{in}} \) and \( E_{\text{out}} \) such that:

- Any \( f \in E_{\text{in}} \) is allowed as input.
- Output: a solution iff \( \exists \) solution \( y \in E_{\text{out}} \).

Risch: \( E_{\text{in}} = E_{\text{out}} = \{ \text{elementary functions} \} \)

\[
= \{ \text{expressions with } \mathbb{C}(x) \exp \log + - \cdot \div \text{ composition and algebraic extensions} \}.
\]
Previous slide: A closed form solution is an expression for an exact solution with only a finite amount of data.

Risch algorithm finds (if it exists) a closed form solution $y$ for:

$$y' = f$$

To make that well-defined, specify which expressions are allowed:

Define $E_{\text{in}}$ and $E_{\text{out}}$ such that:

- Any $f \in E_{\text{in}}$ is allowed as input.
- Output: a solution iff \( \exists \) solution $y \in E_{\text{out}}$.

Risch: $E_{\text{in}} = E_{\text{out}} = \{\text{elementary functions}\}$

$$= \{\text{expressions with } \mathbb{C}(x) \exp \log + - \cdot \div \text{ composition and algebraic extensions}\}.$$
Liouvillian solutions

Kovacic’ algorithm (1986)

1. Solves homogeneous differential equations of order 2
   
   \[ a_2 y'' + a_1 y' + a_0 y = 0 \]

   (Risch: inhomogeneous equations of order 1)

2. It finds solutions in a larger class:
   
   \[ E_{\text{out}} = \{ \text{Liouvillian functions} \} \supset \{ \text{elementary functions} \} \]

3. but it is more restrictive in the input:
   
   \[ a_0, a_1, a_2 \in \{ \text{rational functions} \} \subset \{ \text{elementary functions} \} \]

Remark

\[ \exists \text{ common functions that are not Liouvillian.} \]

Allow those as closed form \( \rightsquigarrow \) need other solvers.
A non-Liouville example

Let
\[ y := \int_y \exp\left( \frac{t^2 + x + 1}{x - t} \right) \frac{t}{x^2 + x + 1} \, dt \]

**Zeilberger’s telescoping algorithm** \( \leadsto \) an equation for \( y \):

\[(x^4 - x)y'' + (4x^4 + 2x^3 - 3x^2 - 7x + 1)y' + (6x^3 - 9x^2 - 12x + 3)y = 0\]

Closed form solutions were thought to be rare.

But (for order 2) **telescoping equations** often (always?) have closed form solutions:

\[ \exp(-2x) \cdot \left( I_0 \left( 2\sqrt{x^2 + x + 1} \right) - \frac{x + 1}{\sqrt{x^2 + x + 1}} I_1 \left( 2\sqrt{x^2 + x + 1} \right) \right) \]

and

\[ \exp(-2x) \cdot \left( K_0 \left( 2\sqrt{x^2 + x + 1} \right) + \frac{x + 1}{\sqrt{x^2 + x + 1}} K_1 \left( 2\sqrt{x^2 + x + 1} \right) \right) \]
A non-Liouville example

Let

\[ y := \oint_{\gamma} \exp \left( \frac{t^2 + x + 1}{x - t} \right) \frac{t}{x^2 + x + 1} \, dt \]

Zeilberger’s telescoping algorithm \( \leadsto \) an equation for \( y \):

\[ (x^4 - x)y'' + (4x^4 + 2x^3 - 3x^2 - 7x + 1)y' + (6x^3 - 9x^2 - 12x + 3)y = 0 \]

Closed form solutions were thought to be rare.

But (for order 2) telescoping equations often (always?) have closed form solutions:

\[
\exp(-2x) \cdot \left( I_0 \left( 2\sqrt{x^2 + x + 1} \right) - \frac{x + 1}{\sqrt{x^2 + x + 1}} I_1 \left( 2\sqrt{x^2 + x + 1} \right) \right)
\]

and

\[
\exp(-2x) \cdot \left( K_0 \left( 2\sqrt{x^2 + x + 1} \right) + \frac{x + 1}{\sqrt{x^2 + x + 1}} K_1 \left( 2\sqrt{x^2 + x + 1} \right) \right)
\]
Diagonals of rational functions

Take a rational function in several variables, for example:

\[ F = \frac{1}{1 - x - 2y - 3z - 4yz - 5xyz - 6xyz^2} \in \mathbb{Q}(x, y, z) \]

and write it as a multivariate power series

\[ F = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} x^i y^j z^k \in \mathbb{Q}[[x, y, z]] \]

The diagonal is

\[ \text{diag}(F) = \sum_{i=0}^{\infty} a_{iii} x^i \in \mathbb{Q}[[x]] \]

**Fact:** Diagonals of rational functions are **D-finite** (holonomic); they satisfy a homogeneous linear differential equation with polynomial coefficients.

**Conjecture:** If such an equation has order 2 then it has closed form solutions.
Diagonals of rational functions

For the example, the diagonal \( D = \text{diag}(F) \) satisfies:

\[
a_2 D'' + a_1 D' + a_0 D = 0
\]

where

\[
a_2 = x(2x + 1)(40x^3 + 4x^2 - 72x - 5)(500x^4 - 149x^3 + 939x^2 - 1061x + 5)
\]
\[
a_1 = 120000x^8 + 32160x^7 - 288416x^6 - 74344x^5 + 135372x^3 + 93397x^2 + 10510x - 25
\]
\[
a_0 = 40000x^7 + 10240x^6 - 210464x^5 + 4944x^4 - 58610x^3 + 15752x^2 - 3715x + 1225
\]

To solve order 2 equations with \( a_0, a_1, a_2 \in \mathbb{Q}[x] \) download:

www.math.fsu.edu/~eimamogl/hypergeometricsols

Result:

\[
\binom{1}{12} \binom{5}{12} \cdot (625x^4 + 140x^3 + 1158x^2 - 196x + 1)^{-1/4}
\]

where

\[
f = \frac{x^3(2x + 1)^2(500x^4 - 149x^3 + 939x^2 - 1061x + 5)}{(625x^4 + 140x^3 + 1158x^2 - 196x + 1)^3}
\]
Random equations rarely have closed form solutions.

So-called **globally bounded** equations are common in:
- combinatorics (Mishna’s tutorial)
- physics (Ising model, Feynman integrals, etc.)
- Period integrals, creative telescoping, diagonals.

**Conjecture**

*Globally bounded equations (of order 2) have closed form solutions.*

In other words: **Closed form solutions are common**
(for non-random equations of order 2)

If Maple + Mathematica don’t find them  ➞ download our code!
Local to global strategy

**Risch:** Given elementary function $f$, solve:

$$y' = f$$

\{poles of $y$\} $\subseteq$ \{poles of $f$\} = known

**Kovacic:** Given polynomials $a_0, a_1, a_2 \in \mathbb{C}[x]$, solve:

$$a_2y'' + a_1y' + a_0y = 0$$

\{poles of $y$\} $\subseteq$ \{roots of $a_2$\} = known

Local to global strategy

\{poles of $y$\} $+$ \{terms in polar parts\} ($+$ other data) $\leadsto$ $y$
Example:

\[ y = \exp(r) \text{ where } r = \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x-1} + 5 + 7x \]

\( y \) has **essential singularities** at the poles of \( r \).

**Definition**

\( y_1, y_2 \neq 0 \) have an **equivalent singularity** at \( x = p \)
when \( y_1/y_2 \) is meromorphic at \( x = p \).

Equivalence class of \( y \) at \( x = 0, \ x = 1, \ x = \infty \)  (local data)

\[ \rightsquigarrow \text{Polar part of } r \text{ at } x = 0, \ x = 1, \ x = \infty \]  (local data)

\[ \rightsquigarrow \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x-1} + 7x \]  (local data)

\[ \rightsquigarrow r \text{ (up to a constant term)} \]  (global data)

\[ \rightsquigarrow y \text{ (up to a constant factor)} \]  (global data)
Reconstructing solutions from local data

Recall: \( y_1, y_2 \) have **equivalent singularity** at \( x = p \)
if \( y_1/y_2 \) is meromorphic at \( x = p \).

Hence:

\[
y_1, y_2 \text{ equivalent at every } p \in \mathbb{C} \cup \{\infty\} \iff \ y_1/y_2 \text{ meromorphic at every } p \in \mathbb{C} \cup \{\infty\} \iff \ y_1/y_2 \in \mathbb{C}(x)
\]

Hence:

\[
\{\text{Eq. class of } y \text{ at all } p\} \iff y \text{ up to a rational factor}
\]

For a differential equation \( L \) **can compute**:

\[
\{\text{generalized exponents} \text{ of } L \text{ at } p\} \approx \{\text{Eq. classes of solutions}\}
\]

Choose the right one at each \( p \) \( \sim \) a solution (up to \( \sim \))
Example: let $L$ have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp\left(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x - 3} + \frac{e_{4,1}}{x - 4}\right)$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp\left(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x - 3} + \frac{e_{4,2}}{x - 4}\right)$$

where $e_{p,i} \in \mathbb{C}\left[\frac{1}{x-p}\right]$ encodes the polar part $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the generalized exponents of $L$ at $x = p$ and can be computed from $L$:

$$E_0 = \{e_{0,1}, e_{0,2}\}, \quad E_3 = \{e_{3,1}, e_{3,2}\}, \quad E_4 = \{e_{4,1}, e_{4,2}\}$$

To find $y_1$ we need to choose the correct element of each $E_p$.

The example has $2^3 = 8$ combinations.
One combination $\leadsto y_1$, another $\leadsto y_2$, other six $\leadsto$ nothing.

Can reduce #combinations (e.g. Fuchs’ relation)
Example: generalized exponents

Example: let $L$ have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp\left(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x - 3} + \frac{e_{4,1}}{x - 4}\right)$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp\left(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x - 3} + \frac{e_{4,2}}{x - 4}\right)$$

where $e_{p,i} \in \mathbb{C}[\frac{1}{x-p}]$ encodes the polar part $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the generalized exponents of $L$ at $x = p$ and can be computed from $L$:

$$E_0 = \{e_{0,1}, e_{0,2}\}, \quad E_3 = \{e_{3,1}, e_{3,2}\}, \quad E_4 = \{e_{4,1}, e_{4,2}\}$$

To find $y_1$ we need to choose the correct element of each $E_p$.

The example has $2^3 = 8$ combinations.

One combination $\rightsquigarrow y_1$, another $\rightsquigarrow y_2$, other six $\rightsquigarrow$ nothing.

Can reduce #combinations (e.g. Fuchs’ relation)
Example: let $L$ have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x - 3} + \frac{e_{4,1}}{x - 4})$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x - 3} + \frac{e_{4,2}}{x - 4})$$

where $e_{p,i} \in \mathbb{C}[\frac{1}{x-p}]$ encodes the polar part $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the generalized exponents of $L$ at $x = p$ and can be computed from $L$:

$$E_0 = \{e_{0,1}, e_{0,2}\}, \quad E_3 = \{e_{3,1}, e_{3,2}\}, \quad E_4 = \{e_{4,1}, e_{4,2}\}$$

To find $y_1$ we need to choose the correct element of each $E_p$.

The example has $2^3 = 8$ combinations.
One combination $\leadsto y_1$, another $\leadsto y_2$, other six $\leadsto$ nothing.

Can reduce #combinations (e.g. Fuchs’ relation)
Generalized exponents $\sim$ hyper-exponential solutions:

Let $a_0, \ldots, a_n \in \mathbb{C}[x]$ and $L(y) := a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$.

Hyper-exponential solution: $y = \exp(\int r)$ for some $r \in \mathbb{C}(x)$.

\{generalized exponent of such $y$ at all singularities $p$ of $L$\} $\sim$

$y$ up to a polynomial factor (generalized exponent $\approx$ eq. class)

**Algorithm hyper-exponential solutions:**

1. Compute generalized exponents $\{e_{p,1}, \ldots, e_{p,n}\}$ at each singularity $p \in \mathbb{C} \cup \{\infty\}$ of $L$.

2. For each combination $e_p \in \{e_{p,1}, \ldots, e_{p,n}\}$ (for all $p$) compute polynomial solutions of a related equation.
Let \( a_0, \ldots, a_n \in \mathbb{C}[x] \) and \( L(y) := a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \).

Hyper-exponential solution: \( y = \exp(\int r) \) for some \( r \in \mathbb{C}(x) \).

\{\text{generalized exponent of such } y \text{ at all singularities } p \text{ of } L\} \leadsto \text{y up to a polynomial factor} \quad \text{(generalized exponent } \approx \text{ eq. class)}

Algorithm hyper-exponential solutions:

1. Compute generalized exponents \( \{e_{p,1}, \ldots, e_{p,n}\} \) at each singularity \( p \in \mathbb{C} \cup \{\infty\} \) of \( L \).
2. For each combination \( e_p \in \{e_{p,1}, \ldots, e_{p,n}\} \) (for all \( p \)) compute polynomial solutions of a related equation.
Combine \textbf{generalized exponents} $\rightsquigarrow$ hyper-exponential solutions.

To do the same for \textbf{difference equations} we need the difference analogue of generalized exponents:

\textbf{Difference case:} $p = \infty$ is similar to the differential case.

But a finite singularity is not an element $p \in \mathbb{C}$.

Instead it is an element of $\mathbb{C}/\mathbb{Z}$ because

$$y(x) \text{ singular at } p \iff y(x + 1) \text{ singular at } p$$

is only true for $p = \infty$.

(1997): Generalized exponents

(1999): Difference case analogue:

- generalized exponents at $p = \infty$ and
- valuation growths at $p \in \mathbb{C}/\mathbb{Z}$

$\rightsquigarrow$ Algorithm for \textbf{hypergeometric solutions}. 

\[14 / 29\]
Combine **generalized exponents** \( \leadsto \) hyper-exponential solutions.

To do the same for **difference equations** we need the difference analogue of generalized exponents:

**Difference case:** \( p = \infty \) is similar to the differential case.

But a finite singularity is not an element \( p \in \mathbb{C} \).

Instead it is an element of \( \mathbb{C}/\mathbb{Z} \) because

\[
y(x) \text{ singular at } p \iff y(x + 1) \text{ singular at } p
\]

is only true for \( p = \infty \).

(1997): Generalized exponents

(1999): Difference case analogue:

- generalized exponents at \( p = \infty \) and
- **valuation growths** at \( p \in \mathbb{C}/\mathbb{Z} \)

\( \leadsto \) Algorithm for **hypergeometric solutions**.
Closed form solutions of linear differential equations:

**Goal:** define, then find, **closed form solutions** of:

\[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{with} \quad a_0, \ldots, a_n \in \mathbb{C}(x). \quad (1) \]

The **order** is \( n \) (we assume \( a_n \neq 0 \)).

Consider **closed form** expressions in terms of **functions** that are:

- **well known**, and
- **D-finite**: satisfies an equation of form (1).

**D-finite of order 1** = hyper-exponential function.

**Well known D-finite functions of order 2:**

- Airy functions, Bessel functions, Kummer, Whittaker, \ldots
- Gauss hypergeometric function \( \, _2 F_1(a, b; c \mid x) \)

Klein’s theorem: Liouvillian solutions are \( \, _2 F_1 \) expressible.
Goal: define, then find, **closed form solutions** of:

\[ a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{with} \quad a_0, \ldots, a_n \in \mathbb{C}(x). \quad (1) \]

The **order** is \( n \) (we assume \( a_n \neq 0 \)).

Consider **closed form** expressions in terms of **functions** that are:

1. well known, and
2. D-finite: satisfies an equation of form (1).

**D-finite of order 1 = hyper-exponential function.**

**Well known D-finite functions of order 2:**

- Airy functions, Bessel functions, Kummer, Whittaker, \ldots
- Gauss hypergeometric function \( \text{$_2F_1$(a, b; c | x)} \)

  Klein’s theorem: Liouvillian solutions are \( \text{$_2F_1$} \) expressible.
Idea for constructing a Bessel-solver:

- Bessel functions have an essential singularity at $x = \infty$.
- Just like the function $\exp(x)$.
- So the strategy for hyper-exponential solutions may work for Bessel-type solutions as well.
- It also works for Airy, Kummer, Whittaker, and hypergeometric $p F_q$ functions if $p + 1 \neq q$.

Later: Other strategies for Gauss hypergeometric $2F_1$ function (to solve globally bounded equations of order 2).

Question: Which Bessel expressions should the solver look for? Which Bessel expressions are D-finite?
Idea for constructing a Bessel-solver:

- Bessel functions have an essential singularity at $x = \infty$.
- Just like the function $\exp(x)$.
- So the strategy for hyper-exponential solutions may work for Bessel-type solutions as well.
- It also works for Airy, Kummer, Whittaker, and hypergeometric $pF_q$ functions if $p + 1 \neq q$.

Later: Other strategies for Gauss hypergeometric $2F_1$ function (to solve **globally bounded** equations of order 2).

Question: Which Bessel expressions should the solver look for? Which Bessel expressions are D-finite?
Bessel type closed form expressions

Let $B_\nu(x)$ be one of the Bessel functions, with parameter $\nu$.

**Bessel type closed form expressions** should allow:
- algebraic functions
- exp and log
- composition
- field operations
- differentiation and integration
- and of course $B_\nu(x)$.

**Example:** $B_0(\exp(x))$ is a Bessel type closed form expression but is not relevant for (1) since it is not D-finite.

**Question:** which Bessel type expressions are D-finite?
D-finite functions:

A function $y = y(x)$ is **D-finite of order** $n$ if it satisfies a differential equation of order $n$ with rational function coefficients.

**Operations that don’t increase the order:**
1. $y(x) \mapsto y(f)$ for some $f \in \mathbb{C}(x)$ called pullback function.
2. $y \mapsto r_0y + r_1y' + \cdots + r_{n-1}y^{(n-1)}$ for some $r_i \in \mathbb{C}(x)$.
3. $y \mapsto \exp(\int r) \cdot y$ for some $r \in \mathbb{C}(x)$.

**Operations that can increase the order:**
4. Same as (1),(2),(3) but with algebraic functions $f$, $r_i$, $r$.
5. $y_1, y_2 \mapsto y_1 + y_2$ order $n_1, n_2 \mapsto$ order $\leq n_1 + n_2$
6. $y_1, y_2 \mapsto y_1 \cdot y_2$ order $n_1, n_2 \mapsto$ order $\leq n_1 \cdot n_2$
   Special case: $y \mapsto y^2$ order $n \mapsto$ order $\leq \frac{n(n+1)}{2}$

Have algorithms to recover any combination of: (2), (3), (5), and part of (6).
D-finite functions:

A function \( y = y(x) \) is **D-finite of order** \( n \) if it satisfies a differential equation of order \( n \) with rational function coefficients.

**Operations that don’t increase the order:**

1. \( y(x) \mapsto y(f) \) for some \( f \in \mathbb{C}(x) \) called pullback function.
2. \( y \mapsto r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)} \) for some \( r_i \in \mathbb{C}(x) \).
3. \( y \mapsto \exp(\int r) \cdot y \) for some \( r \in \mathbb{C}(x) \).

**Operations that can increase the order:**

4. Same as (1),(2),(3) but with algebraic functions \( f, r_i, r \).
5. \( y_1, y_2 \mapsto y_1 + y_2 \) order \( n_1, n_2 \mapsto \text{order} \leq n_1 + n_2 \)
6. \( y_1, y_2 \mapsto y_1 \cdot y_2 \) order \( n_1, n_2 \mapsto \text{order} \leq n_1 \cdot n_2 \)
   Special case: \( y \mapsto y^2 \) order \( n \mapsto \text{order} \leq \frac{n(n+1)}{2} \)

Have algorithms to recover any combination of: (2), (3), (5), and part of (6).
Let $B_\nu(x)$ be one of the Bessel functions. $B_\nu(\sqrt{x})$ is D-finite of order 2. Transformations (1), (2), (3) $\leadsto$

$$\exp\left(\int r \cdot \left( r_0 \cdot B_\nu(\sqrt{f}) + r_1 \cdot B_\nu(\sqrt{f})' \right) \right)$$

is D-finite of order 2 for any $r, r_0, r_1, f \in \mathbb{C}(x)$.

**Theorem (Quan Yuan 2012)**

Let $k$ be a subfield of $\mathbb{C}$ and let $L$ be a linear homogeneous differential equation over $k(x)$ of order 2.

If $\exists$ solution of form (2) with **algebraic functions** $r, r_0, r_1, f$ then $\exists$ solution with **rational functions** $r, r_0, r_1, f \in k(x)$.

**Bessel-type solutions of higher order equations:**

$\leadsto$ Add transformations (4),(5),(6).
Finding Bessel type solutions

\[ a_2y'' + a_1y' + a_0y = 0 \quad \text{where} \quad a_0, a_1, a_2 \in \mathbb{C}[x]. \]

**Goal:** Find Bessel-type solutions.

**Idea:** Recover the pullback function \( f \) in transformation (1) from data that is invariant under transformations (2),(3).

**Hyper-exponential solutions:**
Generalized exponents \( \mapsto \{ \text{polar parts of } f \} \mapsto f \)

**Bessel-type solutions:**
Generalized exponents \( \mapsto \{ \text{[half]} \text{ of terms of polar parts of } f \} \mapsto \text{need more data to find } f. \)

**More data:** regular singularities \( \mapsto \text{roots of order } \notin \text{denom}(\nu) \cdot \mathbb{Z} \)

**Combine data** \( \mapsto f \) **except in one case:** \( \text{denom}(\nu) = 2 \)
that “happens” to be solvable with Kovacic
Finding Bessel type solutions

\[ a_2y'' + a_1y' + a_0y = 0 \quad \text{where} \quad a_0, a_1, a_2 \in \mathbb{C}[x]. \]

**Goal:** Find Bessel-type solutions.

**Idea:** Recover the pullback function \( f \) in transformation (1) from data that is **invariant** under transformations (2),(3).

**Hyper-exponential solutions:**
Generalized exponents \( \rightsquigarrow \{ \text{polar parts of } f \} \rightsquigarrow f \)

**Bessel-type solutions:**
Generalized exponents \( \rightsquigarrow \{ \lfloor \text{half} \rfloor \text{ of terms of polar parts of } f \} \)
\( \rightsquigarrow \) need **more data** to find \( f \).

**More data:** regular singularities \( \rightsquigarrow \) roots of order \( \not\in \text{denom}(\nu) \cdot \mathbb{Z} \)

**Combine data** \( \rightsquigarrow f \quad \text{except in one case:} \quad \text{denom}(\nu) = 2 \)
that “happens” to be **solvable with Kovacic**
Local to global strategy for difference equations

Use local data that is **invariant** under the difference analogue of transformations (2),(3):

- Giles Levy (Ph.D 2009)
- Yongjae Cha (Ph.D 2010)

**Example:** oeis.org/A000179 (Ménage numbers)

**Recurrence operator:**

\[(\tau + 1) \circ (n\tau^2 - (n^2 + 2n)\tau - n - 2)\]

_Where \(\tau\) is the shift-operator._

**Solver**  
\[c_1 \cdot n \cdot I_n(-2) + c_2 \cdot n \cdot K_n(2) + c_3 \cdot \epsilon(n)\]

_Where \(I_n(x)\) and \(K_n(x)\) are Bessel functions and \(\epsilon(n)\) is a complicated expression that converges to 0 as \(n \to \infty\)._

**Result:**

\[A000179(n) = \text{round} \left( \frac{2n}{e^2} \cdot K_n(2) \right) \quad (\text{for } n > 0)\]
The **Gauss hypergeometric function** is:

\[
_{2}F_{1}\left(\begin{array}{c} a, b \\ c \\
\end{array} \bigg| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_nn!} x^n
\]

where \((a)_n = a \cdot (a + 1) \cdots (a + n - 1)\).

If \(L(y) = 0\) is a **globally bounded** equation of order 2 then it conjecturally has algebraic or \(_{2}F_{1}\)-type solutions:

\[
y = \exp\left(\int r \cdot \left( r_0 \cdot _{2}F_{1} \left( \begin{array}{c} a, b \\ c \\
\end{array} \bigg| f \right) + r_1 \cdot _{2}F_{1} \left( \begin{array}{c} a, b \\ c \\
\end{array} \bigg| f \right) ' \right) \right)
\]

**Problem**: The local to global strategy:

\[
\text{invariant local data} \leadsto \text{pullback function } f \leadsto y
\]

works for many functions, but \(_{2}F_{1}\) can be problematic because \(f\) can be large even if the amount of local data is small.
The Gauss hypergeometric function is:

\[ _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n \]

where \((a)_n = a \cdot (a + 1) \cdots (a + n - 1)\).

If \( L(y) = 0 \) is a globally bounded equation of order 2 then it conjecturally has algebraic or \(_2F_1\)-type solutions:

\[ y = \exp \left( \int r \cdot \left( r_0 \cdot _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| f \right) + r_1 \cdot _2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \middle| f \right) \right) \right) \]

**Problem**: The local to global strategy:

- invariant local data \( \leadsto \) pullback function \( f \) \( \leadsto \) \( y \)

works for many functions, but \(_2F_1\) can be problematic because \( f \) can be large even if the amount of local data is small.
Small equation:

\[ 4x(x^2 - 34x + 1)y'' + (8x^2 - 204x + 4)y' + (x - 10)y = 0 \]

The smallest solution:

\[ \frac{\sqrt{3 - 3x - \sqrt{x^2 - 34x + 1}}}{x + 1} \cdot {}_2F_1 \left( \frac{1}{3}, \frac{2}{3} \mid f \right) \]

has

\[ f = \frac{(x^3 + 30x^2 - 24x + 1) - (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x + 1)^3} \]

How to construct \( f \) from a small amount of invariant local data:

- Exponent-differences: 0, 0, \( \frac{1}{2} \) (mod \( \mathbb{Z} \))
- at the singularities: \( x = 0, \ x = \infty, \ x^2 - 34x + 1 = 0 \)
$2F_1$-type solutions and related topics:

- Tingting Fang (Ph.D 2012)
  - Compute $D$-module automorphisms $\rightsquigarrow$ descent.
  - (also useful for non $2F_1$ cases and for order $> 2$)

- Vijay Kunwar (Ph.D 2014)
  - Small $f$: Construct from invariant local data.
  - Large $f$: Tabulate and use combinatorial objects (such as dessins d’enfant) to prove completeness.

- Erdal Imamoglu (Ph.D 2017)
  - If transformation (2) is not needed: quotient method.
  - Otherwise: Differential analogue of POLRED $\rightsquigarrow$ simpler equations. Then use quotient method.

- Wen Xu (Ph.D in progress)
  - Multivariate generalizations of $2F_1$ such as Appell $F_1$. 
Algebraic computations often lead to an equation:

\[ f(x) = 0 \]

for some irreducible \( f \in \mathbb{Q}[x] \). Such \( f \) defines a number field:

\[ K = \mathbb{Q}[x]/(f) \]

In many computations there is no reason to assume that \( f \) is the simplest polynomial that defines \( K \).

**Algorithm POLRED**

**Input:** Irreducible \( f \in \mathbb{Q}[x] \).

**Output:** Monic \( g \in \mathbb{Z}[x] \) for the same field:

\[ K \cong \mathbb{Q}[x]/(g) \]

with near-optimal size for \( \max(\text{abs(coefficients of } g)) \).
Differential analogue of POLRED

The following equation came from lattice path combinatorics \( \Rightarrow \) \textbf{globally bounded}, conjecturally implies \( \exists \, 2F_1\)-type solutions

\[
x(8x^2 - 1)(8x^2 + 1)(896x^5 - 512x^4 + 832x^3 - 127x^2 - 6x - 12) \cdot y'' - (8x^2 + 1)(71680x^7 - 36864x^6 + 46080x^5 - 3528x^4 - 5280x^3 + 155x^2 + 24x + 36) \cdot y' + (1720320x^8 - 786432x^7 + 1078272x^6 - 183360x^5 + 48384x^4 - 12464x^3 - 4560x^2 - 928x - 96) \cdot y = 0
\]

www.math.fsu.edu/~eimamogl/hypergeometricsols

**Finds smaller equation by imitating POLRED:**

- Take the differential module for this equation.
- Compute its integral basis.
- Construct integral element \( Y \) with \textbf{minimal degree} at infinity.
- Then \( Y \) satisfies a small equivalent equation:

\[
x(8x^2 - 1)(8x^2 + 1) \cdot Y'' + (320x^4 - 1) \cdot Y' + 192x^3 \cdot Y = 0
\]

- Quotient method \( \leadsto \) closed form for \( Y \) \( \leadsto \) closed form for \( y \).
The following equation came from lattice path combinatorics \( \Longrightarrow \) **globally bounded**, conjecturally implies \( \exists 2F_1\)-type solutions

\[
x(8x^2 - 1)(8x^2 + 1)(896x^5 - 512x^4 + 832x^3 - 127x^2 - 6x - 12) \cdot y'' - (8x^2 + 1)(71680x^7 - 36864x^6 + 46080x^5 - 3528x^4 - 5280x^3 + 155x^2 + 24x + 36) \cdot y' + (1720320x^8 - 786432x^7 + 1078272x^6 - 183360x^5 + 48384x^4 - 12464x^3 - 4560x^2 - 928x - 96) \cdot y = 0
\]

www.math.fsu.edu/~eimamogl/hypergeometricsols

**Finds smaller equation by imitating POLRED:**

- Take the **differential module** for this equation.
- Compute its **integral basis**.
- Construct integral element \( Y \) with **minimal degree** at infinity.
- Then \( Y \) satisfies a small equivalent equation:

\[
x(8x^2 - 1)(8x^2 + 1) \cdot Y'' + (320x^4 - 1) \cdot Y' + 192x^3 \cdot Y = 0
\]

- Quotient method \( \rightsquigarrow \) closed form for \( Y \) \( \rightsquigarrow \) closed form for \( y \).
Example from the Ising model

\[ x(1-11x)(1+4x^2)y''+(1-22x+8x^2-132x^3)y'+(-3+x-33x^2)y = 0 \]

Solutions near \( x = 0 \):
\[ S_1 = 1 + 3x + \frac{37}{2}x^2 + \cdots \]
\[ S_2 = \ln x + (3 \ln x + 5)x + \left(\frac{37}{2} \ln x + \frac{73}{2}\right)x^2 + \cdots \]

Solutions near \( x = 1/11 \):
\[ T_1 = 1 - \frac{77}{25}(x - \frac{1}{11}) + \cdots \]
\[ T_2 = \ln \left(\frac{1}{11} - x\right) - \left(\frac{77}{25} \ln \left(\frac{1}{11} - x\right) + \frac{649}{125}\right)(x - \frac{1}{11}) + \cdots \]

Analytic continuation from \( x = 0 \) to \( x = 1/11 \) sends \( S_1 \) to a linear combination of \( T_1, T_2 \). Which linear combination? Can always find an approximate answer (e.g. by evaluating at intermediate points)

www.math.fsu.edu/~eimamogl/hypergeometricsols

\( \mapsto \) closed form solution \( \mapsto \) exact answer.
If globally bounded equations of order 2 have $2F_1$-type solutions, what about higher order?

**Univariate** generalization of $2F_1$: hypergeometric $pF_q$ functions.

Globally bounded order 3 equations need not be $pF_q$-solvable.
Can construct a univariate example from multivariate hypergeometric functions (substitution $\mapsto$ univariate).

There are many multivariate hypergeometric functions. A particle zoo of functions?

Fortunately, they have been organized in terms of polytopes:

A-hypergeometric functions
Gelfand, Kapranov, Zelevinsky (1990)
Beukers (ISSAC’2012 invited talk and recent papers)

Are globally bounded equations solvable in terms of such functions?
If globally bounded equations of order 2 have $2F_1$-type solutions, what about higher order?

**Univariate** generalization of $2F_1$: hypergeometric $pF_q$ functions.

Globally bounded order 3 equations need not be $pF_q$-solvable. Can construct a **univariate** example from **multivariate** hypergeometric functions (substitution $\leadsto$ **univariate**).

There are many multivariate hypergeometric functions. A **particle zoo** of functions?

Fortunately, they have been organized in terms of polytopes:

**A-hypergeometric functions**
Gelfand, Kapranov, Zelevinsky (1990)
Beukers (ISSAC’2012 invited talk and recent papers)

Are globally bounded equations solvable in terms of such functions?
Trying to solve order $> 2$ equations in terms of such functions leads to many questions, for instance: how they relate to each other? Do we need reducible $A$-hypergeometric systems?

**Example:** The Horn $G_3$ function satisfies a bivariate system of order 3. In the **reducible case** $a = 1 - 2b$ this function

$$G_3(1 - 2b, b \mid x, y)$$

satisfies the same bivariate differential equations as:

$$(1 + 3y)^{3/2} b^{-1} y^{1-2b} \cdot _2F_1 \left( \begin{array}{c} \frac{1}{3} - b \frac{1}{2}, \frac{2}{3} - b \frac{1}{2} \end{array} \left| \frac{(27xy^2 - 9y - 2)^2}{4(1 + 3y)^3} \right. \right)$$

Found similar formulas for other reducible order 3 systems.

Thank you