Factoring Linear Recurrence Operators

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May 13, 2019

[1]Supported by NSF 1618657

Recurrence operators with rational function coefficients

Let $a_i(n) \in \mathbb{Q}(n)$ be rational functions in n.

Recurrence relation:

$$a_k(n)u(n+k) + \cdots + a_1(n)u(n+1) + a_0(n)u(n) = 0.$$

Solutions $u(n)$ are viewed as functions on subsets of $\mathbb{C}$.

Recurrence operator: write the recurrence relation as $L(u) = 0$ where

$$L = a_k\tau^k + \cdots + a_0\tau^0 \in \mathbb{Q}(n)[\tau]$$

Here τ is the shift operator. It sends $u(n)$ to $u(n+1)$.

Recurrence relations come from many sources:

Zeilberger's algorithm, walks, QFT computations, OEIS, etc.

Goal: factoring recurrence operators

Factoring: if possible, write L as a composition $L_1 \circ L_2$ of lower order operators.

Computing first order right-factors:

> Same as computing hypergeometric solutions, there are algorithms (Petkovšek 1992, vH 1999) and implementations.

Goal: compute right-factors of order $d > 1$.

Method 1: Hypergeometric solutions of a system of order $\binom{k}{d}$.

Method 2: Construct factors from special solutions.

Example: Entry A025184 in OEIS

$$\begin{aligned} L(u) = {} & 33n(3n-1)(3n-2)u(n) \\ & +11(2047n^3 - 10725n^2 + 17192n - 8520)u(n-1) \\ & -9(4397n^3 + 10169n^2 - 110500n + 145368)u(n-2) \\ & -54(2n-5)(5353n^2 - 33313n + 53904)u(n-3) \\ & -115668(n-4)(2n-5)(2n-7)u(n-4) = 0. \end{aligned}$$

$L \in \mathbb{Q}(n)[\tau^{-1}]$ has order 4 and ***n*-degree 3**.

Our implementation finds a right-hand factor R where $R(u) =$

$$\begin{aligned} & 3n(3n-1)(3n-2)(221n^2 - 723n + 574)u(n) \\ -2(2n-1)(7735n^4 - 33040n^3 + 48239n^2 - 27998n + 5280)u(n-1) \\ & -36(n-2)(2n-1)(2n-3)(221n^2 - 281n + 72)u(n-2) \end{aligned}$$

R order 2 but ***n*-degree 5** which is more than L!

(Explanation: R has 3 true and 2 apparent singularities).

Gauss' lemma does not hold for $\mathbb{Q}[n][\tau] \subset \mathbb{Q}(n)[\tau]$

Gauss' lemma does not hold for difference operators:

1. Reducible operators in $\mathbb{Q}(n)[\tau]$ are often irreducible in $\mathbb{Q}[n][\tau]$.
2. L can have a right-factor R with higher n-degree than L (after clearing denominators).

This means:

1. It is not enough to factor in the τ-Weyl algebra $\mathbb{Q}[n][\tau]$.
2. Bounding n-degrees of right-factors is non-trivial.

Method 1: Reduce order-d factors to order-1 factors

Beke (1894) gave a method to reduce:

- order-d factors of a differential operator of order k

to

- order-1 factors of several operators of order $\binom{k}{d}$.

Bronstein (ISSAC'1994) gave significant improvements:

1. Use only one system of order $\binom{k}{d}$

   (instead of several operators of that order, whose factors had to be combined with a potentially costly computation)
2. This system has much smaller coefficients, which improves performance as well.

Beke 1894 / Bronstein 1994 works for recurrence operators as well.

Method 1: Reduce to order 1

Let $\mathcal{D} := \mathbb{Q}(n)[\tau]$.
Let $L \in \mathcal{D}$ have order k.
Suppose L has a right-factor R of order d.

Consider the $\mathcal{D}$-modules

$$M_L := \mathcal{D}/\mathcal{D}L \quad \text{and} \quad M_R := \mathcal{D}/\mathcal{D}R$$

and homomorphism:

$$\phi : \bigwedge^d M_L \;\to\; \bigwedge^d M_R$$

Over $\mathbb{Q}(n)$:

$$\dim\left(\bigwedge^d M_L\right) = \binom{k}{d} \;\text{ and }\; \dim\left(\bigwedge^d M_R\right) = \binom{d}{d} = 1$$

Hence:

$$\phi \;\rightsquigarrow\; \text{a hypergeometric solution of the system for } \bigwedge^d M_L$$

System for $\bigwedge^d M_L$: Example with $k = 4$ and $d = 2$.

Let $L = \tau^4 + a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ and $M_L := \mathcal{D}/\mathcal{D}L$.

Action of τ on basis of $\bigwedge^2 M_L$ is:

$$
\begin{array}{lcl}
b_1 := \tau^0 \wedge \tau^1 & \mapsto & \tau^1 \wedge \tau^2 = b_4 \\
b_2 := \tau^0 \wedge \tau^2 & \mapsto & \tau^1 \wedge \tau^3 = b_5 \\
b_3 := \tau^0 \wedge \tau^3 & \mapsto & \tau^1 \wedge \tau^4 = a_0 b_1 - a_2 b_4 - a_3 b_5 \\
b_4 := \tau^1 \wedge \tau^2 & \mapsto & \tau^2 \wedge \tau^3 = b_6 \\
b_5 := \tau^1 \wedge \tau^3 & \mapsto & \tau^2 \wedge \tau^4 = a_0 b_2 + a_1 b_4 - a_3 b_6 \\
b_6 := \tau^2 \wedge \tau^3 & \mapsto & \tau^3 \wedge \tau^4 = a_0 b_3 + a_1 b_5 + a_2 b_6
\end{array}
$$

($\tau^4 = -a_0\tau^0 - a_1\tau^1 - a_2\tau^2 - a_3\tau^3$ in M_L)

System: $AY = \tau(Y)$ where $A = \begin{pmatrix} 0 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_0 \\ 1 & 0 & -a_2 & 0 & a_1 & 0 \\ 0 & 1 & -a_3 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 1 & -a_3 & a_2 \end{pmatrix}$

Hypergeometric solutions of systems

Suppose L has order k and a right-factor R of order d.

Let $N = \binom{k}{d}$ and A the $N \times N$ matrix as in the previous slide.

Then

$$AY = \tau(Y)$$

must have a hypergeometric solution:

$$Y = \lambda \begin{pmatrix} P_1 \\ \vdots \\ P_N \end{pmatrix} \quad \text{with} \quad P_i \in \mathbb{Q}[n] \quad \text{and} \quad r := \frac{\tau(\lambda)}{\lambda} \in \mathbb{Q}(n)$$

Bronstein found (similar to Petkovšek's algorithm) that one can write $r = c\frac{a}{b}$ with $c \in \mathbb{Q}^*$ and $a, b \in \mathbb{Q}[n]$ monic with:

$$b \,|\, \mathrm{denom}(A) \quad \text{and} \quad a \,|\, \mathrm{denom}(A^{-1})$$

$\rightsquigarrow$ almost an algorithm (still need c)

Algorithms

Computing c, improvements, implementation: Barkatou + vH.

More work in progress: Barkatou + vH + Middeke + Schneider.

If L has high order then $AY = \tau(Y)$ has high dimension $N = \binom{k}{d}$.

There is a faster method that works remarkably often even though it is not proved to work.

Another way to factor

LLL algorithm to factor $L \in \mathbb{Q}[x]$ in polynomial time:

1. Compute a p-adic solution α of L.
2. Find $M \in \mathbb{Z}[x]$ of lower degree with $M(\alpha) = 0$ if it exists.
3. If no such M exists, then L is irreducible, otherwise $\gcd(L, M)$ is a non-trivial factor.

In order for this to work for $L \in \mathbb{Q}(n)[\tau]$, the solution in Step 1 must meet this requirement:

Definition

A solution u of L is **order-special** if it satisfies an operator M of lower order.

Unlike the polynomial case, most solutions of most reducible operators are not order-special.

Factoring with a special solution

If L is reducible and u is order-special then write:

$$R := \sum_{i=0}^{k-1} \left(\sum_{j=0}^{\text{Degree bound}} c_{ij}\, n^j \right) \tau^i$$

Then

$$R(u) = 0 \quad \rightsquigarrow \quad \text{equations for } c_{ij} \quad \rightsquigarrow \quad R$$

We need:

1. Special solutions
2. Degree bound
   (How to bound the number of apparent singularities?).

Example: Special solutions

$$L(u) = 33n(3n-1)(3n-2)u(n)$$
$$+\cdots$$
$$-115668(n-4)(2n-5)(2n-7)u(n-4) = 0.$$

$L(u) = 0$ determines $u(n)$ in terms of $u(n-1), \ldots, u(n-4)$ except if n is a root of the leading coefficient.

Take $q \in \{0, \frac{1}{3}, \frac{2}{3}\}$. Define $u : q + \mathbb{Z} \to \mathbb{C}$ with:

$$L(u) = 0, \qquad u(n) = 0 \text{ for all } n < q, \qquad u(q) = 1.$$

Then u is called a leading-special solution. Likewise:

Roots of the trailing coefficient $\rightsquigarrow$ trailing-special solutions.

(Leading/trailing)-special solutions are frequently order-special!

Leading/trailing vs order special solutions

(Leading/trailing)-special solutions are frequently order-special.

We can only explain that for certain cases:

Suppose L is a Least-Common-Left-Multiple of L_1 and L_2.

Suppose L_1 and L_2 do not have the same valuation growths at some $q + \mathbb{Z}$ for some $q \in \mathbb{C}$.

Then a (leading/trailing)-special solution[2] is order-special.

Valuation-growth: the valuation (root/pole order) at $q + \mathrm{large}\ n$ minus the valuation at $q - \mathrm{large}\ n$.

[2] of L or its desingularization

Degree bound (with Yi Zhou)

Due to apparent singularities, a right-factor R of L can have higher n-degree than L.

A bound can be computed from generalized exponents.

Generalized exponents $\approx$ asymptotic behavior of solutions.

Example: $L = \tau - r$ with $r = 7n^3(1 + 8n^{-1} + \cdots n^{-2} + \cdots)$.
The dominant part of r is $e = 7n^3(1 + 8n^{-1})$.
This e encodes the dominant part of the solution

$$u(n) = 7^n \, \Gamma(n)^3 \, n^8 \, (1 + \cdots n^{-1} + \cdots n^{-2} + \cdots)$$

Definition

Let $e = c \cdot n^v \cdot (1 + c_1 n^{-1/r} + c_2 n^{-2/r} + \cdots + c_r n^{-1})$.
Then e is called a generalized exponent of L if:

> The operator obtained by dividing solutions of L by $\mathrm{Sol}(\tau - e)$ has an indicial equation with 0 as a root.

Degree bound (with Yi Zhou)

Let $R = r_d\tau^d + \cdots + r_0\tau^0$ be a right-factor of L in $\mathbb{Q}(n)[\tau]$.

$$\det(R) \;:=\; (-1)^d \frac{r_0}{r_d} \;\in \mathbb{Q}(n)$$

$$=\; c\, n^v(1 + c_1 n^{-1} + c_2 n^{-2} + \cdots) \in \mathbb{Q}((n^{-1}))$$

Dominant part:

$$\mathrm{dom}(\det(R)) \;=\; c\, n^v(1 + c_1 n^{-1})$$

c_1 = number of apparent singularities of R (with multiplicity)
$+$ a term that comes from $\{$true singularities of $R\}$
$\subseteq \{$true singularities of $L\}$

$\{$gen. exp. of $L\} \supseteq \{$gen. exp. of $R\}$ $\rightsquigarrow$ $\mathrm{dom}(\det(R))$ $\rightsquigarrow$ c_1

$\rightsquigarrow$ bound(number apparent singularities) $\rightsquigarrow$ degree bound

Irreducibility proof

Except for special cases, method 2 does not prove that the factors it finds are irreducible.

Suppose L is not factored by method 2.

Idea:

- Gen. exponents $\rightsquigarrow$ finite set of potential $\mathrm{dom}(\det(R))$
- p-curvature $\rightsquigarrow$ conditions mod p for $\mathrm{dom}(\det(R))$
- Incompatible? $\rightsquigarrow$ L is irreducible.

Overview:

1. Factor with method 2.
2. Apply the above idea to the factors.
3. Any factor not proved irreducible: fall back on method 1.