Third Order A-hypergeometric Functions

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Motivation

How to solve a differential equation L_{input}?

- To solve it: find a sequence of transformations, from an equation L with known solutions, to L_{input}.

Example

Suppose L_{input} has a solution $(x^{\frac{1}{4}} \cdot B_v(x^2+1))'$ where B_v is the Bessel function. Finding such solutions = finding transformations:

$$\begin{array}{ccccccc} L_{Bessel} & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & L_{input} \\ \updownarrow & (i) & \updownarrow & (ii) & \updownarrow & (iii) & \updownarrow \\ B_v(x) & \rightarrow & B_v(x^2+1) & \rightarrow & x^{\frac{1}{4}} \cdot B_v(x^2+1) & \rightarrow & (x^{\frac{1}{4}} \cdot B_v(x^2+1))' \end{array}$$

- (i) change of variables transformation: $S(x) \mapsto S(x^2+1)$.
- (ii) exp-product transformation $S(x) \mapsto x^{\frac{1}{4}} \cdot S(x)$.
- (iii) gauge transformation $S(x) \mapsto S(x)'$.

Transformations

Transformations preserving the order

- (i) Change of variables transformation: $S(x) \mapsto S(f)$ with $f \in \mathbb{C}(x) - \mathbb{C}$. (Pullback function).
- (ii) Exp-product transformation: $S \mapsto \exp(\int r) \cdot S$ with $r \in \mathbb{C}(x)$.
- (iii) Gauge transformation: $S \mapsto G(S)$ where $\mathrm{GCRD}(G, L) = 1$ and $G(y) = r_0 y + r_1 y' + \ldots + r_{n-1} y^{(n-1)}$ with r_0, r_1,..., $r_{n-1} \in \mathbb{C}(x)$.

- There are transformations that may increase the order and we have algorithm for them.

Transformations (i)(ii)(iii) send S to:

$$\exp(\int r \mathrm{d}x)(r_0 S(f) + r_1 S(f)' + \ldots + r_{n-1} S(f)^{(n-1)}).$$

Solve differential equations in terms of well-known functions S.

Globally Bounded Equations

Globally Bounded

Let $y \in \mathbb{C}[[x]] - \{0\}$, if y has a positive radius of convergence and there exist $c_1, c_2 \in \mathbb{C} - \{0\}$ such that $c_1 \cdot y(c_2 x) \in \mathbb{Z}[[x]]$, then $y(x)$ is called *globally bounded*. An irreducible differential equation is called globally bounded if it has a globally bounded solution.

Globally bounded order 2 equations are very common, and so far they all turn out to have ${}_2F_1$-type solutions. $\rightsquigarrow$ Conjecture

Conjecture 1 (van Hoeij, Kunwar)

Every globally bounded order 2 equation has a ${}_2F_1$-type solution or an algebraic solution.

Hypergeometric Functions

The ${}_2F_1$ function, also called Gauss hypergeometric function, is:

$${}_2F_1(a, b; c \,|\, x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k$$

with $(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1)$. It satisfies:

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0.$$

The univariate generalization of ${}_2F_1$ is ${}_pF_q$:

$${}_pF_q\left(\begin{matrix} \alpha_1 \ldots \alpha_p \\ \beta_1 \ldots \beta_q \end{matrix} \,|x\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k k!} x^k.$$

- $p = q+1$: the equation is regular singular. With suitable parameters, ${}_pF_q$ function $\rightsquigarrow$ globally bounded functions.
- $p \neq q+1$: the equation is irregular singular. Bessel, Airy, Kummer, Whittaker: solvers from Quan Yuan.

Preliminaries: A-hypergeometric Functions

Appell's F_1 function is a multivariate generalization of ${}_2F_1$:

$$F_1(a, b_1, b_2, c \mid x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m! n!} x^m y^n.$$

It satisfies following equations:

$$x(1-x)\partial_x^2 F_1 + y(1-x)\partial_x\partial_y F_1 + (c-(a+b_1+1)x)\partial_x F_1 - b_1 y \partial_y F_1 - ab_1 F_1 = 0,$$
$$y(1-y)\partial_y^2 F_1 + x(1-y)\partial_x\partial_y F_1 + (c-(a+b_2+1)y)\partial_y F_1 - b_2 x \partial_x F_1 - ab_2 F_1 = 0.$$

- There are many other multivariate generalizations.
- Fortunately, there is a framework to classify all of them in terms of polytopes – called "GKZ" or "A-hypergeometric functions".

Question

- Closed form solutions are very common $\rightsquigarrow$ good solvers are useful $\rightsquigarrow$ Fang, Kunwar, Imamoglu: ${}_2F_1$-solver.
- Is this conjecture also true for higher order equations? How about order 3 globally bounded equations?
- The univariate generalization of ${}_2F_1$ is ${}_3F_2$ $\rightsquigarrow$ Question 1:

Question 1

Does every globally bounded order 3 equation have a ${}_3F_2$-type solution, or a square of ${}_2F_1$-type solution, or an algebraic solution?

To find such solutions,

- Algebraic solution $\rightsquigarrow$ Liouvillian solution
- ${}_2F_1$-type solutions: ${}_2F_1$-solver from Imamoglu.
- ${}_3F_2$-type solutions $\rightsquigarrow$ ${}_3F_2$-solver (partial) $\rightsquigarrow$ degree 1 pullback functions.

Question 1: Solving order 3 globally bounded equations

- From OEIS, we didn't find a counter example of Question 1.
- We constructed a counter example of Question 1 using Appell's F_1 function, a multivariate hypergeometric function (substitution $\rightsquigarrow$ univariate)

So:

- Univariate hypergeometric functions are not enough for solving globally bounded univariate equations of order 3.
- To correct Question 1, at least F_1 should be added $\rightsquigarrow$ a partial F_1-solver (degree 1 pullback functions).
- Other functions for order 3 may be needed as well:

Univariate	Multivariate
${}_3F_2$	$F_1,\ G_1,\ G_2,\ G_3,\ H_3,\ H_6$

Table: A-Hypergeometric Functions of Order 3

A-Hypergeometric Functions

- Do we need to add all A-hypergeometric functions in the table to Question 1?
- Do we need solvers for each of them?

Investigate their relations $\rightsquigarrow$ Develop multivariate tools:

- recover transformation (iii) between two multivariate systems
- recover transformation (ii)+(iii)

More Questions

- Question 2: are globally bounded equations solvable in terms of A-hypergeometric functions?
- Question 3: can we restrict Question 2 to irreducible systems? (are factors of reducible A-hypergeometric systems again A-hypergeometric?)
  - With ${}_2F_1$-solver and multivariate tools, we verified Question 3 for order 3 systems.
  - To study order 4 systems, one can use ${}_3F_2$-solver and F_1-solver.

Outline

- Preliminaries
- ${}_3F_2$-solver
- F_1-solver
- Multivariate tools
- Applications

Preliminaries: Recover one type of Transformations

Let $L_1, L_2 \in \mathbb{C}(x)[\partial]$ with $\partial = \frac{d}{dx}$. If L_1, L_2 differ by just one transformation, (i),(ii) or (iii), then recovering this transformation is:

(i)	Easy
(ii)	Trivial
(iii)	DEtools[Homomorphisms]

Table: Recover $L_1 \rightarrow L_2$

But if L_1, L_2 differ by two transformations, $L_1 \rightarrow\rightarrow L_2$, then recovering the transformations is much more difficult than using Table twice because the intermediate operator is not known.

Projective Algorithm

Operators $L_1, L_2 \in \mathbb{C}(x)[\partial]$ are projectively equivalent if there exist some $r \in \mathbb{C}(x)$ and $G \in \mathbb{C}(x)[\partial]$ s.t. $y \mapsto e^{\int r} \cdot G(y)$ sends solutions of L_1 to solutions of L_2:

$$L_1 \xrightarrow{(ii)} \xrightarrow{(iii)} L_2.$$

For order 3, to recover (ii)+(iii):

- First recover (ii) (r)
- Then the intermediate operator M is known: $L_1 \xrightarrow{(ii)} M \xrightarrow{(iii)} L_2$.
- Use "Homomorphisms" in Maple to compute (iii).

To recover $r \rightsquigarrow$ use data invariant under (iii) $\rightsquigarrow$ generalized exponents modulo "integers"

Let L_2 be an irreducible order 3 operator in $\mathbb{C}(x)[\partial]$. To find its ${}_3F_2$-type solutions $e^{\int r} \cdot G({}_3F_2(a_1, a_2, a_3, b_1, b_2 \mid f))$ is to find parameters a_1, a_2, a_3, b_1, b_2 in ${}_3F_2$ and transformations from L_1 to L_2:

$$L_1 \xrightarrow{(i)} \xrightarrow{(ii)+(iii)} L_2$$

where L_1 is the minimal operator of ${}_3F_2(a_1, a_2, a_3, b_1, b_2 \mid x)$.

- It suffices to recover f (degree 1) in (i) and parameters in ${}_3F_2$.
- $L_1 \xrightarrow[f]{(i)} \xrightarrow{(ii),(iii)} L_2$ sends non-removable singularities of L_1 $(0, 1, \infty)$ to roots of $\{f, 1-f, \frac{1}{f}\}$: non-removable singularities of L_2.
- Non-removable singularities of L_2 are known if L_2 is known.

To recover a_1, a_2, a_3, b_1, b_2

- $\xrightarrow{(i)}$ preserves generalized exponents when $\deg(f) = 1$.
- $\xrightarrow{(ii)(iii)}$ preserves the generalized exponents differences modulo integers.

To recover a_1, a_2, a_3, b_1, b_2 in ${}_3F_2 \rightsquigarrow$ data invariant under (i)+(ii)+(iii) $\rightsquigarrow$ generalized exponents differences mod $\mathbb{Z}$ at non-removable singularities.

F_1-solver

Let L_2 be an irreducible order 3 operator in $\mathbb{C}(x)[\partial]$. To find its F_1-type solution $e^{\int r} \cdot F_1(a, b_1, b_2, c \mid u, v)$ is to find parameters a, b_1, b_2, c in F_1 and the transformation from L_1 to L_2:

$$L_1 \xrightarrow{(ii)} L_2$$

where L_1 is the minimal operator of $F_1(a, b_1, b_2, c \mid u, v)$ with two pullback functions $u, v \in \mathbb{C}(x)$. We restrict u, v to degree 1.

To recover pullback functions $u, v \rightsquigarrow$ Non-removable singularities.

- Non-removable singularities of L_1:
  roots of $\{u, 1-u, \frac{1}{u}, v, 1-v, \frac{1}{v}, u-v\}$.

Suppose the set of non-removable singularities of L_2 is A, then

- Candidates for u and v:
  Mobius transformation f with $\{0, 1, \infty\} \subseteq f(A)$.
- Candidates for pairs $[u, v]$:
  Roots of $\{u, 1-u, \frac{1}{u}, v, 1-v, \frac{1}{v}, u-v\} = A$.

Divide pairs $[u, v]$ into orbits

Following functions satisfy the same differential equations

- $F_1(a, b_1, b_2, c \mid u, v)$
- $r_1 \cdot F_1(c - a, b_1, b_2, c \mid \frac{u}{u-1}, \frac{v}{v-1})$
- $r_2 \cdot F_1(a, c - b_1 - b_2, b_2, c \mid \frac{u}{u-1}, \frac{v-u}{1-u})$
- $r_3 \cdot F_1(a, b_1, c - b_1 - b_2, c \mid \frac{v-u}{v-1}, \frac{v}{v-1})$
- $F_1(a, b_1, b_2, b_1 + b_2 + a + 1 - c \mid 1 - u, 1 - v)$

where $r_1 = (1-u)^{-b_1}(1-v)^{-b_2}$, $r_2 = (1-u)^{-a}$, $r_3 = (1-v)^{-a}$.

Let $G =< R_1, R_2, R_3, R_4 >\subseteq \mathrm{Aut}(\mathbb{Q}(u, v))$ where:

- $R_1([u, v]) = [\frac{u}{u-1}, \frac{v}{v-1}]$
- $R_2([u, v]) = [\frac{u}{u-1}, \frac{v-u}{1-u}]$
- $R_3([u, v]) = [\frac{v-u}{v-1}, \frac{v}{v-1}]$
- $R_4([u, v]) = [1 - u, 1 - v]$.

Divide all pairs $[u, v]$ into orbits under group $G \cong S_5$.

Recover parameters in F_1 and $r \rightsquigarrow$ Exponents differences

Now u, v are known, need to find a, b_1, b_2, c in F_1 and transformation:

$$L_1 \xrightarrow{(ii)} L_2$$

where L_1 is the minimal operator of $F_1(a, b_1, b_2, c \mid u, v)$.

To recover a_1, a_2, a_3, b_1, b_2

- As in ${}_3F_2$-solver, match exponents differences of L_1 to L_2 and solve for parameters.
- Different relations of u, v generates different exponents $\rightsquigarrow$ many cases to consider.

(Projective) Homomorphism

Let $D = \mathbb{C}(x)[\partial]$ and $L_1 \in D - \{0\}$ be the minimal operator of y, $Dy := \{L(y) \mid L \in D\}$. Then $Dy \cong D/DL_1$ is a left D-module.

- Transformation $\xrightarrow{(iii)}$ corresponds to a D-module isomorphism from $D/DL_1 \to D/DL_2$.
- Transformation $L_1 \xrightarrow[r,G]{(ii)(iii)} L_2$ corresponds to an isomorphism from $D/DL_1 \otimes D/D(\partial - r) \to D/DL_2$.

- Task 1. Given D-modules M_1 and M_2, find $\mathrm{Hom}_D(M_1, M_2)$.
- Task 2. Given D-modules M_1 and M_2, find a 1-dimensional module I for which there is a non-trivial homomorphism h from $M_1 \otimes I$ to M_2. (Call h a projective homomorphism)

Task 1: algorithm for univariate case.
Task 2: algorithm for univariate order 2. Have extended it to univariate order 3.
Now want to extend both tasks to multivariate.

D-modules

- All multivariate order 3 A-hypergeometric functions are bivariate, so let $K = \mathbb{C}(x, y)$, $D = K[\partial_x, \partial_y]$.
- A D-module is a finitely dimensional K-vector space on which D acts. To turn K^n into a D-module, take two $n \times n$ matrices M_x and M_y over K and define

$$\partial_x \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = M_x \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \partial_x(a_1) \\ \vdots \\ \partial_x(a_n) \end{bmatrix},$$

$$\partial_y \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = M_y \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} \partial_y(a_1) \\ \vdots \\ \partial_y(a_n) \end{bmatrix}.$$

- Matrices M_x and M_y satisfy "integrability" ($\partial_x \partial_y = \partial_y \partial_x$).
- A-hypergeometric functions satisfy differential equations, which could be presented by D-modules.

Task 1: Homomorphisms between two D-modules

- $K = \mathbb{C}(x, y)$, $D = K[\partial_x, \partial_y]$.
- $D_x = K[\partial_x]$, $D_y = K[\partial_y]$.
- Let M, M' be D-modules. As K-vector spaces, $M \cong K^n$ and $M' \cong K^{n'}$.
- Goal: compute homomorphisms between M and M' as D-modules. Idea: multivariate $\rightarrow$ univariate.

Theorem

$\mathrm{Hom}_D(M, M') = \mathrm{Hom}_{D_x}(M, M') \cap \mathrm{Hom}_{D_y}(M, M')$.

Theorem reduces our goal to two tasks.

- Compute $V_1 := \mathrm{Hom}_{D_x}(M, M')$ and $V_2 := \mathrm{Hom}_{D_y}(M, M')$.
- Given two vector spaces $V_1, V_2 \subseteq \mathrm{Mat}_{n',n}(K)$, where V_1 is a $\mathbb{C}(y)$-vector space and V_2 is a $\mathbb{C}(x)$-vector space, compute their intersection $V_1 \cap V_2$ (a $\mathbb{C}$-vector space).

Compute $\mathrm{Hom}_{D_x}(M, M')$

- Every $G \in$ Homomorphisms(L'_x, L_x) in Maple corresponds a homomorphism $\phi \in \mathrm{Hom}_{D_x}(M, M')$.
- An element $m \in M$ is a *cyclic vector* w.r.t. x if $D_x m = M$. Cyclic vector theorem ensures the existence of cyclic vectors.
- Given a D-module and a variable, "CycVec" returns a cyclic vector and its minimal operator w.r.t that variable (L_x, L'_x).
- $\mathrm{Hom}_{D_x}(M, M') \subseteq \mathrm{Hom}_K(M, M') = \mathrm{Mat}_{n',n}(K)$ could be computed from G applying on a basis of M' and the change of basis matrix.
- $\mathrm{Hom}_{D_x}(M, M')$ is a $\mathbb{C}(y)$-vector space.
- $\mathrm{Hom}_{D_y}(M, M')$ is a $\mathbb{C}(x)$-vector space.

$\mathrm{Hom}_{D_x}(M, M') \cap \mathrm{Hom}_{D_y}(M, M')$ – Algorithm "Hom"

Let M be the D-module of $F_1(1, b_1, b_2, c \mid x, y)$, I be the D-module of $(y-x)^{1-b_1-b_2}(1-y)^{b_1-2}(1-x)^{c-b_1}x^{b_1+b_2-c}$ and M' be the D-module of ${}_2F_1(1-b_1, c-b_1-b_2, c-b_1 \mid \frac{y(x-1)}{x(y-1)})$ tensor I.
Then the $h_1 \in \mathrm{Hom}_{D_x}(M, M')$ and $h_2 \in \mathrm{Hom}_{D_y}(M, M')$ are:

$$h_1 = \begin{bmatrix} \frac{1}{x-1} & -\frac{((b_1+b_2-1)y-c+b_1+1)x+(c-2b_1-b_2)y}{b_1(x-1)^2(x-y)} & \frac{((b_1+b_2+1)y-b_2)x+(1-b_1)y}{b_2(x-y)(x-1)(y-1)} \\ 0 & -\frac{(b_1-1)(b_1+b_2-c)y}{b_1(b_1-c)x(x-1)(y-1)} & \frac{(b_1-1)(b_1+b_2-c)y}{b_2(b_1-c)x(y-1)^2} \end{bmatrix},$$

$$h_2 = \begin{bmatrix} y-1 & -\frac{(y-1)\cdot(((b_1+b_2-1)y-c+b_1+1)x+(c-2b_1-b_2)y)}{b_1(x-1)(x-y)} & \frac{((b_1+b_2+1)y-b_2)x+(1-b_1)y}{b_2(x-y)} \\ 0 & -\frac{(b_1-1)(b_1+b_2-c)y}{b_1(b_1-c)x} & \frac{(b_1-1)(b_1+b_2-c)(x-1)y}{b_2(b_1-c)x(y-1)} \end{bmatrix}.$$

Now $(y-1)\cdot h_1 = \frac{h_2}{x-1}$, recall h_1 is unique up to $\mathbb{C}(y)$ and h_2 is unique up to $\mathbb{C}(x)$. So $(y-1)\cdot h_1 \in \mathrm{Hom}_D(M, M')$.

Task 2: Compute Projective Homomorphisms "projHom"

Goal: Input: D-modules M, M'. Find a 1-dimensional module I for which there is a non-trivial homomorphism from $M \otimes I$ to M'.

- Generalized exponents w.r.t x $\rightsquigarrow$ candidates for I as D_x-module.
- Generalized exponents w.r.t y $\rightsquigarrow$ candidates for I as D_y-module.
- A large set of candidates for I: any generalized exponent difference from M to M' is a candidate.
- Options "injective" and/or "surjective" drastically reduce the number. For example, "surjective" implies: for every generalized exponent of M', there must be a generalized exponent of M matching it.

Suppose there exists a projective homomorphism from M_1 to M_2:

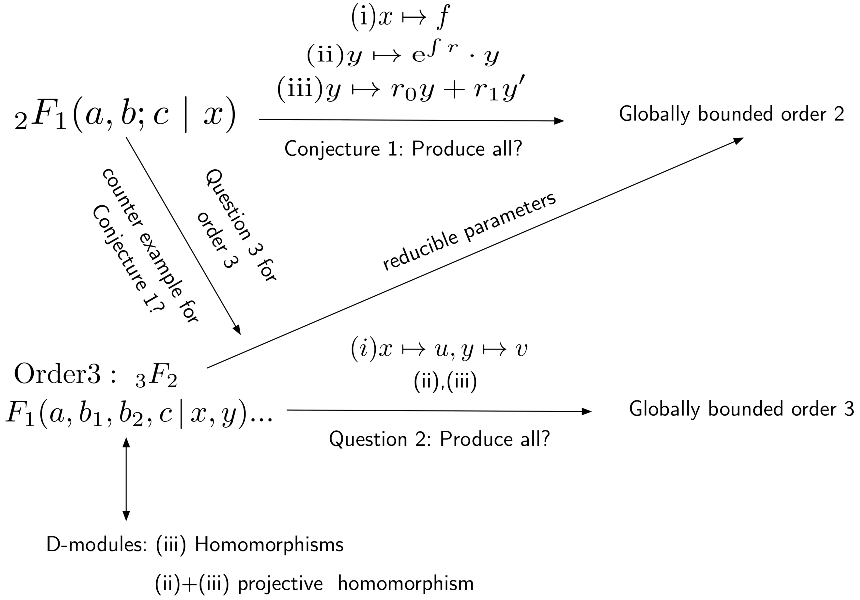
- M_1 irreducible $\rightsquigarrow$ injective; M_2 irreducible $\rightsquigarrow$ surjective.

Example of “projHom”

Example

Let M_1 be the D-module of $F_1(1, b_1, b_2, c \,|\, x, y)$ and M_2 be the D-module of ${}_2F_1(1 - b_1, c - b_1 - b_2, c - b_1 \,|\, \frac{y(x-1)}{x(y-1)})$.

- Algorithm “projHom” gives 243 candidates for I.
- But M_2 is irreducible. The number of candidates I drops to 1 after adding “surjective” option.


(i)$x \mapsto f$
(ii)$y \mapsto e^{\int r} \cdot y$
(iii)$y \mapsto r_0 y + r_1 y'$
${}_2F_1(a, b; c \mid x)$
Globally bounded order 2
Conjecture 1: Produce all?
counter example for Conjecture 1?
Question 3 for order 3
reducible parameters
(i)$x \mapsto u, y \mapsto v$
(ii),(iii)
Order3 : ${}_3F_2$
$F_1(a, b_1, b_2, c \mid x, y)$...
Globally bounded order 3
Question 2: Produce all?
D-modules: (iii) Homomorphisms
(ii)+(iii) projective homomorphism

Reducible F_1

Let $F_1^D(a, b_1, b_2, c \mid x, y)$ be the D-module of $F_1(a, b_1, b_2, c \mid x, y)$.
It is reducible if and only if
$a \in \mathbb{Z}$ or $b_1 \in \mathbb{Z}$ or $b_2 \in \mathbb{Z}$ or $c - a \in \mathbb{Z}$ or $c - b_1 - b_2 \in \mathbb{Z}$.

D-modules of following functions are projectively equivalent

- $F_1(a, b_1, b_2, c \mid u, v)$
- $F_1(c - a, b_1, b_2, c \mid \frac{u}{u-1}, \frac{v}{v-1})$
- $F_1(a, c - b_1 - b_2, b_2, c \mid \frac{u}{u-1}, \frac{v-u}{1-u})$
- $F_1(a, b_1, c - b_1 - b_2, c \mid \frac{v-u}{v-1}, \frac{v}{v-1})$
- $F_1(a, b_1, b_2, b_1 + b_2 + a + 1 - c \mid 1 - u, 1 - v)$

All reducible $F_1^D(a, b_1, b_2, c \mid x, y) \rightsquigarrow F_1^D(a' \in \mathbb{Z}, b_1', b_2', c' \mid x', y')$
with $\mathbb{C}(x', y') = \mathbb{C}(x, y)$.

Reducible F_1

- Algorithm "Hom" gives a homomorphism from $F_1^D(a, b_1, b_2, c \,|\, x, y)$ to $F_1^D(a+1, b_1, b_2, c \,|\, x, y)$:
$$H = \begin{bmatrix} a - b_1 - b_2 & \frac{b_1+b_2-c+1}{x-1} & \frac{b_1+b_2-c+1}{y-1} \\ b_1 & -\frac{a+b_1-c+1}{x-1} & -\frac{b_1}{y-1} \\ b_2 & \frac{-b_2}{x-1} & -\frac{a+b_2-c+1}{y-1} \end{bmatrix}.$$
- H is an isomorphism when $a \neq 0 \Rightarrow F_1^D(a \in \mathbb{Z}, b_1, b_2, c \,|\, x, y)$ $\rightsquigarrow F_1^D(0, b_1, b_2, c \,|\, x, y)$ and $F_1^D(1, b_1, b_2, c \,|\, x, y)$.
- "Hom" gives a homomorphism between the dual module of $F_1(1-a, 1-b_1, 1-b_2, 3-c \,|\, x, y)$ and $F_1^D(a, b_1, b_2, c \,|\, x, y)$.
- $F_1^D(0, b_1, b_2, c \,|\, x, y)$ and $F_1^D(1, b_1, b_2, c \,|\, x, y)$ reduce to each other under the dual.

Reducible F_1

"projHom" gives a projective homomorphism from $F_1^D(1, b_1, b_2, c \mid x, y)$ to the D-module of ${}_2F_1(1 - b_1, c - b_1 - b_2, c - b_1 \mid \frac{y(x-1)}{x(y-1)})$.

Theorem

Any irreducible 2^{nd} order submodule or quotient module of $F_1^D(a, b_1, b_2, c \mid x, y)$ comes from ${}_2F_1$.

A-hypergeometric Horn G_1, G_2, H_3 and H_6 relate to F_1. From their relations, one can obtain the same conclusion of their reducible submodules and quotient modules.

Reducible Horn G_3

One Reducible Case of G_3

The reducible $G_3(1-2b, b \mid x, y)$ satisfies the same differential equations as

$$(3y+1)^{\frac{3b}{2}-1} y^{1-2b} \cdot {}_2F_1\left(\frac{1}{3}-\frac{1}{2}b, \frac{2}{3}-\frac{1}{2}b, \frac{1}{2} \,\middle|\, \frac{(27xy^2-9y-2)^2}{4(3y+1)^3}\right).$$

How did we find this relation?

- Use ${}_2F_1$-solver to recover the pullback function and parameters in ${}_2F_1$.
- Use "projHom" to compute projective homomorphisms between D-modules of G_3 and ${}_2F_1$.

Thank You