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ALGORITHMS FOR SIMPLIFYING DIFFERENTIAL EQUATIONS

By

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LIST OF SYMBOLS

The following list of symbols are used throughout this thesis.

R	a commutative ring with identity
k	a differential field
∂	a derivation, usually $\partial = \frac{d}{dx}$
δ	a derivation that equal to $x\,\partial$
C	the field of constants of the differential field k
$\mathrm{ord}(L)$	the order of the operator L
D, $k[\partial]$	the ring of differential operators over k
$V(L)$	the solution space of L
$G(K/k)$	the differential Galois group of the field extension K of k
$LCLM$	least common left multiple of differential operators
$GCRD$	greatest common right divisor of differential operators
$\xrightarrow{(i)}$	change of variables transformation
$\xrightarrow{(ii)}$	gauge transformation
$\xrightarrow{(iii)}$	exponential product transformation
t_c	the local parameter of c
L_f	the operator obtained from L under a change of variables $x \mapsto f$
$L_{\frac{1}{x}}$	the operator obtained from L under a change of variables $(x, \partial) \mapsto (\frac{1}{x}, x^{-2}\partial)$
v	a valuation map
S_e	the substitution map
$[t_c^0](f)$	the coefficient of t_c^0 in f
$\#S_{ing}(L)$	the number of the singularities of L
$E_{(iii)}(L)$	the equivalence class of L under (iii)-transformation
$NF(L)$	the normal form of L
$Tr(\alpha)$	the trace of α over C
μ	the value of the formal solutions of differential operators
$L[B]$	gauge equivalent to L such $L \xrightarrow{(ii)}_B L[B]$

ABSTRACT

We present three algorithms to reduce homogeneous linear differential equations to their simplest form. Factoring a differential operator reduces a differential equation $L(y) = 0$ to equations of minimal order, but this is not the only simplification one can make. There are three order-preserving transformations that can change the degree of the coefficients. To fully simplify a differential equation, after the order is minimized, we want to find the smallest equation that can be reached under any order-preserving transformations. We design algorithms to find transformations that reduce L to its simplest form under all three transformations. The algorithms use relative invariants and integral bases for differential operators. We give an algorithm to find all relative invariants, and we generalize a prior integral basis algorithm to cover all cases.

CHAPTER 1

INTRODUCTION

Linear homogeneous differential equations with coefficients in $\mathbb{C}(x)$ are commonly used in several fields of science (e.g. mathematics, statistics, physics, etc). On computer algebra systems (Maple, Matlab, Mathematica, etc) there is no general solver that can solve every linear differential equation in closed form. However, many algorithms and solvers are established to find several types of closed form solutions. For example, the algorithms in [14, 20, 31] find Liouvillion solutions, and algorithms in [6, 7, 12, 16, 22, 35] find hypergeometric functions solutions of linear differential equations. Finding solutions of differential equations in terms of solutions of lower order differential equations is a method of solving differential equations [25, 29, 30]. Also, solving irreducible differential equations by simplifying them to smaller equations of the same order (coefficients with minimal degrees) is another method of solving differential equations. For example, in [33, 11] there are algorithms that find Bessel type solutions of differential equations by simplifying equations to Bessel equations, and in [15, 16] there is an algorithm that finds ${}_2F_1$ type solutions of differential equations by simplifying them to ${}_2F_1$ equations. We try to simplify any homogeneous linear differential equation to a smaller equation of the same order, and our work is built on solvers from [7, 15, 16].

The corresponding differential operator of a differential equation $y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=0$ is $\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in \mathbb{C}(x)[\partial]$ where $\partial = \frac{d}{dx}$. Let $L \in \mathbb{C}(x)[\partial]$ be a differential operator of order n. In this thesis, we present three simplifiers that simplify L to an operator $\tilde{L}$ of the same order such that $\tilde{L}$ is close to as simple as possible (minimize the number of singular points, Definition 3.1). If y is a solution of $\tilde{L}$, then the transformations that we use to simplify differential equations send y to a solution of L of this form

$$\exp(\int r\,dx) \cdot (r_0 y(f) + ... + r_{n-1} y^{(n-1)}(f))$$

where f, r, r_0, $\ldots$, $r_{n-1} \in \mathbb{C}(x)$ and f is not constant. Our simplifiers are combination and extension of methods from [3, 4, 7, 15]. In [7], E.S Cheb-Terrab and A.D Roche presented a method that finds solutions of this form

$$\exp(\int r\,dx) \cdot y(f) \tag{1.1}$$

where $f, r \in \mathbb{C}(x)$, and $y = p\mathrm{F}q$, a hypergeometric function, for $0 \leq p \leq 3, q = 2$. The key tools for this method are called *relative invariants*, Definition 6.9. We generalized this method by writing an *algorithm that computes all relative invariants* of any *weight* for differential operators of order 3 and higher, Chapter 6.

In [15, 16, 8, 18], a method is introduced and developed to find solutions of this form

$$r_0\, y + r_1\, y' + r_2 y'' + \cdots + r_{n-1} y^{(n-1)} \text{ for } r_0, r_1, \ldots, r_{n-1} \in \mathbb{C}(x) \tag{1.2}$$

for regular singular differential operators of order n. The key tool for this method is called *integral bases*, see Section 5.1. To compute *integral basis* for differential operators, one must look at the *generalized exponents*, Section 3.3, of differential operators at their singularities. The *generalized exponents* of a regular singular differential operator are constants at all of its singularities; however, if a differential operator has *irregular singular* point, Definition 3.2, then at least one of the *generalized exponents* is not constant which makes technical difficulty to compute integral basis at that point. We treated this problem by improving the algorithms in [15, 16], so we can compute *integral bases* at regular and irregular singular points of differential operators.

1.1 Why It Is Likely that Equations Can Be Simplified

Example 1.1. *Let*

$$f(x) = x^8 - x^6 - 3x^5 - x^4 + 4x^3 + 4x^2 - 2x - 1 \tag{1.3}$$

which is irreducible in $\mathbb{Q}[x]$ *and defines a number field*

$$K = \mathbb{Q}[x]/(f(x)) \tag{1.4}$$

Now pick some arbitrary element of K*. Take for example*

$$G = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + (f(x)) \in K \tag{1.5}$$

Let $F(x) \in \mathbb{Q}[x]$ *be the minimal polynomial of* G*, so* $F(G) = 0 \in K$*. Then*

$$\begin{aligned} F(x) = &x^8 - 96x^7 + 6428x^6 - 135416x^5 + 7826194x^4 + 77174188x^3 \\ &- 109916113x^2 - 1650304228x - 305452239 \end{aligned} \tag{1.6}$$

and thus $K \cong \mathbb{Q}[x]/(F(x))$*.*

This raises a question, given a polynomial like $F(x)$ in (1.6), how to find (if it exists) a smaller polynomial like $f(x)$ in (1.3) that defines the same number field?

We argue that in practice, this type of simplification $F(x) \rightsquigarrow f(x)$ is often possible. The reason is that if one encounters an algebraic number during a computation, for example an element of the field K in (1.4), then there is no reason why the first encountered element of K happens to be an element with the smallest minimal polynomial.

Thus, if K is a number field, and α is the first encountered element of K that generates K, to reduce expression sizes it may be best to search for another generator of K with a smaller minimal polynomial.

There are two ways we can interpret the expression G in (1.5). It is an element of K and thus has a minimal polynomial $F(x)$. But we can also view G as a map $x \mapsto G(x)$ that sends solutions of $f(x)$ to solutions of $F(x)$. In other words: G allows us to write solutions of $F(x)$ in terms of solutions of $f(x)$. This means that G reduces this problem: solve $F(x)$ to this problem solve $f(x)$.

Thus we can view G as a *transformation that simplifies $F(x)$ to $f(x)$*. Starting from a polynomial such as $F(x)$, the algorithm POLRED [9] can find $f(x)$ and G. This means that POLRED simplifies $F(x)$ to $f(x)$.

The goal in this thesis is simplifying differential equations. It turns out there is a differential analogue to the POLRED method. The differential analogue is called a *gauge transformation* which will be defined in more details in Section 2.4. A gauge transformation is not the only transformation that is applicable to differential equations. There are three types of transformations that preserve the order, see Section 2.4 for details, and in this thesis we develop simplifiers for all three types.

The motivation for these simplifiers is as follows: Finding closed form solutions (if they exist) is generally easier when a differential equation has been reduced to its simplest form. But even if there are no closed form solutions, it is more efficient to solve a differential equation with smaller coefficients (the coefficients in this thesis will not be integers like in Example 1.1, instead, they will be polynomials. Smaller coefficients then means smaller degrees).

CHAPTER 2

PRELIMINARIES

2.1 Differential Fields and Differential Operators

Definition 2.1. *Let R be a ring. A derivation on R is a map $\partial : R \to R$ satisfying the following properties*

1. *For all a,$b \in R$, $\partial(a + b) = \partial(a) + \partial(b)$.*
2. *For all $a, b \in R$, $\partial(a \cdot b) = \partial(a) \cdot b + a \cdot \partial(b)$.*

A ring R is called a differential ring if there is a derivation on it. A field k with a derivation ∂ satisfying the above properties is called a differential field.

A field extension K of k is called differential extension of k if it has a derivation $\partial : K \to K$ whose restriction to k is the derivation on k.

In this thesis, k is a field of characteristic 0.

Definition 2.2. *An element $b \in k$ is a constant if $\partial(b) = 0$. The constant field of k with respect to the derivation ∂ is the field that contains all constant elements, $C = \{a \in k|\partial(a) = 0\}$.*

Example 2.3. 1. *The derivation $\partial \equiv 0$ is the trivial derivation for any ring R.*

2. *The field of rational functions with complex coefficients, $\mathbb{C}(x)$, with the*

$$\partial(f(x)) = \frac{df}{dx}$$

is a differential field.

Definition 2.4. *Let (k, ∂) be a differential field. A differential operator L of order n, ord$(L) = n$, is defined by*

$$L = \sum_{i=0}^{n} a_i \partial^i, a_i \in k. \tag{2.1}$$

The ring of all differential operators is denoted by $k[\partial]$ where addition of operators is given by adding the coefficients of the same terms and multiplication is given by composition.

The ring $D := k[\partial]$ has all but one of the properties of a Euclidean domain. It is a noncommutative ring because $\partial f = f\partial + \partial(f)$, for $f \in k$, and $\partial(f)$ is not always 0.

Each differential operator corresponds to a homogenous differential equation $L(y) = 0$. The operator (2.1) gives the equation

$$\sum_{i=0}^{n} a_i y^{(i)} = 0. \tag{2.2}$$

Definition 2.5. *Let k be differential field with a derivation ∂. A universal extension V of k is a minimal differential ring containing k, its field of constants is algebraic closure of C (denoted by $\overline{C}$), and for any operator $L \in k[\partial]$ there are exactly ord(L) linearly independent solutions over $\overline{C}$ in V [34].*

Definition 2.6. *Let L be a differential operator in D. We say y is a solution of L if $L(y) = 0$. The vector space*

$$V(L) = ker(L : V \to V)$$

is called the solution space of the operator L.

The solution space $V(L)$ is a vector space over $\overline{C}$ with dimension ord(L).

2.2 Type of Solutions

Definition 2.7. *Let L be a differential operator of order n in D and $L(y) = 0$.*

1. *If $y \in k$, it is called a rational solution.*
2. *If $y' \in k$, it is called an integral solution.*
3. *If y is algebraic over k, it is called an algebraic solution.*
4. *If $y'/y \in k$, y is called exponential solution.*

Definition 2.8. *Let k be a differential field where its field of constants C is algebraically closed. If K is a differential field extension of k such that the constants field of K is C, and if there exists a tower of fields*

$$k = K_0 \subset K_1 \subset K_2 \subset ... \subset K_n = K$$

such that $K_{i+1} = K_i(y_{i+1})$ for $i = 0, ..., n-1$ and y_{i+1} is either

1. *algebraic over K_i, or*

2. *exponential over K_i, (i.e. $y'_{i+1}/y_{i+1} \in K_i$),or*

3. *an integral over K_i, (i.e. $y'_{i+1} \in K_i$),*

then K is called Liouvillian extension, and y_n is called Liouvillian function.

By using group theory, F.Singer in [24] showed that if $L \in k[\partial]$ has a Liouvillian solution then it also has a solution y with y'/y algebraic over k. J.Kovacic in [20] wrote an algorithm for finding all Liouvillian solutions for second order linear homogeneous differential equations over $\mathbb{C}(x)$. For arbitrary order, Michael F.Singer presented an algorithm to compute a basis for the solution space of a given differential operator over $\mathbb{C}(x)$ if it has Liouvillian solutions [24].

Definition 2.9. [25] *Let K be a differential field extension of k. If there exists a tower of fields*

$$k = K_0 \subset K_1 \subset K_2 \subset ... \subset K_n = K$$

such that

1. *$K_i = K_{i-1}(u_i)$ where u_i is algebraic over K_{i-1},*

2. *$K_i = K_{i-1}(u_i)$ where u'_i is in K_{i-1},*

3. *$K_i = K_{i-1}(u_i)$ where u'_i/u_i is in K_{i-1},or*

4. *$K_i = K_{i-1}(u_i, v_i)$ where u_i and v_i are linearly independent solutions of second order linear homogeneous differential equation over K_{i-1},*

then K is called an eulerian extension of k.

Definition 2.10. *Let $L \in k[\partial]$ of order n. A differential field extension K is a Picard Vessiot extension of k for L if K is the smallest differential field extension of k such that the fields of constants of K and k are the same, K is generated by the solutions of L and their derivations over k, and L has n linearly independent solutions in K.*

Definition 2.11. *Let k be a differential field with derivation ∂. A differential isomorphism between two differential field extensions of k is a field isomorphism which fixes k and commutes with ∂. If K is a differential field extension of k, the differential Galois group $G(K/k)$is the set of all differential automorphisms of K. If K is a Picard Vessiot extension of k for $L \in k[\partial]$, $G(K/k)$ is called differential Galois group of L.*

Theorem 2.12 (Theorem 2.8 [13]). *Let K_1 and K_2 be Picard Vessiot extensions of a differential field k for $L \in k[\partial]$. Then K_1 and K_2 are differential isomorphic over k. Therefore, Picard Vessiot extension is unique up to isomorphism.*

Proposition 2.13 (Proposition 2.3 [25]). *Let M be a Picard Vessiot extension of k, where its field of constants C_k is algebraically closed. If M lies in an eulerian extension of k, the differential Galois group of M over k is eulerian group.*

2.3 Operations with Differential Operators

In this section, $D = k[\partial]$ is the ring of differential operators over k. There are many similar properties that D and polynomial ring in one variable share in.

Lemma 2.14 (Lemma 2.1 [27]). *For $L_1, L_2 \in D$ such that $L_1 \neq 0$. There are unique operators R, Q such that*

$$L_2 = QL_1 + R, \text{ where } ord(R) < ord(L_2).$$

In the above lemma, R is called the right remainder of L_2 by L_1, and Q is called the right quotient of L_2 by L_1. If $R = 0$, L_1 is called a right divisor of L_2.

Corollary 2.15 (Corollary 2.3 [27]). *For every left ideal $I \subset D$, there exists $L \in D$ such that $I := DL$, so left ideals are always principal. Also, for a right ideal J there exists an operator $L \in D$ such that $J := LD$.*

LCLM(L_1, L_2) is defined as the unique monic generator of $DL_1 \cap DL_2$, and it is the operator with minimal order such that its solution solutions space is

$$V(\mathrm{LCLM}(L_1, L_2)) = V(L_1) + V(L_2).$$

Also, GCRD(L_1, L_2) is defined as the unique monic generator of $DL_1 + DL_2$, and it is the minimal operator such that

$$V(\mathrm{GCRD}(L_1, L_2)) = V(L_1) \cap V(L_2).$$

For more details see [27].

Definition 2.16. *Let L be a differential operator of order n in $k[\partial]$. We call that L is reducible, if there exists L_1 and L_2 in $k[\partial]$ of order greater than 0 such that $L = L_1 \cdot L_2$. If it is not reducible, it is called irreducible.*

A linear homogeneous differential equation is reducible if it is associated to a reducible differential operator. The factorization of L is not always unique, but orders of the factors are unique up to reordering.

Definition 2.17. *Let L_1 and L_2 be differential operators, then the symmetric product of L_1 and L_2 is an operator L whose solution space*

$$V(L) = SPAN\{y_1y_2|y_1 \in V(L_1), y_2 \in V(L_2)\},$$

and it is denoted by $L_1 Ⓢ L_2$. If the order of L_1, L_2 and L are n_1, n_2 and n respectively in order,then:

$$n_1 + n_2 - 1 \leq n \leq n_1n_2$$

Definition 2.18. *Let L be a differential operator of order n, and $\{y_1, ..., y_n\}$ be a solutions basis of L. The m'th symmetric power of L is the minimal differential operator where its solutions space is spanned by the monomials of degree m in $y_1, ..., y_n$. It is denoted by $L^{Ⓢm}$.*

Lemma 2.19 (Lemma 3.2 [25])**.** *Let $L \in k[\partial]$ be a differential operator of order n. Then $L^{Ⓢm}$ has order at most $\begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix}$, and if L has order 2 and $\{y_1, y_2\}$ is the basis of its solution space, then*

$$\{y_1^m, y_1^{m-1}y_2, ..., y_1y_2^{m-1}, y_2^m\}$$

is the basis of the solution space of $L^{Ⓢm}$ which has order $m + 1$.

Example 2.20. *Let L be a differential operator of order 2, $L = \partial^2 + a_1\partial + a_0$, and y be a solution of L then*

$$y'' + a_1y' + a_0y = 0. \tag{2.3}$$

from the definition of symmetric power y^2 is a solution of $L^{Ⓢ2}$. Let

$$f = y^2$$

then

$$f' = 2yy',$$

$$\begin{aligned} f'' &= 2(y')^2 + 2yy'' \\ &= 2(y')^2 - 2a_1yy' - 2a_0y^2, \ \ by\ (2.3). \end{aligned}$$

Repeatedly using (2.3) shows

$$f''' = -3a_1f'' - (2a_1^2 + a_1' + 4a_0)f' - (4a_1a_0 + 2a_0')f.$$

Therefore,

$$L^{\circledS 2} = \partial^3 + 3a_1\partial^2 + (2a_1^2 + a_1' + 4a_0)\partial + (4a_1a_0 + 2a_0') \tag{2.4}$$

Definition 2.21. *Let* $L = c_n\partial^n + c_{n-1}\partial^{n-1} + c_{n-2}\partial^{n-2} + ... + c_0 \in k[\partial]$. *Define* normal form *of* L *by*

$$NF(L) = \frac{1}{c_n}\sum_{i=0}^{n} c_i \cdot \big(\partial - \frac{c_{n-1}}{nc_n}\big)^i = \partial^n + 0\partial^{n-1} + ... \tag{2.5}$$

This is $L \circledS (\partial - \frac{c_{n-1}}{nc_n})$ *made monic, so this erase the* ∂^{n-1} *coefficient and makes the* ∂^n *coefficient equal to* 1.

2.4 Transformations

We consider transformations between differential operators $L_1, L_2 \in k[\partial]$ that can be described in terms of solutions. Example 2.20 showed a transformation that increased the order (from 2 to 3). However, There are three types of transformations that preserve k and the order of the operators.

Definition 2.22.

1. *Let* $k = \mathbb{C}(x)$ *and* $f \in k - \mathbb{C}$. *Substituting* $(x, \partial) \to (f, \frac{1}{f'}\partial)$ *defines an endomorphism of* $k[\partial]$. *Suppose it sends* L_1 *to* L_2. *Then* $y(f) \in V(L_2)$ *for each* $y \in V(L_1)$. *This transformation is called* ***change of variables transformation*** *and denoted by* $L_1 \xrightarrow{(i)} L_2$. *The function* f *is called pullback function.*

2. *Let* $L, G \in D$ *with* $GCRD(L, G) = 1$. *Write* $LCLM(L, G) = Q\,G$. *Denote* $L[G]$ *as that operator* Q, *so* $L \to L[G]$ *such that* $y \mapsto G(y)$ *is a map from* $V(L)$ *to* $V(L[G])$. *The Operators* L *and* $L[G]$ *are called* gauge equivalent, *and* G *is called a* ***gauge transformation,*** *denoted by* $\xrightarrow{(ii)}$. *Moreover,* $L[G]$ *can be computed from* L *by computing the right division of* $LCLM(L, G)$ *by* G.

3. *Let* $r \in k$. *The map* $\sum a_i\partial^i \mapsto \sum a_i(\partial - r)^i$ *is a automorphism of* $k[\partial]$, *and called* ***Exponential Product transformation***. *This transformation is denoted by* $L_1 \xrightarrow{(iii)} L_2$ *and* $exp(\int r\, dx) \cdot y \in V(L_2)$ *for each* $y \in V(L_1)$.

Notation 2.23.

- $L_1 \xrightarrow{(iii)} L_2$ *if and only if* $NF(L_1) = NF(L_2)$, *so this transformation is easy to detect (in general, detection* combination *of transformations is much harder than detecting a single transformation). If* $L_1 \xrightarrow{(iii)} L_2$, *then* $L_2 = L_1 ⓢ (\partial - r)$ *such that* $r = \frac{1}{n}(a_{n-1} - b_{n-1})$ *where* n *is the order of* L_1 *and* L_2, *and* a_{n-1} *and* b_{n-1} *are the coefficients of* ∂^{n-1} *of* L_1 *and* L_2 *respectively.*

- L_f *denotes the operator obtained from* L *under a change of variables transformation* $(x, \partial) \mapsto (f, \frac{1}{f'}\partial)$ *where* $f \in k - C$.

Lemma 2.24 ([10]). *Let* L_1 *and* $L_2 \in D$ *be operators of order* n. *Then* $L_1 \xrightarrow{(i)} L_2$, $L_1 \xrightarrow{(ii)} L_2$, *and* $L_1 \xrightarrow{(iii)} L_2$ *are reflexive and transitive. Except change of variables transformation, the others are also symmetric.*

Change of variables transformation is not symmetric because if $L_1 \xrightarrow{(i)} L_f$, where $f \in k$, the inverse of f could be an algebraic function. For example, let $f = x^2$, and to compute L_1 from L_f, need to apply $x \mapsto \sqrt{x} \notin k$ on L_f.

Lemma 2.25 (Lemma 2 [11]). *Let* L_1, L_2 , L *and* $M \in D$ *be differential operators of order* n. *If* $L_1 \xrightarrow{(ii)}_{r_0,r_1,\ldots,r_{n-1}} L \xrightarrow{(iii)}_{r} L_2$, *there exists* $\bar{L} \in D$ *such that* $L_1 \xrightarrow{(iii)}_{\bar{r}} \bar{L} \xrightarrow{(ii)}_{t_0,t_1,\ldots,t_{n-1}} L_2$ *where* $r, r_0, \ldots, \bar{r}, t_0, \ldots, t_{n-1} \in k$. *Also, if* $L_1 \xrightarrow{(iii)}_{r} M \xrightarrow{(ii)}_{r_0,r_1,\ldots,r_{n-1}} L_2$, *there exists an operator* $\bar{M} \in D$ *such that* $L_1 \xrightarrow{(ii)}_{t_0,t_1,\ldots,t_{n-1}} \bar{M} \xrightarrow{(iii)}_{\bar{r}} L_2$.

From the above lemmas, we get this definition:

Definition 2.26. *Differential operators* L_1 *and* $L_2 \in D$ *of order* n *are called* projective equivalent *and denoted by* $L_1 \sim_p L_2$ *if there exists a bijection between* $V(L_1)$ *and* $V(L_2)$ *of the form*

$$y \rightarrow e^{\int r}(r_0 y + \ldots + r_{n-1} y^{(n-1)})$$

with $r, r_0, \ldots, r_{n-1} \in k$.

Theorem 2.27 (Lemma 3 [11]). *Let* L_1 *and* $L_2 \in D$ *be differential operators of order* n. *If* $L_1 \longrightarrow L_2$ *by change of variables, gauge, and exponential product transformations in any order, there exists* M_1 *and* $M_2 \in D$ *such that*

$$L_1 \xrightarrow{(i)}_{f} M_1 \xrightarrow{(iii)}_{r} M_2 \xrightarrow{(ii)}_{r_0,\ldots,r_{n-1}} L_2,$$

where $f, r, r_0, \ldots, r_{n-1} \in k$.

Proof. Since applying gauge transformation after exponential product transformation is equivalent to applying exponential transformation after gauge transformation, we only need to show that:

1. If $L_1 \xrightarrow{(iii)}_r M_1 \xrightarrow{(i)}_f M_2$, there exists $\bar{M}_1 \in D$ such that $L_1 \xrightarrow{(i)}_f \bar{M}_1 \xrightarrow{(iii)}_{r(f)} M_2$.

2. If $L_1 \xrightarrow{(ii)}_{r_0,r_1,\ldots,r_{n-1}} M_3 \xrightarrow{(i)}_f M_2$, there exist $\bar{M}_3 \in D$ such that
$$L_1 \xrightarrow{(i)}_f \bar{M}_3 \xrightarrow{(ii)}_{r_0(f),r_1(f),\ldots,r_{n-1}(f)} M_2.$$

For the first case, let $f, r \in k$, and $y \in V(L_1)$. Then,

$$v = e^{\int r(f)} y(f) \in V(M_2).$$

Also, this solution can be obtained by applying change of variables, $x \mapsto f$, on L_1 and then applying exponential product, $\partial \mapsto \partial - r(f)$ on the result.

In the second case, let $f, r_0, r_1, \ldots, r_{n-1} \in k$. then;

$$v = r_0(f)y(f) + r_1(f)y'(f) + \cdots + r_{n-1}(f)y^{(n-1)}(f) \in V(M_2).$$

Since

$$(y(f))' = y'(f)f' \Rightarrow y'(f) = \frac{(y(f))'}{f'},$$
$$(y(f))'' = y''(f)(f')^2 + y'(f)f'' \Rightarrow y''(f) = \frac{(y(f))''}{(f')^2} - \frac{(y(f))'f''}{(f')^3}$$

and continue until find $y^{(n-1)}(f)$. Substituting the value of $y(f), y'(f), y''(f), \ldots$ and $y^{(n-1)}(f)$ in v gives that

$$v = R_0 y(f) + R_1(y(f))' + R_2(y(f))'' + \cdots + R_{n-1}(y(f))^{(n-1)},$$

where $R_0, \ldots, R_{n-1}$ are functions in $r_i, f, f', \ldots$. Therefore, the solution space $V(M_2)$ also can be obtained from the solution space of L_1 first by applying change of variables then apply gauge transformation. □

Since change of variables is not always symmetric, the converse of (1) and (2) in proof of the last theorem is not true because it may needs algebraic functions.

Definition 2.28. *Let $y(x)$ be a function which satisfies*

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0, \textit{where } a_n, \ldots, a_0 \in \mathbb{C}(x) \textit{ and } a_n \neq 0.$$

Then y is called D-finite of order n.

Let y_1 and y_2 be D-finite functions of order n_1 and n_2 respectively, and let L_1 and L_2 be the differential equations that y_1 and y_2 satisfy respectively. The transformations in the beginning of this section $(i), (ii)$ and (iii) do not increase the order of y_1 and y_2. However, if y is D-finite function and the parameter of the transformations, $f, r, r_0, r_1, ..., r_{n-1}$, are algebraic functions, then order of the D-finite functions that obtained from y by applying any of the above transformations could be higher than the order of y. Also, the least common left multiple of L_1 and L_2, which sending y_1, y_2 to $y_1 + y_2$, can increase the order but $\leq n_1 + n_2$. If $y_1 \in V(L_1)$ and $y_2 \in V(L_2)$, then $y_1 y_2$ satisfies the symmetric product operator of L_1 and L_2, and its order can increase but $\leq n_1 n_2$. Also, y_i^2 satisfies the symmetric square operator L_i, and its order is $\leq \frac{n_i(n_i+1)}{2}$.

Example 2.29. *Let $y_1 = e^{x^3}$ and $y_2 = \cos(x^3)$. Then y_1 is a D-finite function of order 1 because $y_1' - 3x^2 y_1 = 0$, and y_2 is a D-finite function of order 2 because $y_2'' + 9x^4 y_2 = 0$*

1. *Let $f(x) = \frac{1}{\sqrt{x-1}}$. Then $y_1(f(x))$ is a D-finite function and satisfies this differential equation*

$$y'' + \frac{5}{2}\frac{1}{x-1}y' - \frac{9}{4(x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1)}y = 0.$$

   It has order higher than the order of y_1.

2. *The differential equation that $y_1 + y_2$ satisfies is*

$$y''' - \frac{9x^6 + 15x^3 + 1}{x(3x^3+1)}y'' + 9x^4 y' - \frac{27x^3(3x^6 + x^3 - 1)}{3x^3 + 1}y = 0,$$

   where its order is 3.

3. *$y''' + 36x^4 y' + 72x^3$ is the equation that y_2^2 satisfies, and $y'' - 6x^2 y' + 18x^4 - 6x$ is the equation that $y_1 y_2$ satisfies.*

2.5 Differential Module

let k be a differential field with the field of constants C and a derivation ∂. A linear differential equations over k can be presented as a differential module form.

Definition 2.30. *A differential module (M, ∂) is a finite dimensional k-vector space M equipped with an additive map $\partial : M \to M$ satisfying*

$$\partial(fm) = f'm + f\partial(m), \forall f \in k \text{ and } m \in M,$$

which equivalent to saying that M is a $k[\partial]$-module.

Example 2.31. *1. The trivial differential module over k is k itself.*

2. Let $k = \mathbb{C}(x)$ and $\partial = \frac{d}{dx}$. Then, $M = k \cdot e^x$ is a differential module of dimension one. Let $m = e^x \in M$, then $\partial(m) = m$ and

$$\partial(fm) = (f' + f)m, \text{ for any } f \in k.$$

Definition 2.32. *A differential module homomorphism $\phi : (M_1, \partial_1) \to (M_2, \partial_2)$ is a k-linear map such that $\phi(\partial_1(m)) = \partial_2(\phi(m))$ for all $m \in M_1$. If ϕ is a bijective differential module homomorphism, then the differential modules (M_1, ∂_1) and (M_2, ∂_2) are isomorphic.*

An element $e \in M$ is called a cyclic vector if M is generated over k by the elements $e, \partial e, \partial^2 e, \ldots$ The existence of a cyclic vector for a differential modual M is proven in [17].

Theorem 2.33. *[27][Cyclic Vector Theorem] Let k be a differential field with proper constants field $C \neq k$. Every differential module M is isomorphic to a module of the form $k[\partial]/k[\partial]L$ for some $L \in k[\partial]$.*

This theorem associates a scalar differential operator L over k to a differential module M. Here L is not unique because the differential module M has many cyclic vectors.

Lemma 2.34. [*19*, *27*] *Let k be a differential field with proper field of constants C, and let M be a differential module with basis $\{e_1, ..., e_n\}$. Let $c_1, ..., c_n \in k$ be linearly independent over the constants field of k. Then there exists integers $0 \leq b_{i,j} \leq n$, where $0 \leq i, j \leq n$, such that $m = \sum_{i=0}^{n} a_i e_i$ is a cyclic vector for M, where $a_i = \sum_{j=0}^{n} b_{i,j} c_j$.*

Two differential operators L_1 and L_2 are gauge equivalent, as in Section 2.4, if and only if $D/DL_1 \cong D/DL_2$ where $D = k[\partial]$. Let e_1 and e_2 be cyclic vectors for M, and let $\mathrm{Ann}(e_1) = k[\partial]L_1$ and $\mathrm{Ann}(e_2) = k[\partial]L_2$ be their annihilators. Then

$$M \cong k[\partial]/k[\partial]L_1 \text{ and } M \cong k[\partial]/k[\partial]L_2.$$

Therefore,

$$k[\partial]/k[\partial]L_1 \cong k[\partial]/k[\partial]L_2. \tag{2.6}$$

In this case, L_1 and L_2 are called gauge equivalent.

Example 2.35. *1. Let $k = \mathbb{C}(x)$. If $M = \mathbb{C}(x)$, then M is isomorphic to $D/D \cdot \partial$. Therefore, 1 is a basis of this module.*

2. *Let* $k[\partial] = \mathbb{C}(x)[\frac{d}{dx}]$, *and* $M = \mathbb{C}(x) \cdot e^x$. *There is a surjective ring homomorphism from* $k[\partial]$ *to* M *such as*

$$\Phi : k[\partial] \to M,$$

$$\Phi(L) = L(e^x), \text{ for all } L \in k[\partial].$$

The kernel of Φ *is* $L \in k[\partial]$ *such that* $L(e^x) = 0$. *Since* $(\frac{d}{dx} - 1)(e^x) = 0$, *the kernel of* Φ *is* $k[\partial] \cdot (\partial - 1)$ *and* $M \cong k[\partial]/k[\partial] \cdot (\partial - 1)$

3. *If we take* xe^x *instead of* e^x *in the previous example, then* $M \cong k[\partial]/k[\partial] \cdot (\partial - \frac{x+1}{x})$. *Referring to* (2.6), e^x *and* $x \cdot e^x$ *are cyclic vector of* M.

CHAPTER 3

GENERALIZED EXPONENTS

3.1 $C(x)[\delta]$ Ring

Let $C \subset \mathbb{C}$ be a differential field with a derivation $\partial = \frac{d}{dx}$. Define the operator $\delta = x\partial \in C(x)[\partial]$. Since for any $a, b \in C(x)$

$$\delta(a+b) = x\partial(a+b) = x(\partial a + \partial b) = x\partial a + x\partial b = \delta a + \delta b,$$

and

$$\delta(ab) = x\partial(ab) = x(b\partial a + a\partial b) = xb\partial a + xa\partial b = b\delta a + a\delta b,$$

δ is a derivation on $C(x)$, and $C(x)[\delta] = C(x)[\partial]$. Therefore, any differential operator in $C(x)[\partial]$ can be written in terms of δ. For example,

$$\delta = x\partial \Rightarrow \partial = \frac{1}{x}\delta,$$

$$\delta^2 = x^2\partial^2 + x\partial \Rightarrow \partial^2 = \frac{1}{x^2}(\delta^2 - \delta),$$

and so on.

3.2 Singularities

Definition 3.1. *Let $L = a_n\partial^n + a_{n-1}\partial^{n-1} + ... + a_0 \in C(x)[\partial]$, where $C \subseteq \mathbb{C}$ and $\partial = \frac{d}{dx}$. A point $c \in \overline{C}$, the algebraic closure of C, is singular if it is a pole of one of $\frac{a_0}{a_n}, \ldots, \frac{a_{n-1}}{a_n}$. Infinity is a singular point of L if 0 is a singular point of $L_{\frac{1}{x}}$. If c is not a singular point, it is called a regular point.*

Definition 3.2. *Let L as above definition. A singular point c is called a regular singularity if it is not a pole of $(x-c)^i a_{n-i}/a_n$, for $0 \leq i \leq n$, at $x = c$. It is called irregular if it is not regular. The local parameter of c is denoted by t_c, and it is*

$$t_c = \begin{cases} x - c & \text{if } c \neq \infty, \\ \frac{1}{x} & \text{if } c = \infty. \end{cases}$$

Definition 3.3.
- *A singular point $x = c$ is called an apparent singularity of $L \in C(x)[\partial]$ if the solutions of L are analytic at c.*
- *Removable singularities are singular points that are regular points of some projective equivalent , Definition* 2.26*, operator. If a singular point is not removable, it is called a true singular point.*

L is called a regular singular differential operator if all of its singularities are regular, and the corresponding homogeneous differential equation of L is called *Fuchsian.*

Example 3.4. *Suppose that*

$$L_1 = \partial^2 - \frac{36x^5 - 297x^4 + 830x^3 - 841x^2 + 280x - 56}{3(x-2)x(12x^2 - 35x + 7)(x-1)}\partial - \frac{108x^5 - 735x^4 + 1717x^3 - 1703x^2 + 947x - 238}{6(x-2)x(x-1)^2(12x^2 - 35x + 7)}.$$

The operator L_1 defines a D-module $M = D/DL_1$. However, just like in Example 1.1*, there is no reason to assume that L_1 is simplest equation that defines this module M. Indeed M is isomorphic to D/DL_2 where*

$$L_2 = \partial^2 - \frac{3x+2}{3x}\partial - \frac{1}{2x}.$$

The operators L_1 and L_2 are gauge equivalent such that $L_1 = L_2[G]$, defined in Section 2.4. *where*

$$G = \frac{3x^2 - 4x - 2}{3(x-1)x(x-2)}\partial - \frac{2x^2 - 5x + 1}{2x(x-1)(x-2)}.$$

Similar to Example 1.1*, this means that one can use G to write solutions of L_1 in terms of a simpler equation L_2, so finding G and L_2 reduces solving L_1 to solving L_2.*

Notice that $x = 2$, $x = 1$ and all of the roots of $12x^2 - 35x + 7$ are removable singularities of L_1 because they are not to singularities of L_2, see Definition 3.3.

$$y_1 = x^{\frac{5}{3}}\ KummerM(\frac{13}{6}, \frac{8}{3}, x), \ \ and \ \ y_2 = x^{\frac{5}{3}}\ KummerU(\frac{13}{6}, \frac{8}{3}, x)$$

are solutions of L_2. By the definition of gauge transformations $G(y_1)$ and $G(y_2)$ are solutions of L_1, and they are regular at the roots of $12x^2 - 35x + 7$, which means that the roots of $12x^2 - 35x + 7$ are apparent singularities of L_1.

Example 3.5. *Take $L_1 = \partial^2 + \frac{2}{x}\partial + \frac{1}{x^4}$ and $L_2 = \partial^2 + \frac{1}{2x}\partial$, and take*

$$L = LCLM(L_1, L_2) = \partial^4 + \frac{3(85x^2 + 28)}{2x(15x^2 + 4)}\partial^3 + \frac{225x^4 + 123x^2 + 4}{x^4(15x^2 + 4)}\partial^2 + \frac{90x^4 + 75x^2 + 4}{x^5(15x^2 + 4)}\partial.$$

Observe that L has singular points (e.g. $x = c$ where c is a root of $15x^2 + 4$) that are not singular points of L_1 or L_2. Since $x = c$ is regular for L_1, it means that solutions of L_1 are analytic at $x = c$. The same is true for L_2, but then this must also be true for L because solutions of LCLM(L_1, L_2) are sums of solutions of L_1 and L_2.

That raises a question: if all solutions of L are analytic at $x = c$, then why is $x = c$ a singular point? To answer that, we compute the exponents of L at $x = c$ and find: $0, 1, 2, 4$. In other words, there are solutions with a root at $x = c$ of order $0, 1, 2,$ and 4, but no solution has a root of order 3. *At regular points, the exponents would have been* $0, 1, 2, 3$.

In practice, whenever this situation happens for an operator L of order n, the most common scenario is that the exponents are $0, 1, \ldots, n-2, n$ (i.e. $n-1$ is missing). If we now randomly add a solution to L by replacing L with LCLM$(L, \partial - r)$ for some randomly chosen r, then it is likely that the exponents at $x = c$ become $0, 1, \ldots, n-1$ (i.e. $n-1$ is no longer missing) and those are the exponents of an operator that is not singular at $x = c$. This idea leads to the following trick, which experiments show is likely to remove the most common type of apparent singularities (those with a single "gap" in the exponents, like in the example where the exponents were $0, 1, 2, 4$) : If c is a singular point of an operator L but not a singular point of $\bar{L} =$ LCLM$(L, \partial - r)$, for some randomly chosen r, then c must be an apparent singularity of L. For many applications we can discard such singularities. If L is monic, d is its denominator, r is randomly chosen (we take $r \in \mathbb{Z}$), and $\bar{d}$ is the denominator of $\bar{L}$, then any solution of d but not a solution of gcd$(d, \bar{d})$ is an apparent singularity. For example, the denominator of L_1, as in Example 3.4, *and the denominator the LCLM$(L_1, \partial + 1)$ are*

$$d_1 = 6x(x-1)^2(x-2)(12x^2 - 35x + 7),$$

and

$$d_2 = 6x(x-1)(x-2)(144x^6 - 1272x^5 + 4231x^4 - 6421x^3 + 4575x^2 - 1703x + 350),$$

respectively. The roots of $12x^2 - 35x + 7$ are not solutions of gcd$(d_1,\, d_2) = x\,(x-1)\,(x-2)$, and we have already proven that they are apparent singularities of L_1 in Example 3.4.

Our goal is to simplify a given differential operator to an operator with fewer apparent singularities. Therefore, in the following sections we will introduce *generalized exponents,* and the relations between the number of apparent singularities and the generalized exponents.

3.3 Generalized Exponents

Let $r \in \mathbb{C}(x) \subseteq \mathbb{C}((x))$. Write

$$r = \frac{e_1}{x} + h,$$

where $e_1 \in \mathbb{C}[x^{-1}]$, and $h \in \mathbb{C}[[x]]$. Then the solution of $\partial - r$ is

$$\exp(\int r\, dx) = \exp(\int \frac{e_1}{x}\, dx)\big(\exp(\int h\, dx)\big),$$

where the latter factor has neither root nor pole at $x = 0$ (we say it has valuation 0, Definition 3.7). So the first factor $\exp(\int \frac{e_1}{x}\, dx)$ completely determines the asymptotic behavior of $\partial - r$ at $x = 0$. We call this e_1 the *generalized exponent* of $\partial - r$ at $x = 0$. If e_1 is a constant, then $\exp(\int \frac{e_1}{x}\, dx)$ is x^{e_1} in which case e_1 is called an *exponent* of $\partial - r$. The purpose of this section is to extend this to higher order, to define $e_1, \ldots, e_n \in \mathbb{C}[x^{\frac{-1}{r}}]$ of an n'th order differential operator L that characterise the asymptotic behavior of a basis of solutions $y_1, \ldots, y_n$ of L at $x = 0$.

Remark 3.6. *L is regular singular at $x = 0$ if and if all generalized exponents of L at $x = 0$ are constants.*

Definition 3.7. *Let $K = \mathbb{C}((x^{\frac{1}{r}}))$. Define a valuation on K by*

$$v : K \to \frac{1}{r}\mathbb{Z} \cup \{\infty\}$$

such that for any $P \in K$, $v(P)$ is the smallest exponent of x in P with nonzero coefficient, and $v(0) = \infty$. Note that $x^{-v(P)}P \in \mathbb{C}[[x^{\frac{1}{r}}]]$.

Definition 3.8. *A map $v : K \to \mathbb{Q} \cup \mathbb{Z}$ is valuation on K if $\forall a, b \in K$,*

1. *$v(ab) = v(a) + v(b)$,*
2. *$v(a + b) = min(v(a), v(b))$, and*
3. *$v(a) = \infty \iff a = 0$.*

Let

$$E = \bigcup_{r \geq 1} \mathbb{C}[x^{\frac{-1}{r}}]. \tag{3.1}$$

We define $\mathrm{Exp}(e)$, where $e \in E$, to be a nonzero solution of $\delta - e$ on a universal extension V, Definition 2.5. We also denote that by $\exp(\int \frac{e}{x} dx)$. If $e \in \mathbb{C}[x^{\frac{-1}{r}}]$ with r is minimal, then r is called the ramification index of e, which is called unramified if $r = 1$ and ramified otherwise.

Definition 3.9. *Define the substitution map* $S_e : \overline{\mathbb{C}((x))}[\delta] \mapsto \overline{\mathbb{C}((x))}[\delta]$ *by*

$$S_e(\delta) = \delta + e, \tag{3.2}$$

where $e \in \overline{\mathbb{C}((x))}$ *and* $\delta = x\,\partial$, *Section* 3.1. *This map is a ring automorphism, and if* $L \in \mathbb{C}(x)[\delta]$, *then the solution space* $V(L) = Exp(e) \cdot V(S_e(L))$. *This map is almost the same as the exponential product transformation, except that we use* δ *instead of* ∂.

Lemma 3.10. *Let* $L \in \mathbb{C}((x^{\frac{1}{r}}))[\partial]$ *such that* $L \neq 0$. *There exists a constant* c *and a finite set* S *such that for every* $p \in \mathbb{C}((x^{\frac{1}{r}}))$ *with* $v(p) \notin S$ *we have* $v(L(p)) = c + v(p)$.

Proof. Let N be a variable, and

$$L = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + ... + a_0(x).$$

By applying L to x^N,

$$L(x^N) = x^N \cdot (a_n(x)(N(N-1)...(N-n+1)x^{-n} + ... + a_0(x)),$$

so $L(x^N)/x^N$ is an element of $\mathbb{C}[N]((x^{\frac{1}{r}}))$. Let c be the valuation of $L(x^N)/x^N$, and let $P \in \mathbb{C}[N]$ be the coefficient of x^c. Then,

$$L(x^N) = x^N \cdot (Px^c + ...), \tag{3.3}$$

where the dots have valuation $> c$. Therefore, we see that for $N \in \mathbb{Q}$,

$$v(L(x^N)) = N + c,$$

for all $N \in \mathbb{Q}$ except for a finite set; the roots of P. ☐

Definition 3.11. *We denote* $v(L)$ *as that constant* c *in the above Lemma, and* P *in* (3.3) *is called the indicial equation of* L.

Remark 3.12. *If* $x = 0$ *is a regular singular point of* L, *then the Newton polynomial, Definition* 3.1 *[32], of* L *at slope* 0 *is equal to the indicial equation of* L *at* 0, *and its roots are the generalized exponents of* L *at* $x = 0$.

Recall that $\delta = x\,\partial$. For $p \in K = \mathbb{C}((x^{\frac{1}{r}}))$,

$$v(\delta^j(p)) = v(p),$$

for all p except for a finite set of $v(p)$'s, so $v(\delta^j) = 0$ and $v(\partial^j) = -j$.

Corollary 3.13. *$v(L_1 L_2) = v(L_1) + v(L_2)$ because if $L = L_1 L_2$, then*

$$v(L(y)) = v(y) + v(L) = v(y) + v(L_1 L_2)$$

except for $v(y)$ in a finite set, see Lemma 3.10, *and*

$$v(L(y)) = v(L_1(L_2(y))) = v(L_2(y)) + v(L_1) = v(y) + v(L_2) + v(L_1).$$

The valuation of $L = \sum_{i,j} a_{i,j} x^i \delta^j \in \mathbb{C}((x))[\delta]$ is the smallest i such $a_{i,j} \neq 0$ for some j. The indicial equation of L at $x = 0$ is

$$N_0(L) = \sum_j a_{v(L),j} T^j \tag{3.4}$$

where T is used as a variable.

If $L = L_1 L_2$, then

$$N_0(L) = S_{T+v(L_2)} N_0(L_1) N_0(L_2), \tag{3.5}$$

where $v(L_2)$ is the valuation of L_2. This polynomial $S_{T+v(L2)} N_0(L_1)$ means that T is replaced by $T + v(L_2)$. The formula in (3.5) holds for $L_2 \in \overline{\mathbb{C}((x))}[\delta]$, [28, 32].

Definition 3.14 (Definition 3.6 [32])**.** *An element $e \in E$ is called a generalized exponent of $L \in \overline{\mathbb{C}((x))}[\delta]$ if and only if* 0 *is a root of $N_0(S_e(L))$. The multiplicity of the root* 0 *in $N_0(S_e(L))$ is denoted by $u_e(L)$.*

Definition 3.15. *Let $L \in \overline{\mathbb{C}((x))}[\delta]$. Then $e_1, e_2 ..., e_n \in E$ is a list of generalized exponents of L if for all $e \in E$ the number $u_e(L)$ is equal to the number of e_i which equal to e.*

Corollary 3.16. *Let $L \in \overline{\mathbb{C}((x))}[\delta]$.*

$$\sum_{e \in E} u_e(L) = \mathit{order}(L). \tag{3.6}$$

Lemma 3.17. *If $e_1,\, e_2, \ldots,\, e_n$ are the generalized exponents of $L_1 \in \overline{\mathbb{C}((x))}[\delta]$ and $f_1,\, f_2,\, \ldots,\, f_n$ are the generalized exponents of $L_2 \in \overline{\mathbb{C}((x))}[\delta]$. Then*

$$e_1,\, e_2, \ldots,\, e_n,\, f_1 - v(S_{f_1}(L_1)),\, f_2 - v(S_{f_2}(L_1)), \ldots,\, f_n - v(S_{f_n}(L_1))$$

are the generalized exponents of $L_2\, L_1 \in \overline{\mathbb{C}((x))}[\delta]$.

Proof. Let $L = L_2L_1$. Then, for every e_i of the generalized exponents of L_1,

$$N_0(S_{e_i}(L)) = N_0(S_{v(S_{e_i}(L_1))}(S_{e_i}(L_2)))N_0(S_{e_i}(L_1)).$$

Since 0 is a root of $N_0(S_{e_i}(L_1))$, 0 is a root of $N_0(S_{e_i}(L))$. Therefore, e_i is a generalized exponents of L.

Now, for every f_i of the generalized exponents of L_2,

$$\begin{aligned} N_0(S_{f_i}(L)) &= N_0(S_{f_i}(L_2)S_{f_i}(L_1)) \\ &= N_0(S_{v(S_{f_i}(L_1))}(S_{f_i}(L_2)))N_0(S_{f_i}(L_1)) \\ &= S_{T=T+v(S_{f_i}(L_1))}(N_0(S_{f_i}(L_2)))N_0(S_{f_i}(L_1)) \end{aligned}$$

Since 0 is a root of $N_0(S_{f_i}(L_2))$, it is clear that $-v(S_{f_i}(L_1))$ is a root of

$$S_{T=T+v(S_{f_i}(L_1))}(N_0(S_{f_i}(L_2))).$$

Also, 0 is a root of $N_0(S_{f_i+v(S_{f_i}(L_1))}(L))$ because $v(S_{f_i}(L_1)) = v(S_{f_i+v(S_{f_i}(L_1))}(L_1))$. Therefore, $f_i + v(S_{f_i}(L_1))$ is a generalized exponents of L. □

Up to permutation, a list of generalized exponents of every $L \in \overline{\mathbb{C}((x))}[\delta]$ is uniquely defined.

Lemma 3.18 (Lemma 3.1 [28])**.** *Let $L \in \overline{\mathbb{C}((x))}[\delta]$ of order n. Let $e_1, ..., e_n$ be the list of generalized exponents of L, and $e \in E$. Then $e_1 - e, ..., e_n - e$ are the list of generalized exponents for $S_e(L)$.*

Theorem 3.19. *Suppose $L \in C(x)[\partial]$, where $C \subseteq \mathbb{C}$, is a differential operator of order n and r_i is the ramification index of a generalized exponents e_i at $x = 0$, such that $e_i \in \overline{C}[[x^{\frac{-1}{r_i}}]]$. Then there is a basis $\{y_1, ..., y_n\}$ of the solutions space $V(L)$ at the point $x = 0$ such that*

$$y_i(x) = \exp(\int \frac{e_i}{x}\, dx)S_i, \tag{3.7}$$

where $S_i = \sum_{j=0}^{\infty} x^{j/r_i} \cdot P_j$ such that these P_j are in $\bar{C}[\log(x)]$ and $P_0 \neq 0$ for each $i = 1, ..., n$.

For the proof and more details reader can look at [28]. If the list of generalized exponents of L at $x = 0$ is contained in $\overline{C}$, then $x = 0$ is a regular, apparent singularity, or regular singularity, and the solution is in this form

$$y_i(x) = x^{e_i}S_i, \text{ where } e_i \in \overline{C}. \tag{3.8}$$

Notation 3.20. *To compute the generalized exponents of $L \in C(x)[\partial]$ at any point $c \in \overline{C}$, we compute the generalized exponents of $L_{x+c} \in \overline{C}(x)[\partial]$ at 0, followed by replacing x by t_c, Defintion 3.2. Also, the generalized exponents of L at ∞ is equal to the generalized exponents of $L_{\frac{1}{x}} \in C(x)[\partial]$ at 0.*

3.4 The Effects of Transformations on Generalized Exponents

In this section, we will explain how the generalized exponents of a given differential operator are affected by applying exponential and gauge transformations on the operator.

Let $L \in C(x)[\partial]$ of order n such that $e_1, \ldots, e_n$ are the generalized exponents of L at $x = 0$. Let $\tilde{L} = L \circledS (\partial - r)$ for some $r \in C(x)$. If $r = \sum_{i=j}^{\infty} a_i\, x^i$, then $e = \sum_{i=j}^{-1} a_i\, x^{i+1}$ is the generalized exponent of $\partial - r$, as mentioned in the beginning of Section 3.3. Definition 2.17 and Definition 3.9 show that $L \circledS (\partial - r)$ and $S_{-r}(L)$ are the same, except for possibly a left-factor in $C(x)$ (but this does not affect the solution space), so the generalized exponents of $\tilde{L}$ at $x = 0$ are

$$e_1 + e,\ e_2 + e, \ldots,\ e_n + e,$$

by Lemma 3.18. Generally, if $e_1, \ldots, e_n$ are the generalized exponent of L at $c \in \overline{C} \cup \{\infty\}$, then $e_1 + e, \ldots, e_n + e$ are the generalized exponent of $L \circledS (\partial - r)$ at c where $r = \sum_{i=j}^{\infty} a_i\, t_c^i$ and

$$e = \begin{cases} \sum_{i=j}^{-1} a_i\, t_c^{i+1} & \text{if } c \neq \infty, \\ \sum_{i=j}^{1} (-a_i)\, t_\infty^{i-1} & \text{if } c = \infty \end{cases}$$

Lemma 3.21 (Lemma 2.10 [23])**.** *Let $L_1, L_2 \in C(x)[\partial]$ be two linear differential operators such that $L_1 \xrightarrow{(ii)} L_2$. Let e be a generalized exponent of L_1 at $c \in \overline{C} \cup \{\infty\}$ with the ramification index $r_e \in \mathbb{Z}^+$. Then, the operator L_2 has*

$$\bar{e} \in e + \frac{1}{r_e}\mathbb{Z}$$

as a generalized exponent at $x = c$.

3.5 Relation between the Second Highest Order Term and the Generalized Exponents

In this section, we study the relation between the constant part of the generalized exponents of $L \in C(x)[\partial]$, $C \subseteq \mathbb{C}$, at point $c \in \overline{C} \cup \{\infty\}$, the valuation of the exponents difference, and the coefficients of the second high order term.

Let $e_1, e_2, \ldots,$ and $e_n \in E$ be generalized exponents of $L \in C(x)[\partial]$ at point 0. Define the valuation of a generalized exponents e_i of L at 0 by

$$v'(e_i) = \min(v(e_i), 0) = v(\delta - e_i). \tag{3.9}$$

To compute the valuation of exponents of L at $c \in \overline{C} \cup \{\infty\}$, we compute the valuation of exponents of $\tilde{L}$ at 0, where $\tilde{L}$ is obtained from L by moving c to 0.

Notation 3.22. *Let $f = \sum_{i=-j}^{\infty} a_j x^j$. Then define*

$$[x^0](f) = a_0.$$

For any $c \in \bar{C} \cup \{\infty\}$, $[t_c^0](f)$ is the coefficient of t_c^0 when f is written as a power series in t_c which is the local parameter of c, Definition 3.2.

Lemma 3.23. *Let $L = \partial^n + a_{n-1}\partial^{n-1} + ... + a_0 \in C(x)[\partial]$, and let $e_1, e_2, ..., e_{n-1}$ and e_n be the generalized exponents of L at point 0 such that $e_i \in \overline{C}[[x^{\frac{-1}{r_i}}]]$. Let $\delta + b_{n-1}\delta^{n-1} + ... + b_0 \in C(x)[\delta]$ be $x^n L$ rewritten in terms of δ. Then, by writing the coefficients of $\tilde{L}$ as Laurent series at point 0,*

$$[x^0](b_{n-1}) = -\big([x^0](e_1 + e_2 + ... + e_n) + \sum_{i=1}^{n-1}\sum_{i<j}^{n} v'(e_i - e_j)\big).$$

For the proof of this Lemma and more details reader can look at [28] [Lemma 3.4].

Now we want to show the relation between the generalized exponents and the residue of a differential operator $L \in C(x)[\partial]$.

Definition 3.24. *Let $L = \partial^n + a_{n-1}\partial^{n-1} + ... + a_0 \in C(x)[\partial]$. Define the residue of L at $c \in \overline{C}$ as the residue of a_{n-1} at c, denoted by $res_c(L)$, and the residue of L at ∞ as the residue of $-x^2 a_{n-1}$ at ∞. The local residue of $x^n L = \delta + b_{n-1}\delta^{n-1} + ... + b_0 \in C((x))[\delta]$ is the constant part of $b_{n-1} \in C((x))$, and it is denoted by $lres(x^n L)$.*

Lemma 3.25. *Let $L = \partial^n + a_{n-1}\partial^{n-1} + ... + a_0 \in C(x)[\partial]$ and $e_1, e_2, \ldots, e_n$ be the list of generalized exponents of L at 0. Then*

$$\begin{aligned} res_0(L) &= lres(x^n L) + 1 + 2 + ... + (n-1) \\ &= -\big([x^0](e_1 + e_2 + ... + e_n) + \sum_{i=1}^{n-1}\sum_{i<j}^{n} v'(e_i - e_j)\big) + 1 + 2 + ... + (n-1). \end{aligned} \tag{3.10}$$

At $x = \infty$, $res_\infty(L) = lres(x^n L_{\frac{1}{x}}) - (1 + 2 + ... + (n-1))$.

Proof. Writing L in terms of δ leads to

$$\begin{aligned} L &= \partial^n + a_{n-1}\partial^{n-1} + ... + a_0 = (\frac{1}{x}\delta)^n + a_{n-1}(\frac{1}{x}\delta)^{n-1} + ... + a_0 \\ &= \frac{1}{x^n}\delta^n + \big(\frac{a_{n-1}}{x^{n-1}} - \frac{(1 + 2 + ... + (n-1))}{x^n}\big)\delta^{n-1} + ... + b_0 \\ &= \delta^n + \big(a_{n-1}x - (1 + 2 + ... + (n-1))\big)\delta^{n-1} + ... + b_0. \end{aligned}$$

Since $\operatorname{res}_0(a_{n-1}) = [x^0](a_{n-1}x)$, and the constant of the coefficient of δ^{n-1} is constant $(a_{n-1}x) - (1+2+...+(n-1))$,

$$\begin{aligned}\operatorname{res}_0(L) &= [x^0](a_{n-1}x)\\ &= [x^0](a_{n-1}x) - (1+2+...+(n-1)) + (1+2+...+(n-1))\\ &= \operatorname{lres}(x^n L) + (1+2+...+(n-1)).\end{aligned}$$

The proof of second quality is followed from the definition of local residue, Definition 3.24, and Lemma 3.23.

To compute the residue of L at ∞, we compute the residue of $L_{\frac{1}{x}}$ at 0. Since

$$\begin{aligned}L_{\frac{1}{x}} &= (-1)^n x^{2n}\partial^n + \big((-1)^{n-1}x^{2(n-1)}a_{n-1_{|_{x=1/x}}}\\ &\quad + (-1)^n \cdot 2\cdot(1+2+...+(n-1))x^{2n-1}\big)\partial^{n-1} + ...\\ &= \partial^n + \big((-1)a_{n-1_{|_{x=1/x}}}x^{-2} + 2(1+2+...+(n-1))x^{-1}\big)\partial^{n-1} + ...,\end{aligned}$$

then

$$\begin{aligned}\operatorname{res}_0(L_{\frac{1}{x}}) &= \operatorname{res}_0((-1)a_{n-1_{|_{x=1/x}}}x^{-2} + 2(1+2+...+(n-1))x^{-1})\\ &= \operatorname{res}_0((-1)a_{n-1_{|_{x=1/x}}}x^{-2}) + 2(1+2+...+(n-1))\\ &= \operatorname{res}_\infty(-1x^2 a_{n-1}) + 2(1+2+...+(n-1))\\ &= \operatorname{res}_\infty(L) + 2(1+2+...+(n-1)).\end{aligned}$$

Since $\operatorname{res}_0(L_{\frac{1}{x}}) = \operatorname{lres}(x^n L_{\frac{1}{x}}) + (1+2+...+(n-1))$,

$$\operatorname{res}_\infty(L) = \operatorname{lres}(x^n L_{\frac{1}{x}}) - (1+2+...+(n-1)).$$

□

The $\operatorname{res}_c(L)$ and $\operatorname{lres}_c(x^n L)$ at any point $c \in \overline{C}$ are equal to the residue of L_{x+c}, see Notation 2.23, at 0 and the local residue of $x^n L_{x+c}$.

Lemma 3.26 (Fuchs's Relation)**.** *Let $L \in C(x)[\partial]$ of order n. Then*

$$\sum_{c\in\overline{C}\bigcup\infty} res_c(L) = 0, \tag{3.11}$$

that leads to the following relation

$$\sum_{c\in\overline{C}\cup\{\infty\}}\Big(-[t_c^0](e_{c,1}+e_{c,2}+...+e_{c,n}) - \sum_{i=1}^{n-1}\sum_{i<j}^{n} v'(e_{c,i}-e_{c,j}) + \frac{n(n-1)}{2}\Big) = n(n-1), \tag{3.12}$$

where $e_{c,1}, \ldots, e_{c,n}$ are the generalized exponents at c. The relation in (3.12) *is called Fuchs' relation.*

Proof. To prove (3.11), we directly use Cauchy's Residue Theorem and the residue at infinity.

If c is not singular point, then the generalized exponents of L at c are $0, \ldots, n-1$. Substituting them in (3.10) shows that $\mathrm{res}_c(L) = 0$. Now, use Lemma 3.25 to compute the residues of L at the singularities of L. Then, substitute the residues in (3.11) and add $n(n-1)$ in both sides to proves (3.12). $\square$

3.6 A Bound for Number of Apparent Singularities

Definition 3.27. *Let $L \in C(x)[\partial]$ of order n. Suppose all solutions of L at $x = c$ are analytic. Then the generalized exponents $e_1, \ldots, e_n$ of L at $x = c$ are distinct non-negative integers, so define the multiplicity of c by*

$$N_c = \sum_{i=1}^{n} e_i - (0 + 1 + \cdots + (n-1)).$$

At a regular point, the generalized exponents are $0, 1, \ldots, n-1$, making $N_c = 0$. If the point $x = c$ is an apparent singularity, Definition 3.3, then $N_c > 0$ and N_c is called the multiplicity of c.

Comment 3.28. *If we compute LCLM of randomly chosen $L_1, L_2 \in \mathbb{C}(x)[\partial]$, then the apparent singularities usually have multiplicity 1.*

We denote the set of the apparent and the true singularities of L, Definition 3.3, by $ApSing$ and $TrSing$, respectively.

Definition 3.29. *Define the number of the apparent singularities of L with their multiplicities by*

$$N_{app} = \sum_{c \in ApSing} N_c, \tag{3.13}$$

and without multiplicities by $n_{app} = \#ApSing \leq N_{app}$.

By using Fuchs's relation, (3.12), and (3.13), we can show that

$$N_{app} = -n(n-1) + \sum_{c \in TrSing \cup \{\infty\}} \Big(\sum_{i=1}^{n} -[t_c^0](e_{c,i}) - \sum_{i<j}^{n} v'(e_{c,i} - e_{c,j}) + \frac{n(n-1)}{2} \Big). \tag{3.14}$$

Comment 3.30. *Since the left hand side $N_{app} \in \mathbb{N} \subseteq \mathbb{R}$, the right hand side must stay the same if we replace each $[t_c^0](e_i)$ by its real part $Re([t_c^0](e_i))$, which is what we will do from here on.*

Definition 3.31. *We say that $L \in C(x)[\partial]$ is integral if $Re([t_c^0](e_i)) \geq 0$ for any generalized exponent e_i at any point $c \in \overline{C}$. If L is integral, then the degree of L at ∞ is the smallest $d \in \mathbb{Z}$ such that*

$$min\{Re([t_\infty^0](e_i)) \,|\, e_1, \ldots, e_i \text{ are generalized exponents at } \infty\} \geq -d.$$

Lemma 3.18 and Lemma 3.21 show that if L is not integral, we can use gauge and exponential product transformations to find integral operators which is projectively equivalent to L. Therefore, our goal is to find an integral operator with as few as possible singularities, which we will use in Lemma 3.33 below.

For $a \in \mathbb{R}$, let $\{a\} \in [0, 1)$ denoted its fractional part (so that $a - \{a\} \in \mathbb{Z}$).

Definition 3.32. *Let $e \in \overline{C}[[t_c^{\frac{-1}{r_i}}]]$ be a generalized exponent of L at c with the ramification index r_c. Define the fractional part of e by*

$$Frac(e) = \frac{1}{r_c}\{r_c\, Re([t_c^0](e))\}.$$

Lemma 3.33. *Let $L \in C(x)[\partial]$ be integral of order n with degree d at ∞. Then the bound of its apparent singularities is*

$$\begin{aligned} n_{app} \leq & N_{app} \leq -n(n-1) + d\,n \\ & + \sum_{c \in TrSing \cup \{\infty\}} \Big(\sum_{i=1}^{n} -Frac(e_{c,i}) - \sum_{i<j}^{n} v_c'(e_{c,i} - e_{c,j}) + \frac{n(n-1)}{2}\Big). \end{aligned} \tag{3.15}$$

Notation 3.34. *Notice that in* (3.15) *the only term on the right hand side that can change under gauge transformations is the term $d\,n$.*

Proof (*Lemma* 3.33). Since L is integral, we have $\mathrm{Re}([t_c^0](e_{c,i})) \geq 0$ at any finite singular point c of L, Definition 3.31. Therefore,

$$\mathrm{Frac}(\, e_{c,i}) \leq \mathrm{Re}([t_c^0](e_{c,i})), \tag{3.16}$$

Definition 3.31 shows that for any $e_{\infty,i}$, a generalized exponent of L at ∞, $d \geq -\mathrm{Re}([t_\infty^0](e_{\infty,i}))$, so

$$\mathrm{Frac}(e_{\infty,i}) - d \leq \mathrm{Re}([t_\infty^0](e_{\infty,i}).$$

Then

$$\sum_{i=1}^{n} -\mathrm{Re}([t_\infty^0](e_{\infty,i})) \leq \sum_{i=1}^{n} -\mathrm{Frac}(e_{\infty,i}) + d\,n. \tag{3.17}$$

Now, use (3.14), (3.16), and (3.17) to prove (3.15), so

$$
\begin{aligned}
&n_{app} \leq N_{app}\\
&\quad = -n(n-1) + \sum_{c \in TrSing \cup \{\infty\}} \Big(\sum_{i=1}^{n} -\mathrm{Re}([t_c^0](e_{c,i})) - \sum_{i<j}^{n} v_c'(e_{c,i} - e_{c,j}) + \frac{n(n-1)}{2} \Big)\\
&\quad \leq -n(n-1) + d\,n + \sum_{i=1}^{n} -\mathrm{Frac}(e_{\infty,i}) - \sum_{i<j}^{n} v_\infty'(e_{\infty,i} - e_{\infty,j}) + \frac{n(n-1)}{2}\\
&\quad + \sum_{c \in TrSing} \Big(\sum_{i=1}^{n} -\mathrm{Frac}(e_{c,i}) - \sum_{i<j}^{n} v_c'(e_{c,i} - e_{c,j}) + \frac{n(n-1)}{2} \Big)\\
&\quad = -n(n-1) + d\,n\\
&\quad + \sum_{c \in TrSing \cup \{\infty\}} \Big(\sum_{i=1}^{n} -\mathrm{Frac}(e_{c,i}) - \sum_{i<j}^{n} v_c'(e_{c,i} - e_{c,j}) + \frac{n(n-1)}{2} \Big).
\end{aligned}
$$

□

To simplify a differential operator, we need to find a projectively equivalent differential operator that has the smallest bound of apparent singularities. That means: minimize d, see Notation 3.34. In the following sections, we will present methods that remove most of the removable singularities, Definition 3.3.

CHAPTER 4

OVERVIEW OF THE MAIN SIMPLIFICATION ALGORITHM

In this chapter, k is a differential field with a derivation ∂. We start this chapter by this definition:

Definition 4.1. *Let $L_1, L_2 \in k[\partial]$ of order n. Notations as in Definition 2.22*

1. *L_2 immediately descends to L_1 if $L_1 \xrightarrow{(i)} L_2$ for some $f \in k$ and deg$(f) > 1$. This means $QL_2 \in \mathbb{C}(f)[\partial_f] \subsetneq \mathbb{C}(x)[\partial]$ for some $Q \in \mathbb{C}(x) - \{0\}$. It also means that the $\mathbb{C}(x)[\partial]$-module*

$$\mathbb{C}(x)[\partial]/\mathbb{C}(x)[\partial]\, L_1$$

can be viewed as coming from a $\mathbb{C}(f)[\partial_f]$-module

$$\mathbb{C}(f)[\partial_f]/\mathbb{C}(f)[\partial_f]\, L_2.$$

2. *L_2 directly descends to L_1 if $L_1 \xrightarrow{(i)+(iii)} L_2$ for some $f, r \in k$ and deg$(f) > 1$. This means that*

$$\mathbb{C}(x)[\partial]/\mathbb{C}(x)[\partial]\, L_2$$

comes from

$$\mathbb{C}(f)[\partial_f]/\mathbb{C}(f)[\partial_f]\, L_1$$

after tensoring with a 1-dimensional module.

3. *L_2 descends to L_1 if $L_1 \xrightarrow{(i)+(ii)+(iii)} L_2$ for some $f, r, r_0, ..., r_{n-1} \in k$ and deg$(f) > 1$.*

Example 4.2. *Let*

$$L_1 = \partial^2 - \frac{2x-1}{2x}\,\partial - \frac{1}{3x}.$$

Apply change of variables transformation $x \mapsto f$ on L_1 with $f = \frac{x^2+1}{x-5}$. Then

$$L_2 = \partial^2 - \frac{2x^6 - 41x^5 + 223x^4 - 94x^3 + 148x^2 + 243x - 513}{2(x-5)^2(x^2-10x-1)(x^2+1)}\partial - \frac{(x^2-10x-1)^2}{3(x-5)^3(x^2+1)}.$$

Now L_2 immediately descends to L_1. Observe that L_2 has more singularities than L_1.

Suppose that L_2 descends to L_1. Theorem 2.27 shows that no matter by which transformation we start with, the transformations can be organized in the following orders

$$L_1 \xrightarrow{(i)} M_1 \xrightarrow{(ii)} M_2 \xrightarrow{(iii)} L_2$$

and

$$L_1 \xrightarrow{(i)} \bar{M}_1 \xrightarrow{(iii)} \bar{M}_2 \xrightarrow{(ii)} L_2$$

for some M_1, M_2, $\bar{M}_1$, $\bar{M}_2 \in k[\partial]$. If L_2 descends to L_1, then the solutions of L_2 can be expressed in terms of solutions of L_1 as following: If $y \in V(L_1)$, then

$$y \mapsto e^{\int r}(r_0 y(f) + ... + r_{n-1}(y(f))^{(n-1)}) \in V(L_2),$$

where $f, r, r_0, ..., r_{n-1} \in k$ and $\deg(f) > 1$. From the definition above, we can say that M_1 immediately descends to L_1, and M_2 descends to L_1 where the parameter $r = 0$. The operator L_2 is projectively equivalent to M_1.

Given $L \in k[\partial]$ of order n, we want to find a simpler operator, L_{simp}, such that L descends, directly descends, immediately descends, or is projectively equivalent to L_{simp} and the number of singularities of L_{simp} is less than that in L. In other words, $\#S_{ing}(L_{\text{simp}}) \leq \#S_{ing}(L)$. Our method of simplifying differential operators goes through three phases:

- **Phase I:** Optimize L using only transformation (iii). The purpose is a quick size reduction that will make Phases II and III more efficient. For details see Section 4.1.
- **Phase II:** Optimize L using only transformation (ii). The key tool that is needed for this is called integral basis. For details see Chapter 5.
- **Phase III:** Optimize L using only transformations $(i)+(iii)$, in particular, decide if L allows a direct descent. The key tool needed for this are invariants, for details see Chapter 6.

4.1 Phase I

We will explain the idea of this section in a commutative example: Let $L \in \mathbb{C}(x)[y]$, and suppose L factors in $\mathbb{C}((x))[y]$ like

$$\begin{aligned} L &= (y - \frac{3}{x^2} + \frac{2}{x} + 5 + \cdots)\,(y - \frac{2}{x^2} + \frac{3}{x} + 3 + \cdots)\,(y - \frac{2}{x^2} + \frac{1}{x} + 1 + \cdots) \\ &= y^3 + \frac{-7 + 6x + 9x^2 + \cdots}{x^2} y^2 + \frac{16 - 28x - 29x^2 + \cdots}{x^4} y + \frac{-12 + 32x + 19x^2 + \cdots}{x^6}. \end{aligned}$$

Substituting $y \mapsto y + \frac{3}{x^2}$ makes the denominator of L smaller (namely x^5), but substituting $y \mapsto y + \frac{2}{x^2}$ is better because the denominator becomes x^4. Even better if we substitute $y \mapsto y + \frac{2}{x^2} - \frac{3}{x}$ or $y \mapsto y + \frac{2}{x^2} - \frac{1}{x}$ because the denominator becomes x^3. In this section we do somethings similar for differential operators.

Let $L \in \mathbb{C}(x)[\partial]$. Define the equivalence class of L under (iii)-transformation by

$$E_{(iii)}(L) = \{L \circledS\ (\partial - r) |\ r \in \mathbb{C}(x)\}.$$

We want to find an operator in $E_{(iii)}(L)$ such that it has the smallest denominator's degrees.

Let $L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in \mathbb{C}(x)[\partial]$, and let $\tilde{L} = L \circledS (\partial - r)$ for some $r \in \mathbb{C}(x)$. Then

$$\tilde{L} = \partial^n + (a_{n-1} - nr)\partial^{n-1} + \cdots ,$$

The normal form of L, $NF(L)$ which is obtained by taking $r = \frac{a_{n-1}}{n}$, is a unique representative in $E_{(iii)}(L)$. However, in most examples it is not the operator that has smallest denominator's degree.

Example 4.3. *Let* $L = \partial^3 - \frac{1}{x}\partial^2 + \partial + 1$. *Then*

$$NF(L) = \partial^3 - \frac{3x^2 - 4}{3x^2}\partial + \frac{27x^3 + 9x^2 + 16}{27x^3}.$$

Note that $x = 0$ *is a pole of order* 3 *in* $NF(L)$ *while its order is* 1 *in* L. *This shows that the degree of the denominator of* $NF(L)$ *can be up to* ord(L) *times larger than that of* L. *That can also make numerators larger.*

Computing the normal form of a given differential operator removes those singularities that can be removed by exponential product transformation (transformation (iii)), but the denominator need not be minimal. To optimize the denominator, we explain some of the properties of regular singularities.

Lemma 4.4. *Let* $L = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in \mathbb{C}(x)[\partial]$, *and let* $x = 0$ *be a regular singular point of* L. *Then,* $x = 0$ *is a pole of order at most* $n - 1$ *if and only if* 0 *is among the generalized exponents of* L.

Proof. Definition 3.2 shows that if $x = 0$ is a regular singular point of L, then it is a pole of a_i of order at most $n - i$, for $0 \le i \le n - 1$. Now, we need to prove that if 0 is a generalized exponent of L at $x = 0$, then the pole order of a_0 is less than n.

Let P be the indicial equation of L at $x = 0$, Definition 3.11. Then

$$P = N(N-1)(N-2)\cdots(N-n+1) + q_{n-1}N(N-1)(N-2)\cdots(N-n+2) + \cdots + q_0,$$

where N is a variable, and $q_{n-1}, \ldots, q_0$ are equal to $[x^0](x\, a_{n-1}), \ldots, [x^0](x^n\, a_0)$, respectively.

Assume that 0 is a generalized exponent of L at $x = 0$, so it is a root of P, Remark 3.12. Since $P(0) = 0$, then $q_0 = 0$, which is the coefficient of x^{-n} in a_0. Therefore, $x = 0$ is a pole of a_0 of order at most $n - 1$. $\square$

Remark 4.5. *let $x = c$ be a regular singular point of L, then a generalized exponent e of L at c is a constant, Remark 3.6. If we take $r = \frac{-e}{t_c}$, then $\tilde{e} = e + (-e) = 0$ is a generalized exponent of $\tilde{L} = L \circledS (\partial - r)$, see Section 3.4. Therefore, $x = c$ is a pole of $\tilde{L}$ of order at most ord$(\tilde{L}) - 1$, Lemma 4.4.*

Example 4.6. *Suppose we want a rational function r with residue $e = \sqrt{2} + \frac{1}{2}$ at the point $x = \sqrt{2}$. The simplest example of such a function is: $r = e/(x - \sqrt{2})$. Now suppose that this problem arose for some differential operator L in $C(x)[\partial]$ where the field of constants C was equal to $\mathbb{Q}$. For instance, $x = \sqrt{2}$ could have been a singularity of L, e could have been an exponent, Definition 3.14, of L, and we may want to apply an exponential product transformation, Definition 2.22, to L in order to send e to 0 since this may make L smaller, Lemma 4.4.*

The problem with using r is that L is defined over $\mathbb{Q}$, but as soon as we use r we end up with coefficients in $\mathbb{Q}(\sqrt{2})$. In situations like this we take the trace *$Tr(r)$. In this example, the trace is taken over the field extension $\mathbb{Q}(x, \sqrt{2})/\mathbb{Q}(x)$. and $Tr(r)$ equals r plus its conjugate:*

$$Tr(r) = r + \overline{r} = \frac{e}{x - \sqrt{2}} + \frac{-\sqrt{2} + \frac{1}{2}}{x + \sqrt{2}} = \frac{x + 4}{x^2 - 2} \quad .$$

Now $Tr(r)$ is in Q(x) and it still has residue e at the point $x = \sqrt{2}$. Also, it has residue $\overline{e} = -\sqrt{2} + \frac{1}{2}$ at the conjugate point $x = -\sqrt{2}$. This way, we can construct r with this particular residue at $x = \sqrt{2}$ without actually introducing $\sqrt{2}$ into the field of constants.

By the same idea in Lemma 4.4, we can find an exponential equivalent operator (if it exists) to L with minimum pole's order at irregular singular points. Let $x = c$ be an irregular singular point of L, and let $e = \sum_{i=j}^{-1} a_i t_c^{i+1}$ be one of the generalized exponents of L at c. Then the operator $L \circledS (\partial - \sum_{i=j}^{-1} r_i t_c^i)$, where $r_i = -a_i$ or $r_i = 0$, may have lower pole's order at c.

Notation 4.7. *Let p be irreducible in $k[x]$.* Root(p) *means: Let k' be k extended with a root of p (computationally $k' = k[x]/(p)$) and let* Root(p) *denote that root in k'.*

Example 4.8. *Let*

$$L = \partial^2 + \frac{(5x^6 + 33x^4 + 70x^2 + 30)}{x(x^4 + 6x^2 + 10)(x^2 + 3)}\,\partial + \frac{A}{(x^4 + 6x^2 + 10)(x^2 + 3)^6 x^2},$$

where

$$A = 3x^{16} + 60x^{14} + 505x^{12} + 2292x^{10} + 5851x^8 + 7404x^6 + 815x^4 - 8748x^2 - 7290.$$

In this example, we will explain how to find $\tilde{L} \in E_{(iii)}(L)$ such that the denominator's degree of $\tilde{L}$ is small as much as possible.

First, find the normal form of L, so

$$NF(L) = \partial^2 - \frac{B}{4(x^2 + 3)^6 (x^4 + 6x^2 + 10)^2 x^2}$$

$$\begin{aligned} B =& 3x^{20} + 90x^{18} + 1229x^{16} + 10008x^{14} + 53685x^{12} + 197874x^{10} + 506703x^8 \\ &+ 889020x^6 + 1022992x^4 + 699840x^2 + 218700. \end{aligned}$$

Note that, ∞ and the roots of x, $x^2 + 3$, and $x^4 + 6x^2 + 10$ are the singularities of $NF(L)$.

The generalized exponents of $NF(L)$ at $x = 0$ are $e_{0,1} = \frac{-1}{2}$ and $e_{0,2} = \frac{3}{2}$. Remark 4.5 shows that the pole order of $x = 0$ of $NF(L) \circledS (\partial - \frac{-e_{0,i}}{x})$ is less than that in $NF(L)$, for $i = 1, 2$.

Let β be a root of $x^4 + 6x^2 + 10$. It is a regular singular point of $NF(L)$, and the generalized exponents of $NF(L)$ at β are $e_{\beta,1} = \frac{-1}{4}$ and $e_{\beta,2} = \frac{5}{4}$, which also are the generalized exponents of the other roots of $x^4 + 6x^2 + 10$. Therefore, we take

$$r_{\beta,1} = Tr\Big(\frac{-e_{\beta,1}}{t_\beta}\Big) = Tr\Big(\frac{1}{4(x - \beta)}\Big) = \frac{x(x^2 + 3)}{x^4 + 6x^2 + 10}, \tag{4.1}$$

and

$$r_{\beta,2} = Tr\Big(\frac{-e_{\beta,2}}{t_\beta}\Big) = Tr\Big(\frac{-5}{4(x - \beta)}\Big) = \frac{-5x(x^2 + 3)}{x^4 + 6x^2 + 10}. \tag{4.2}$$

to find an exponential transformation for all of the roots of $x^4 + 6x^2 + 10$. The pole order of β of $NF(L) \circledS (\partial - r_{\beta,i})$ is 1, for $i = 1, 2$.

Now, let α be a root of $x^2 + 3$. It is irregular singular point of $NF(L)$, and the generalized exponents of $NF(L)$ at α are

$$e_{\alpha,1} = -\frac{1}{12t_\alpha^2} - \frac{\alpha}{72t_\alpha} + 2, \quad \textit{and} \quad e_{\alpha,2} = \frac{1}{12t_\alpha^2} + \frac{\alpha}{72t_\alpha} + 1.$$

In $NF(L)$, The pole order of α is 6, so we will try the same method above to check if the pole order can be minimized. For $e_{\alpha,1}$, we apply the method in Remark 4.5 on each terms of $e_{\alpha,1}$ to show how they reduce the pole order of α. Let

$$r_{\alpha,2} = -Tr\big(\frac{-1}{12t_\alpha^3}\big),\ r_{\alpha,1} = -Tr\big(\frac{-\alpha}{72t_\alpha^2}\big),\ \ and\ r_{\alpha,0} = -Tr\big(\frac{2}{t_\alpha}\big). \tag{4.3}$$

The pole order of α of $\tilde{L} = NF(L) \circledS (\partial - r_{\alpha,2})$ is 5, $\tilde{L} = NF(L) \circledS (\partial - r_{\alpha,2} - r_{\alpha,1})$ is 4, and

$$\tilde{L} = NF(L) \circledS (\partial - r_{\alpha,2} - r_{\alpha,1} - r_{\alpha,0}) \tag{4.4}$$

is 3. Note that, since $e_{\alpha,i}$ is unramified, we can take the trace of $\frac{-e_{\alpha,1}}{t_\alpha}$ instead of taking the trace of each term of $e_{\alpha,i}$ separately.

Therefore, one of the operators in $E_{(iii)}(L)$ that have lowest denominator's degree is

$$\begin{aligned}\tilde{L} &= NF(L) \circledS (\partial - (\frac{1}{2x} + r_{\beta,1} + r_{\alpha,2} + r_{\alpha,1} + r_{\alpha,0})\\ &= \partial^2 + \frac{5x^{10} + 57x^8 + 245x^6 + 441x^4 + 166x^2 - 270}{(x^2+3)^3(x^4+6x^2+10)x}\partial + \frac{x^2(3x^4+18x^2+28}{(x^4+6x^2+10)(x^2+3)^2},\end{aligned}$$

and its denominator's degree is 11.

Comment 4.9. *Let $e \in \mathbb{C}[t_c^{\frac{-1}{n}}]$ be a generalized exponent of L with ramification index $n \neq 1$. To reduce the pole order of c in L by using the generalized exponent e, we take the first unramified terms of e. For example, if*

$$e = \frac{2}{x^5} - \frac{3}{x^4} + \frac{1}{x^{\frac{5}{2}}} - \frac{5}{x^2} + \frac{2}{x} + 2,$$

we only take $\frac{2}{x^5} - \frac{3}{x^4}$ and compute that

$$L \circledS (\partial - \frac{-(\frac{2}{x^5} - \frac{3}{x^4})}{x}).$$

To compute an exponential equivalent operator to $L \in \mathbb{C}(x)[\partial]$ with the smallest denominator degrees and minimum coefficients' numerators degrees, we wrote the following algorithms:

Algorithm 4.10 (Phase I)**.**

- *Input: $L \in \mathbb{C}(x)[\partial]$ of order n.*
- *Output: An operator that is exponential equivalent to L, its denominators has the lowest degree with respect to x, and the maximum degree of the numerators of its coefficients is less than the maximum degree of the numerators of L's coefficients.*

1. *Compute $NF(L)$ because it removes all removable singularities that are removable by exponential product transformation.*

2. *Take $P = p_1 ... p_s$ to be the denominator of $NF(L)$. Note that the roots of each p_i are singular points of $NF(L)$.*

3. *For i from 1 to s do*

   (a) *Compute the generalized exponents, $e_1, ..., e_n$, of $NF(L)$ at $x = c =$ Root(p_i), Notation 4.7, that presents a root of p_i.*

   (b) *For each $e_l = \sum_{m=j}^{0} a_m t_c^{m/n_l}$, where n_l is the ramification index of e_l, see Theorem 3.19, $Q_l = \{R_m = \mathrm{Tr}(\frac{a_m t_c^{m/n_l}}{t_c}) \mid m/n_l \in \mathbb{Z},\ a_m \in k' = k(c),\ \ j \leq m \leq 0\}$, where Tr as in Example 4.6.*

   (c) *Put $Z_l = 0$. For m from j to 0*

      - *If the denominator degree, d_j, of $\tilde{L} = L \circledS (\partial + Z_l + R_m)$ is less than the denominator degree of L, then $Z_l = Z_l + R_m$, and $d_l = d_j$.*

   (d) *Take $S_i = \{[Z_l, d_l] |\ 1 \leq l \leq n\}$.*

   (e) *Make $d_i = min(\{d_1, ..., d_n\})$ and $H_i = \{Z_l \mid d_i = d_l,\ for\ 1 \leq l \leq n\}$.*

4. *Take M to be the set of all combinations of choosing only one elements from each H_i, so $M = \{T_1, ..., T_q\}$ such that $T_j = \{Z_{1,j}, ..., Z_{s,j}\}$.*

5. *For each $T_j \in M$,*

   - *Compute*
   $$L_j = NF(L) \circledS (\partial + Z_{1,j} + ... + Z_{s,j}).$$

   - *Apply* Equ_at_inf, *Algorithm 4.11. on L_j which gives $[\overline{L}_j, d_j]$ where d_j is the degree of the denominator of $\overline{L}_j$ with respect to x.*

6. *Take $W = \{\overline{L}_1, ..., \overline{L}_q\}$ such that the operators in W have the smallest denominator's degree.*

7. *Return $\overline{L}_i$ which has the shortest length in W.*

Algorithm 4.11 (*Equ_at_inf*)**.**

- *Input: A differential operator L of order n.*

- *Output: An exponential equivalent operator such that its denominator's degree is equal to that in L, the maximum degree of the numerators of its coefficients is less than or equal to the maximum degree of the numerators of L's coefficients.*

1. *Compute* $e_i, \ldots, e_n$, *the generalized exponents of* L *at* ∞.

2. *Note that for each* $s = 1, \ldots, n$, $e_s = \sum_{i=j}^{0} r_i t_\infty^{i/n_\infty}$, *where* n_∞ *is the ramification index at* ∞ *and* j *is an integer number* ≤ 0.

3. *For each* e_s *compute* $a_s = \{R_z = \mathrm{Tr}(\frac{r_z t_\infty^z}{t_\infty}) \,|\, z = i/n_\infty \in \mathbb{Z},\, j \leq i \leq -1\}$, *so* a_s *is all of* e_s*'s terms with integers power.*

4. *For* s *from* 1 *to* n *do*

   (a) *Take* $\overline{L}_s = L$, $Z_s = 0$, *and put* $d =$ *the maximum degree of the numerators of* $\overline{L_s}$*'s coefficients.*

   (b) *For* z *from* j *to* -1, *with* j *as in step* 3

      i. *Compute* $\tilde{L} = L \circledS (\partial - Z_s - R_z)$.

      ii. *Compute the maximum degree of the numerators of* $\tilde{L}$*'s coefficients, and call it* d_s.

      iii. *If* $d_s < d$, *take* $Z_s = Z_s + R_z$, *and* $\overline{L_s} = \tilde{L}$.

5. $M = \{[\overline{L_1}, d_1], \ldots, [\overline{L_n}, d_n]\}$.

6. *From* M, *return* $[\overline{L_s}, d_s]$ *such that* d_s *is the smallest one and* $\overline{L_s}$ *has the shortest length* (*bitsize*).

Example 4.12 (Example 4.11 [26])**.** *Applying* Phase I *on*

$$L = \partial^3 + \frac{3(3x^2-1)}{x(x-1)(x+1)}\partial^2 + \frac{221x^4 - 206x^2 + 5}{12x^2(x-1)^2(x+1)^2}\partial + \frac{374x^6 - 673x^4 + 254x^2 + 5}{54x^3(x-1)^3(x+1)^3}, \tag{4.5}$$

gives this exponential equivalent differential operator

$$\tilde{L} = \partial^3 + \frac{5(3x^2-1)}{2x(x-1)(x+1)}\partial^2 + \frac{35}{3(x-1)(x+1)}\partial + \frac{70}{27x(x-1)(x+1)},$$

such that $L = \tilde{L} \circledS (\partial - \frac{-3x^2+1}{6x^3-6x})$.

CHAPTER 5

PHASE II: INTEGRAL BASIS METHOD

Definition 5.1. *Let $L \in k[\partial]$. We call $L_1 \in k[\partial]$ is a N_p-simplified differential operator to L if $L \sim_p L_1$, Definition 2.26, and $\#S_{ing}(L_1) \leq \#S_{ing}(L_2)$ for any $L_2 \in k[\partial]$ such that $L_2 \sim_p L$.*

Goal 5.2. *Given $L \in \mathbb{C}(x)[\partial]$. We want to find a N_p-simplified L_1 to L, an operator $G \in \mathbb{C}(x)[\partial]$, and $r \in \mathbb{C}(x)$ such that $exp(\int r\, dx) \cdot G(y)$ is a solution of L for any solution y of L_1. (Notation $L_1 \xrightarrow{(ii)+(iii)}_{G,\,r} L$).*

Example 5.3. *Consider*

$$L = \partial^3 + \frac{2(9x-5)}{x(2x+5)(4x-1)}\partial^2 + \frac{120x^2-322x-15}{16x^2(2x+5)(4x-1)}\partial - \frac{32x^2+166x-5}{4x^2(4x-1)(2x+5)}$$

It has three algebraic singular points at $x = 0, \frac{-5}{2}, \frac{-1}{4}$, and the denominator degree is 4. However, it is projectively equivalent to

$$\tilde{L} = \partial^3 - \frac{1}{x}\partial^2 - \frac{33}{16x^2}\partial - \frac{1}{x^2},$$

such that $\tilde{L} \xrightarrow{(ii)+(iii)}_{G,\,r} L$ where

$$G = \frac{16x^2(254x-85)}{61(2x+5)(4x-1)}\partial^2 + \frac{16x(32x^2-394x+75)}{61(2x+5)(4x-1)}\partial - \frac{2560x^2-4658x+1155}{61(2x+5)(4x-1)},$$

and $r = 0$. It can be seen that $\tilde{L}$ has only one algebraic singular point at $x = 0$. By using Maple Software,

$$y = {}_0F_2([], [\frac{-11}{4}, \frac{3}{4}], x)$$

is a solution of $\tilde{L}$, so $G(y)$ is a solution of L, see Section 2.4.

In this chapter, we are trying to find a N_p-simplified operators of a given one by moving the apparent singularities to the irremovable singularities. For regular singular differential operators, Imamoglu and Van Hoeij [16] implemented an algorithm which tries to reduce a given differential operator L to a gauge equivalent differential operator $\bar{L}$ with a fewer apparent singularities. This works well for regular singular operators of order 2.

To extend the work in [16] to irregular singular operators we generalized a key tool: Integral basis for irregular singular differential operators. This tool produces a finite dimension vector space V of candidate gauge transformations. To extend [16] to order > 2, we are developing a new way to find the "best" element of V.

The definition and the algorithms of computing an integral basis of a given differential operator are given in Sections 5.1, 5.2, 5.3, 5.4. Normalizing a global integral basis at infinity is studied in Section 5.5, and using normalized integral basis at infinity to compute gauge equivalent differential operators for a given one is explained in Section 5.6.

5.1 Integral Basis for Differential Operators

Here, we study and introduce an integral basis for a given linear differential operator. Also, we improve the algorithms in [15, 16] to compute an integral basis for a given linear differential operator whether it is regular or irregular.

Integral bases for regular singular differential operators first introduced by Kauers and Koutschan in [18]. For the same kind of differential operators, Imamoglu and Van Hoeij in [15, 16] improved the algorithm to be faster and gave an algorithm to normalize the basis at infinity. Our goal in this section is to extend the algorithm to *irregular singular* linear differential operators.

Let $L \in \mathbb{C}(x)[\partial]$ of order n. At $x = 0$, Theorem 3.19 shows that there are n independent solutions as in (3.7),

$$y_i(x) = \exp\left(\int \frac{e_i}{x} dx\right) S_i, \tag{5.1}$$

where $v(S_i) = 0$ and $e_i \in \mathbb{C}[[x^{\frac{-1}{r_i}}]]$.

Definition 5.4. *Let y_i, as in* (5.1), *be a solution of $L \in \mathbb{C}(x)[\partial]$ at $x = 0$. Define the value of y_i as the real term of the constant part of the generalized exponent e_i, Definition* 3.14. *It is denoted by $\mu(y_i)$, so*

$$\mu(y_i) = Re([x^0](e_i)). \tag{5.2}$$

Notation 5.5. *a point $x = c$ is* regular singular *when every generalized exponent e_i at $x = c$ is a constant. In this case $\mu(y_i) = Re(e_i)$, where y_i is as in* (3.8).

Remark 5.6. *Let*

$$y = \exp\left(\int \frac{e}{x}\, dx\right) \cdot S,$$

where $e \in \mathbb{C}[[x^{\frac{-1}{r_i}}]]$ *and* $S = \sum_j^\infty \left(x^{j/r_i} \cdot P_j\right)$ *such that* $P_j \in \mathbb{C}[\log(x)]$*. Then*

$$\mu(y) = Re([x^0](e)) + v(S).$$

It can be proven by writing y *in this form*

$$y = \exp(\int \frac{e + v(S)}{x}\,dx) \cdot \frac{S}{x^{v(S)}},$$

which satisfies Definition 5.4 *because* $v(\frac{S}{x^{v(S)}}) = 0$.

Lemma 5.7. *Let* y_i *be a formal solution, see Theorem* 3.19*, of* $L \in \mathbb{C}(x)[\partial]$ *at* $x = 0$*. The value of* y_i' *(assume* $y_i' \neq 0$*) is*

$$\mu(y_i') \geq \mu(y_i) - 1 + v'(e_i),$$

where $e_i \in \mathbb{C}[[x^{\frac{-1}{r_i}}]]$ *is the generalized exponent of* y_i*. Therefore, the pole order increases by at most* $1 - v'(e_i)$ *under each derivation.*

Proof. Since y is a formal solution, it can be written in this form

$$y_i = \exp(\int \frac{e_i}{x} dx) S_i,$$

as in Theorem 3.19, where $v(S_i) = 0$. The derivative of y is

$$\begin{aligned} y_i' &= \frac{e_i}{x} \cdot \left(\exp(\int \frac{e_i}{x} dx) \cdot S_i\right) + \exp(\int \frac{e_i}{x} dx) \cdot S_i' \\ &= \exp(\int \frac{e_i}{x} dx) \cdot (\frac{e_i}{x} S_i + S_i'), \end{aligned}$$

so

$$\begin{aligned} \mu(y_i') &= Re([x^0](e_i)) + v(\frac{e_i}{x} S_i + S_i') \\ &\geq \mu(y_i) + \min(v(\frac{e_i}{x} S_i), v(S_i')) \text{ Definition 3.8)} \\ &= \mu(y_i) + \min(v(\frac{e_i}{x}) + v(S_i), v(S_i')) \text{ (Definition 3.8)} \\ &\geq \mu(y_i) + \min(v(\frac{e_i}{x}), -1) \\ &= \mu(y_i) + \min(v(e_i) - v(x), -1) \text{ (Definition 3.8)} \\ &\geq \mu(y_i) + \min(v'(e_i) - 1, -1) \text{ (see (3.9))} \\ &= \mu(y_i) + v'(e_i) - 1 \end{aligned}$$

□

For $x = c$, the situation is reduced to $x = 0$ with a change of variables transformation, see Notation 3.20, so we denote the value and the valuation at $x = c$ by μ_c and v_c respectively.

Definition 5.8 (Definition 4.2.2, Definition 4.2.3. in [15])**.** *A differential operator $G \in \mathbb{C}(x)[\partial]$ is integral for $L \in \mathbb{C}(x)[\partial]$ at point $x = c$ if $\mu_c(G) \geq 0$ where*

$$\mu_c(G) = \mathit{inf}\{\mu_c(G(y))|\ y,\ \text{as in (5.1), is a solution of as } L \text{ at } x = c\}.$$

Also, G is called integral for L if it is integral at all point $c \in \mathbb{C}$.

If $\operatorname{ord}(G) \geq \operatorname{ord}(L)$, then by using division with remainder, G can be written as $G = QL + R$, where $\operatorname{ord}(R) < \operatorname{ord}(L)$, and $\mu_c(G) = \mu_c(R)$ since $G(y) = R(y)$, where y is a solution of L. Therefore, we may assume that $\operatorname{ord}(G) < \operatorname{ord}(L)$.

Definition 5.9 (Definition 4.2.4. in [15])**.** *A* global integral basis *for a differential operator $L \in \mathbb{C}(x)[\partial]$ is a basis of*

$$O_L = \{G \in \mathbb{C}(x)[\partial]|\ G \text{ is integral for } L,\ \text{and } ord(G) < ord(L)\},$$

as $\mathbb{C}[x]$-module.

Definition 5.10 (Definition 4.2.5. in [15])**.** *Let $P \in \mathbb{C}[x]$. A set $\{b_1, ..., b_n\}$ is a* local integral basis *for L at P if each b_i is integral at all roots of P, and every G that is integral at all roots of P is a $\mathbb{C}[x]_{(P)}$-linear combination of $b_1, ..., b_n$, where $\mathbb{C}[x]_{(P)}$ is the* localization *of $\mathbb{C}[x]$ at $P \in \mathbb{C}[x]$,*

$$\mathbb{C}[x]_{(P)} = \{\frac{A}{B}|\ gcd(P, B) = 1\}$$

A local integral basis at $c \in \mathbb{C}$ is an integral basis at $x - c$.

Lemma 5.11. *Let $\{b_1, ..., b_n\}$ be an integral basis for $L \in \mathbb{C}(x)[\partial]$ at $c \in \mathbb{C}$ and $f_1, ..., f_n \in \mathbb{C}(x)$. Then $\{f_1 b_1, ..., f_n b_n\}$ is an integral basis for L at c if $v_c(f_i) = 0$ for all $i = 1, ..., n$.*

The proof of this lemma is clear from the solution's value definition. $\mu_c(f_i b_i) = \mu_c(f_i) + \mu_c(b_i) = \mu_c(b_i)$.

Theorem 5.12 (Theorem 4.2.2 [15])**.**

(i) $\{b_1, ..., b_n\}$ is a local integral basis for $L \in \mathbb{C}(x)[\partial]$ at 0 if and only if

(ii) for $1 \leq i \leq n$ *and* $1 \leq j \leq n$, $\mu_0(b_i(y_j)) \geq 0$, *where* $\{y_1, ..., y_n\}$ *is the formal solution basis for* L *at* 0, *and*

(iii) for all $(f_1, ..., f_n) \in \mathbb{C}^n \backslash (0, ..., 0)$, *there is* $j \in \{1, ..., n\}$ *such that*

$$\frac{1}{x}(f_1 b_1 + ... + f_n b_n)(y_j) < 1.$$

For the proof look at [15]. For more information about integral basis reader can look at [15, 16, 18].

5.2 Local Integral Basis at a Singular Point

In [15], there is an algorithm that computes a local integral basis for *regular singular* differential operators. Here, the algorithm is extended to *irregular singular* differential operators.

A local integral basis for a differential operator L of order n at a point $x = c$ can be computed by computing a local integral basis at $x = 0$ for L_{x+c}, see Notation 2.23. Then, apply the change of variable transformation $x \to x - c$ on the basis. Therefore, we concentrate on a procedure that computes a local integral basis for a differential operator at 0.

If $x = 0$ is a regular singular point, then the generalized exponents at point 0 are constants, Remark 3.6. As mentioned in Remark 5.5, the pole order increases by one under each derivation. However, if $x = 0$ is an irregular singular point, the generalized exponents at 0 are in $\mathbb{C}[[x^{\frac{-1}{r}}]]$. So they are not always constants.

Let $x = 0$ be irregular singular point of L, and let y_i be a formal solution for L at 0 such that e_i is the generalized exponent of y_i. Let $Y_i = \partial^j(y_i)$. Then

$$\mu_0(Y_i) \geq \mu_0(y_i) + (v_0'(e_i) - 1)j \tag{5.3}$$

by Lemma 5.7.

Remark 5.13. *To computes an integral basis at* $x = 0$, *we wrote the below algorithm, Algorithm* 5.14, *which is similar to Algorithm* 4.2.1 [*16*]. *The changes are in steps* (a) *and* (c) *which needed adjustments to handle the irregular singular case.*

Algorithm 5.14 (*Local_Basis_at_0*)**.**

- ***Input:*** $L \in \mathbb{Q}(x)[\partial]$ *a linear differential operator.*

- ***Output:***
  - *A local integral basis for L at* 0.
  - *The degree of the coefficients of ∂^i of b_i, where $1 \leq i \leq n$,*
  - *d as defined in* (5.5).

  Note: *each step will be explained and illustrated in Example* 5.15 *below.*

1. *Find $Y = \{y_1, ..., y_n\}$, a basis of the formal solutions of L at* 0.
2. *Compute the values of the formal solutions in Y, $\mu_0(y_j), \ldots, \mu_0(y_n)$. Then, make*
$$m = -\lfloor min(\{\mu_0(y_j)|1 \leq j \leq n\})\rfloor. \tag{5.4}$$
3. *Compute the valuations of generalized exponents of L at $x = 0$. Then, take*
$$d = -\lfloor min\{v_0'(e_i)|e_i \text{ are generalized exponents at } 0\}\rfloor. \tag{5.5}$$
4. *Make $b_1 = x^m$.*
5. *For i from 2 to n do*
   (a) *$b_i = x^{d+1}\partial \cdot b_{i-1}$. Note that $b_1, ..., b_i$ are integral and linearly independent.*
   (b) *Now, use part (iii) of Theorem 5.12 to obtain an appropriate local integral basis for L at* 0*, so create this ansatz*
   $$\Phi = \frac{1}{x}(f_1 b_1 + ... + f_{i-1}b_{i-1} + b_i), \tag{5.6}$$
   where $f_1, ..., f_{i-1}$ are unknown coefficients.
   (c) *Apply Φ on $\{y_1, ..., y_n\}$.*
   (d) *Collect the coefficients of negative value, μ_0, terms of $\Phi(y_j)$, where $1 \leq j \leq n$, and equate them to zero.*
   (e) *If there is a solution of the system, apply the solution on Φ, and replace b_i by Φ.*
   (f) *Repeat the process until the system does not have a solution. If there is no more solution go to the next i.*
6. *Return $\{b_1, b_2, ..., b_n\}$, the degrees of the leading coefficients of each b_i, and d.*

Example 5.15. *Let $L = \partial^2 + \frac{(5x+3)}{3x^2}\partial - \frac{1}{2x^3}$. In this example, we will explain how the Algorithm* 5.14 *works.*

1. *First, find Y a basis of the formal solutions of L at* 0,

$$Y = \{y_1 = x^{\frac{1}{2}}(1 - \frac{7x}{12} + \frac{91x^2}{96} + ...), y_2 = x^{\frac{-1}{6}} \exp(\frac{1}{x})(1 - \frac{x}{12} - \frac{5x^2}{96} + ...)\} \tag{5.7}$$

   The generalized exponents of y_1, y_2 are $e_1 = \frac{1}{2}, e_2 = \frac{-1}{6} - \frac{1}{x}$ respectively.

2. *Compute the value of the solutions in* (5.7), *so*

$$\mu_0(y_1) = 1/2, \textit{ and } \mu_0(y_2) = -1/6. \tag{5.8}$$

   Therefore, $m = -\lfloor min(\{\frac{1}{2}, \frac{-1}{6}\})\rfloor = 1$ as (5.4).

3. *The valuation of the generalized exponents of L at 0 are $v_0(e_1) = 0$ and $v_0(e_2) = -1$, so $d = 1$, as in* (5.5).

4. *Now, we compute the initial integral operators*

$$b_1 = x^m = x.$$

5. (a) *Take*

$$b_2 = x^{d+1}\partial \cdot b_1 = x^2(x\partial + 1).$$

   (b) *Make*

$$\begin{aligned}\Phi &= \frac{1}{x}(f_1 b_1 + b_2)\\ &= \frac{1}{x}(f_1 x + x^2(x\partial + 1)).\end{aligned}$$

   where f_1 is an unknown coefficient.

   (c) *Applying Φ on the solutions in* (5.7) *gives these equations*

$$\Phi(y_1) = f_1 x^{\frac{1}{2}} + (\frac{3}{2} - \frac{7f_1}{12})x^{\frac{3}{2}} + (\frac{-35}{24} + \frac{91f_1}{96})x^{\frac{5}{2}} + \frac{637x^{\frac{7}{3}}}{192} + ...,$$

   and

$$\Phi(y_2) = \exp(1/x)(\frac{f_1 - 1}{x^{\frac{1}{6}}} - (\frac{15f_1 + 29}{288})x^{\frac{11}{6}} + \frac{(11 - f_1)}{12}x^{\frac{5}{6}} + ...).$$

   (d) *There is only one term of negative value, μ_0, of $\Phi(y_1)$ and $\Phi(y_1)$ and its coefficient is $f_1 - 1$.*

   (e) *Substituting its solution in Φ, and replacing b_2 by this new an integral operator. Then*

$$b_2 = x^2\partial + x + 1.$$

(f) We need to repeat the process above, so

$$\Phi = \frac{1}{x}(f_1 x + x^2\partial + x + 1).$$

Applying the new Φ *on the solutions in* (5.7) *and collecting the coefficients of the negative terms give this system*

$$\{\frac{-1}{2}, \frac{f_1+5}{6}\},$$

which does not have a solution.

6. Therefore, we end with this integral basis

$$\{x, x^2\partial + x + 1\}$$

as a result of Theorem 5.12.

Remark 5.16. *The* Local_Basis_at_0 *computes a local integral basis for* L *at* $x = 0$ *if* 0 *is a regular or irregular singularity. Computing a local integral basis at apparent singularities similar to the algorithm mentioned in [15].*

5.3 Local Integral Basis at Algebraic Singularities

Let $L \in \mathbb{C}(x)[\partial]$ of order n, and let $x = c$ be an algebraic regular singular point of L such that $[\mathbb{Q}(c) : \mathbb{Q}] = l$, and $P \in \mathbb{Q}[x]$ is its minimal polynomial. Kauers and Koutschan in [18] wrote an algorithm which computes a local integral basis at c and at each conjugate of c. Also, in [15], Imamoglu and van Hoeij implemented an algorithm which computes a local integral basis at algebraic singularity $x = c$ for a given differential operator, and modify this integral basis to become an integral basis at c and at each conjugate of c. We conducted some adjustments at step (b) in Algorithm 4.2.2, in [15], to include irregular singularities.

To compute an integral basis at algebraic singularities, we follow these steps. First, use the algorithm 5.14 and change of variable transformation to compute a local integral basis $\{b_1, b_2, ..., b_n\} \subset \mathbb{Q}(c)[x, \frac{1}{x-c}][\partial]$ at c. Then, scale b_i to

$$B_i = f_i b_i,$$

for each $1 \leq i \leq n$, such that

$$f_i = \left(\frac{P}{x-c}\right)^{m+1+(i-1)(1+d)},$$

where m as in (5.4) and d as in (5.5). Note that $\{B_1, ..., B_n\}$ is still a local integral basis at c, Lemma 5.11, and the value of $B_i(y_j)$, for each $j \in \{1, ..., n\}$, is at least 1 at each root of $P/(x-c)$. Finally, taking the *trace*, see Example 4.6, of $\{B_1, ..., B_n\}$ over $\mathbb{Q}(c)/\mathbb{Q}$ leads to that

$$\{\mathrm{Tr}(B_1), ..., \mathrm{Tr}(B_n)\} \subseteq \mathbb{Q}(x)[\partial],$$

which is a local integral basis at $x = c$ and its conjugates by Theorem 5.12.

Remark 5.17. *Similar to the adjustments that we have done in the algorithm 5.14 and in the previous section, we modify* local_basis_minpoly *in* [*15*].

5.4 Combining Two Local Integral Bases and Global Integral Basis

The way of combing two local integral bases at irregular singularities is similar to the way of combing two local integral bases at regular singularities which mentioned in [15] because the effects of irregularity is treated in the previous algorithms in Section 5.2 and Section 5.3. Therefore, we use Algorithm 4.2.3 [15] without any adjustment.

A global integral basis for L is achieved by combining all integral bases at the minimal polynomials of the singularities of L. We use global_integral_basis, Algorithm 4.2.4 [15] to compute a global integral basis for a given differential operator without any adjustment.

5.5 Normalized Integral Basis at Infinity

Definition 5.18. *Let $L \in \mathbb{C}(x)[\partial]$. The degree of $G \in \mathbb{C}(x)[\partial]$ at ∞ for L is deg$(G) = -\lfloor \mu_\infty(G) \rfloor$, see Definition* 5.8. *It is the smallest integer number m such that $\frac{1}{x^m}G$ is integral for L at ∞, so G is integral at ∞ if and only if deg$(G) \leq 0$.*

Let $\{B_1, \ldots, B_n\}$ be a global integral basis for L that computed by the algorithms in the previous sections. If $B \in \mathbb{C}[x]B_1 + \cdots + \mathbb{C}[x]B_n$, then B is *integral* for L, Definition 5.8. Then $L[B]$ is *integral*, Definition 3.31, so we can use the bound in Theorem 3.33. This bound depends on the degree of B at ∞, Definition 5.18. To minimize the degree of B, we need an integral basis $B_1, \ldots, B_n$ that is *normaized at* ∞ :

Definition 5.19. *Let $L \in \mathbb{C}(x)[\partial]$ of order n and $B = \{b_1, b_2, ..., b_n\}$ be a set of differential operators over $\mathbb{C}(x)$. B is called normalized at $x = c$ if there exist rational functions $a_i \in \mathbb{C}(x)$, $1 \leq i \leq n$, such that $\{a_1 b_1, ..., a_n b_n\}$ is a local integral basis for L at $x = c$.*

To normalize an integral basis, we use Algorithm 4.3.1 in [15], normalize_at_infinity, without any adjustment because the effects of irregular singularities are treated in the algorithms in previous sections.

Algorithm 5.20 (normalize_at_infinity, Algorithm 4.3.1 in [15])**.**

Input:

- $[B_1, B_2, \ldots, B_n]$, *a global integral basis for L.*
- $[b_1, b_2, \ldots, b_n]$, *a local integral basis for L at ∞.*

Output:

- $[\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n]$, *a normalized integral basis at ∞.*
- $[d_1, d_2, \ldots, d_n]$, *d_i is degree of $\tilde{B}_i$ at ∞.*

In the previous section, we explained how to compute a global integral basis. To compute a local integral basis for L at ∞, compute a local integral basis for $L_{\frac{1}{x}}$, see Notation 2.23, at 0, then apply change of variables transformation, $(x, \partial) \to (\frac{1}{x}, -x^2\partial)$, on the basis. Note that, $[\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_n]$ is still integral basis for L because each $\tilde{B}_i = sB_i + \sum_{l \neq i} a_l B_l$, where s is nonzero rational number and $a_l \in \mathbb{Q}[x]$.

The goal of normalizing a global integral basis for L at ∞ is to get an integral basis with minimum degree at ∞.

Remark 5.21 (see [15], Section 4.3)**.** *Let $B = \{b_1, ..., b_n\}$. If $G = a_1 b_1 + ... + a_n b_n$, then $deg(G) \leq max(deg(a_i b_i))$. If B is normalized integral basis, $deg(G) = max(deg(a_i b_i))$.*

Example 5.22. *Let*

$$L = \partial^2 - \frac{13(2x^{13} - x^6 + 2)}{2x(2x^{13} + 13x^6 - 2)}\partial - \frac{4x^{14} - 56x^{13} + 78x^7 + 273x^6 - 4x + 30}{2x^2(2x^{13} + 13x^6 - 2)}.$$

By the algorithm in the previous section, a global integral basis for L at algebraic singularities is

$$GB = [b_1 = x^3, b_2 = \frac{x^4}{4(2x^{13} + 13x^6 - 2)}\partial + \frac{x^3(x^7 + 3)}{4(2x^{13} + 13x^6 - 2)}].$$

To compute their degrees at ∞, we use Lemma 5.7 and Definition 3.8. Let $\{y_1, y_2\}$ be a basis of the formal solutions of L, then the degree of b_i at ∞ is

$$d_{b_i} = min(-\lfloor \mu_\infty(b_i(y_1)) \rfloor, -\lfloor \mu_\infty(b_i(y_2)) \rfloor).$$

For b_1, $-\lfloor \mu_\infty(b_1(y_i)) \rfloor = -\lfloor \mu_\infty(x^3 y_i) \rfloor = -\lfloor \mu_\infty(y_i) + v'_\infty(x^3) \rfloor = -\lfloor \frac{-7}{2} + (-3) \rfloor = 7$ because the generalized exponents of L at ∞ are $\{e_1 = \frac{-2}{x^{1/2}} - \frac{7}{2},\ e_2 = \frac{2}{x^{1/2}} - \frac{7}{2}\}$, and $d_{b_2} = 1$. To compute a basis with minimum degree at ∞, apply Algorithm 5.20 on GB. It gives this basis

$$\widetilde{GB} = [\tilde{b}_1 = \frac{-2x^{10}}{2x^{13} + 13x^6 - 2}\partial + \frac{x^3(7x^6 - 2)}{2x^{13} + 13x^6 - 2},$$
$$\tilde{b}_2 = \frac{x^4}{4(2x^{13} + 13x^6 - 2)}\partial + \frac{x^3(x^7 + 3)}{4(2x^{13} + 13x^6 - 2)}],$$

where their degrees at ∞ are $[0, 1]$.

If $B = b_1$, b_2, $\tilde{b}_1$ or $\tilde{b}_2$, then $L[B]$ is integral. Without computing $L[B]$, we can compute the bound of its apparent singularities by computing the generalized exponents of L at its true singularities as the following: First, the trick in Example 3.5 shows that the true singularities of L are 0 and ∞, and the generalized exponents of L at $x = 0$ are $\{-3, \frac{-5}{2}\}$. The ramification index at $x = 0$ is 1 and at $x = \infty$ is $\frac{-1}{2}$, so by Lemma 3.21, at $x = 0$,

$$-Frac(\tilde{e}_1) = 0,\ -Frac(\tilde{e}_2) = -\frac{1}{2},\ \text{ and }\ v'_0(\tilde{e}_1 - \tilde{e}_2) = 0.$$

At $x = \infty$,

$$-Frac(\tilde{e}_1) = 0,\ -Frac(\tilde{e}_2) = 0,\ \text{ and }\ v'_0(\tilde{e}_1 - \tilde{e}_2) = \frac{-1}{2}.$$

By using (3.15), the bound of the apparent singularities of $L[B]$ is

$$n_{app}(L[B]) \leq -1 + 2d - (-(0) - (0) + \frac{-1}{2}) + \big(-(0) - (\frac{1}{2}) - 0 + 1 \big) = 2d$$

where d is the degree of B at ∞, so

$$n_{app}(L[b_1]) \leq 14, \quad n_{app}(L[b_2]) \leq 2, \quad n_{app}(L[\tilde{b}_1]) \leq 0, \quad n_{app}(L[\tilde{b}_2]) \leq 2.$$

The actual numbers of apparent singularities for

$$L[b_1],\ L[b_2],\ L[\tilde{b}_1],\ L[\tilde{b}_2]\ \text{ are } 13,\ 0,\ 0,\ 0,$$

respectively.

In the previous example, we see how normalizing integral basis at infinity reduce the bound of the apparent singularities.

5.6 Phase II, Using a Normalized Integral Basis

The N_p-simplified differential operators of a given one are obtained by applying appropriate exponential product and gauge transformations. In Phase I, we explained how to minimize $\#S_{ing}$ by (iii)-transformations. In this section, we will try to minimize the number of apparent singularities of a given differential operator by applying appropriate gauge transformation on it.

Example 5.23 (Continue Example 5.22)**.** *Differential operators often appear in* normal form, *Definition* 2.21*. The normal form of L is*

$$NF(L) = \partial^2 - A,$$

where $A \in \mathbb{C}(x)$ has denominator x^2P where $P = (2x^{13} + 13x^6 - 2)^2$. The generalized exponents of L at P are 0, 2, *and the generalized exponents of $NF(L)$ at P are $\frac{-1}{2}$, $\frac{3}{2}$. Note that, there is no gauge transformation applied to $NF(L)$ can remove P because the generalized exponents at the roots of P are not in $\mathbb{Z}$, and Lemma* 3.21 *shows that gauge transformations shift the generalized exponents of L at the roots of P by an integer number. Therefore, Phase I was designed to move the generalized exponents of the apparent singularities to $\mathbb{Z}$, so it is better to use Phase I before applying Phase II.*

Let $L \in \mathbb{C}(x)[\partial]$ of order n. Let $[b_1, b_2, ..., b_n]$ be a normalized integral basis for L, and let $[d_1, d_2, ..., d_n]$ be their degrees at ∞, Definition 5.18.

Definition 5.24. *Let L be as above and d be a non-negative integer. The d-search space, V_d, is the set of all integral for L with degree $\leq d$ at ∞. The basis V_d is*

$$\{x^j b_i \,|\, 0 \leq j \leq d - d_i,\, 1 \leq i \leq n\}. \tag{5.9}$$

Let d be some integer ≥ 0. Then

$$B = \sum_{i=1}^{n} \sum_{j=0}^{d-d_i} c_{i,j}\, x^j\, b_i \tag{5.10}$$

is a generic element of V_d, where $c_{i,j}$ are unknown coefficients.

Example 5.25. *Let*

$$L = \partial^3 + \frac{72x^2 + 234x - 85}{4x(2x+5)(4x-1)}\partial^2 + \frac{24x^2 - 26x - 105}{8(2x+5)x^2(4x-1)}\partial - \frac{8x^2 + 34x + 13}{x^2(2x+5)(4x-1)}.$$

By the same method in Example 5.22, *we find that the bound of the apparent singularities of* $L[B]$, *where* $B \in V_d - \{0\}$, *is*

$$n_{app}(L[B]) \leq 3d.$$

For L, *Algorithm* 5.20 *gives the following normalized integral basis at* ∞

$$\begin{aligned}&[[b_1 = \frac{x^2}{2048(2x+5)(4x-1)}\partial^2 + \frac{x(4x+9)}{8192(2x+5)(4x-1)}\partial \\ &+ \frac{8x^2+18x+3}{16384(2x+5)(4x-1)}, d_1 = 0], [b_2 = \frac{-x^2}{512(2x+5)(4x-1)} \\ &- \frac{5x^2}{4096(2x+5)(4x-1)}\partial - \frac{x^4}{512(2x+5)(4x-1)}\partial^2, d_2 = 1], [b_3 = x, d_3 = 1]],\end{aligned}$$

If $d = 0$, *then we only have one choice which is* $B = b_1$, *so*

$$L[B] = \partial^3 + \frac{9}{4x}\partial^2 + \frac{3}{8x^2}\partial - \frac{1}{x^2}, \tag{5.11}$$

where $n_{app}(L[B]) = 0$. *If* $d = 1$, *the basis of all integral operator of degree* ≤ 1 *is*

$$\{b_1,\, xb_1,\, b_2,\, b_3\},$$

so we can write the following linear combination

$$B = (c_{1,0} + c_{1,1}x)b_1 + c_{2,1}b_2 + c_{3,1}b_3.$$

If we take $c_{1,1} = c_{2,1} = c_{3,1} = 0$ *and* $c_{1,0} = 1$, *then* $B = b_1$ *which leads to* (5.11). *If we take* $c_{1,0} = 20480\, c_{3,1}$, $c_{1,1} = \frac{-409600}{9}\, c_{3,1}$, $c_{2,1} = \frac{-65536}{3} c_{3,1}$, *and* $c_{3,1} = c_{3,1}$ (*see next section*), *then*

$$L[B] = \partial^3 - \frac{3}{4x}\partial^2 - \frac{9}{8x^2}\partial - \frac{1}{x^2}. \tag{5.12}$$

The operators in (5.11) *and* (5.12) *have less number of apparent singularities because some of the apparent singularities of* L *move to other singular points.*

The advantage of taking a large number search space, V_d, is that more chance of computing optimal differential operators, and the disadvantage is more computations are needed. Our method of moving some singular points to others is explained in the following example.

Example 5.26. *Let*

$$L = \partial^2 - \frac{(x-1)(x+1)}{(x^2+1)x}\partial + \frac{x^2+2x+3}{x(x^2+1)}.$$

By the same method in Example 5.22*, if B is an integral operator of L, then the bound of the apparent singularities of $L[B]$ is*

$$n_{app}(L[B]) \leq 2d,$$

where d is the degree of B at ∞. To compute an appropriate integral operator B, apply Algorithm 5.20 *on L to compute a normalized integral basis with their degrees at ∞,*

$$[b_1 = \frac{x}{x^2+1}\partial + \frac{x-1}{x^2+1},\, b_2 = 1],\, [d_1 = 0, d_2 = 1].$$

For any d, this date allow us to compute a basis of all integral operator for L with degree $\leq d$. If $d = 0$, then V_d has basis $\{b_1\}$. Therefore, $n_{app}(L[B]) = 0$, and

$$L[b_1] = \partial^2 + \frac{1}{x}\partial + \frac{1}{x}. \tag{5.13}$$

Now, if $d = 1$, then V_d has basis:

$$\{b_1,\ xb_1,\ b_2\}.$$

Now, write

$$B = (c_1 + c_2x)b_1 + c_3b_2,$$

so

$$L[B] = \partial^2 + \frac{A_1}{S}\partial + \frac{A_2}{S}$$

where

$$\begin{aligned} S &= ((c_2^2 + 2c_2c_3 + c_3^2)x^2 + (2c_1c_2 + 2c_1c_3 - c_2c_3)x + c_1^2 - 2c_1c_3 + c_3^2)x \\ &= S_{param} \cdot S_{fixed}. \end{aligned}$$

such that $S_{fixed} = x$ and $S_{param} = (c_2^2 + 2c_2c_3 + c_3^2)x^2 + (2c_1c_2 + 2c_1c_3 - c_2c_3)x + c_1^2 - 2c_1c_3 + c_3^2$. The apparent singularities of $L[B]$ are depend on the value of the parameters, so there are two free singular points. Note: If $c_1 = c_2 = 0$, $c_3 = 1$, then $B = 1$ that leads to $L[B] = L$.

In this example, we will explain our strategy of how to compute the best values of the unknown coefficients of B that makes the number of the apparent singularities of $L[B]$ fewer than that in L by moving some or all of the apparent singularities to the true singularities, which are $x = 0$ and $x = \infty$.

Case 1 (*moving the apparent singularities to* $x = 0$) : *When the coefficients of* x^0 *and* x^1 S_{param} *are vanished, the roots of* S_{param} *move to* 0. *Therefore, there are two solutions for this case,*

$$\{c_1 = 0, c_2 = c_2, c_3 = 0\}, \quad \textit{and} \quad \{c_1 = c_1, c_2 = -2c_1, c_3 = c_1\},$$

and these solutions lead to the following operators

$$L[B] = \partial^2 - \frac{1}{x}\partial + \frac{x+1}{x^2}, \quad \textit{and} \;\; L[B] = \partial^2 - \frac{1}{x}\partial + \frac{1}{x}.$$

It is obvious that both solutions move the apparent singularities to $x = 0$.

Case 2 (*moving the apparent singularities to* $x = \infty$) : *In this case, we solve the coefficients of* x^2 *and* x^1 *of* S_{param}, *so there is only one solution, which is*

$$c_1 = c_1, c_2 = 0, c_3 = 0.$$

This solution leads to (5.13), *obtained from the case* $d = 0$.

Case 3 (*moving one of the apparent singularities to* $x = 0$ *and the other to* $x = \infty$) : *In this case, we solve the coefficients of* x^2 *and* x^0 *of* S_{param}. *There is only one solution for this case which is* $c_1 = -c_2,\ c_2 = c_2,\ c_3 = -c_2$, *and this solution leads to this operator*

$$L[B] = \partial^2 + \frac{1}{x}. \tag{5.14}$$

which is the optimal operator. Note that, large b leads to large search space V_d, *so move chance of finding optimal simple operator.*

If there are more true singularities and more parameters, the computation will not be that easy like in the previous example. Therefore, we implement a new method that compute the best value of the parameters that move the apparent singularities of a given differential operator to true singularities.

5.6.1 Computing Apparent Singularities of $L[B]$

When B has many parameters, such as $c_{1,1}, \ldots, c_{1,j_1}, c_{2,1} \ldots, c_{n,1}, \ldots, c_{n,j_n}$ where $j_i = d - d_i$ and d, d_i as in Definition 5.24, then $L[B] \in D = \mathbb{C}(c_{i,j}, x)[\partial]$ is very large. Fortunately, we can compute the apparent singularities of $L[B]$ without computing $L[B]$ itself.

Let $L[B] = \sum_{i=0}^{n} a_i \partial^i$ where $a_n = 1$. To compute the value of the parameters of B, we should annihilate B in $D/D\,L$. Let R_i be the image of $\partial^i \cdot B$ in $D/D\,L$ (using division with reminder, Lemma

2.14). Then $\sum_{i=0}^{n-1} R_i = 0$ in $D/D\,L$ because of the annihilating of B in $D/D\,L$. Now, each coefficient of ∂^i, for $i = 0, \ldots, n-1$, gives one equation, so we can find the value of the of the parameters and thus $L[B]$ by solving linear equations. Trying to solve the linear equations consumes a lot of time because $L[B]$ is very large, and we only need the denominator of $L[B]$ because its roots are the singularities of $L[B]$.

Let M be a matrix where its entries are the coefficients of $R_0, \ldots, R_{n-1}$, so if $R_i = r_{i,n-1}\partial^{i-1} + r_{i,n-2}\partial^{n-2} + \cdots + r_{i,0}$, then

$$M = \begin{bmatrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots & r_{0,n-1} \\ r_{1,0} & r_{1,1} & r_{1,2} & \cdots & r_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n-1,0} & r_{n-1,1} & r_{n-1,2} & \cdots & r_{n-1,n1} \end{bmatrix}. \tag{5.15}$$

By *Cramer's Rule,* $\det(M)$ is the denominator of the system's solutions, so the singularities of $L[B]$ are roots of $\det(M)$.

Remark 5.27. *Taking the primitive part of* $\det(M)$, *with respect to the unknown coefficients* $c_{1,1}, \ldots, c_{1,j_1}, c_{2,1} \ldots, c_{n,1}, \ldots, c_{n,j_n}$ *means that delete its factor that do not have those unknown coefficients.*

Let $\mathfrak{J}$ be the primitive part of $\det(M)$ with respect to $\{c_{i,j} | 1 \le i \le n,\, 1 \le j \le d - d_i\}$. Then, the apparent singularities of $L[B]$ are the roots of $\mathfrak{J}$, which is similar to S_{param} in Example 5.26.

5.6.2 Compute and Organize All of the Choices of the Linear Combinations of the Unknown Coefficients

We use the method that is explained in Example 5.26 to compute the best elements of the search space of the unknown coefficients that move the apparent singularities to true singularities.

Example 5.28. *Let* L *as in Example* 5.25, *and let* B *interpret all integral operators for* L *of degree* ≤ 1. *Then*

$$B = (c_{1,0} + c_{1,1}x)b_1 + c_{2,1}b_2 + c_{3,1}b_3. \tag{5.16}$$

To compute $\mathfrak{J}$, we first compute R_1, R_2, R_3, which are the image of $\partial^i \cdot B$ in $D/D\,L$. Then, find the primitive part of det(M), as in (5.15). Therefore,

$$\begin{aligned}\mathfrak{J} =&4(c_{1,1}+16384c_{3,1})^3x^3+(12c_{1,0}c_{1,1}^2+393216c_{1,0}c_{1,1}c_{3,1}+3221225472c_{1,0}c_{3,1}^2\\&-33c_{1,1}^2c_{2,1}+147456c_{1,1}^2c_{3,1}+224c_{1,1}c_{2,1}^2-1867776c_{1,1}c_{2,1}c_{3,1}\\&+4831838208c_{1,1}c_{3,1}^2-256c_{2,1}^3+3670016c_{2,1}^2c_{3,1}-21743271936c_{2,1}c_{3,1}^2\\&+39582418599936c_{3,1}^3)x^2+(12c_{1,0}^2c_{1,1}+196608c_{1,0}^2c_{3,1}+6c_{1,0}c_{1,1}c_{2,1}\\&+98304c_{1,0}c_{1,1}c_{3,1}+32c_{1,0}c_{2,1}^2-688128c_{1,0}c_{2,1}c_{3,1}+1610612736c_{1,0}c_{3,1}^2\\&+9c_{1,1}c_{2,1}^2-122880c_{1,1}c_{2,1}c_{3,1}+402653184c_{1,1}c_{3,1}^2-245760c_{2,1}^2c_{3,1}\\&+3355443200c_{2,1}c_{3,1}^2-10995116277760c_{3,1}^3)x\\&+c_{1,0}(3c_{2,1}-16384c_{3,1}+4c_{1,0})(c_{2,1}-8192c_{3,1}+c_{1,0})\end{aligned} \tag{5.17}$$

By using the same strategy in Example 5.26, we can move all of the apparent singularities of L to 0 by equating the coefficients of x^0, x^1, x^2 of $\mathfrak{J}$ to 0, that gives three equations which have 10 solutions, one of which is

$$S = \{c_{1,0}=0,\, c_{1,1}=\frac{81920}{3}\,c_{3,1},\, c_{2,1}=8192\,c_{3,1},\, c_{3,1}=c_{3,1}\}.$$

Substituting S into $\mathfrak{J}$ gives that

$$\mathfrak{J} = 4(\frac{81920}{3}c_{3,1}+16384c_{3,1})^3x^3+0\cdot x^2+0\cdot x+0,$$

and substituting S into B gives that

$$L[B] = \partial^3-\frac{3}{4x}\partial^2+\frac{9}{8x^2}\partial-\frac{4x+3}{4x^3}.$$

Let $L \in \mathbb{C}(x)[\partial]$. Let **Sing** be a set that contains ∞ and all of the minimal polynomials of the singularities of L, and let $\mathfrak{J}$ be a polynomial, as in Example 5.28, for L. We wrote Algorithm 5.30 which takes $\mathfrak{J}$ and **Sing** as input, and use the strategy that is used in Example 5.26 to compute a list of $\mathbb{Q}$ linear combinations of the $c_{1,1},\ldots,c_{1,j_1}$, $c_{2,1}\ldots,c_{n,1},\ldots,c_{n,j_n}$, where $j_i = d-d_i$ and d, d_i as in Definition 5.24. The best values of the parameters must satisfy most of the linear combinations that are obtained by the algorithm.

Algorithm 5.29 (*Equations_at_one_Sing*)**.**

Input: P, *as* $\mathfrak{J}$ *in Example* 5.28, *and* p *is an irreducible polynomial or* ∞.

Output: *Lists of* $\mathbb{Q}$ *linear combinations of* $c_{1,1}, ..., c_{1,j_1}, c_{2,1}, ..., c_{n,j_n}$ *with their qualities, which is the number of the apparent singularities that are removed by the solution of the linear combination.*

- *Take* d *to be the degree of* P *with respect to* x. *In Example* 5.28, $d = 3$.
- $V = \{c_{1,1}, ..., c_{1,j_i}, c_{2,1}, ..., c_{n,j_n}\}$, *a set of the unknown coefficients of* P. *In Example* 5.28, $V = \{c_{1,0},\, c_{1,1},\, c_{2,1},\, c_{3,1}\}$.
- *If* $d = 0$ *then return* $\{[\ ]\}$.
- *Else*
  - *Let* $R = \{\ \}$.
  - *Let* $E = P|_{x=Root(p)}$ *or be the leading coefficient if* P *is* ∞.
  - *For each factor*[i], a_i, *of* E, *if the degree of* a_i *is* 1 *with respect to* V *do:*
    * *Take* A *to be the primitive part with respect to the variables in* V, *see Remark* 5.27, *of substituting the solution of* a_i [ii] *in* P.
    * *Take* q *to be the difference between the degree of* P *and* A *with respect to* x.
    * $Q = [a_i,\, q]$, (q *is called the quality of the solution of* a_i).
    * $W = \text{Equations\_at\_one\_Sing}(A, p)$.
    * $R = R \cup \{[Q,\ \text{the elements of } w_1], \ldots, [Q,\ \text{the elements of } w_s]\}$, *where* $w_1, \ldots, w_s$ *are the elements of* W.
- *Return* R.

Algorithm 5.30 (*All_Equations*)**.**

Input: $\mathfrak{J}$, *as in Example* 5.28, *and* **Sing**.

Output: *Set of lists of* $\mathbb{Q}$ *linear combinations of* $c_{1,1}, ..., c_{1,j_1}, c_{2,1}, ..., c_{n,j_n}$ *with their qualities.*

- *Take* $E = \{\ \}$.
- *For each* $p \in$ **Sing** *do*
  - $E = E \cup$ *Equations_at_one_Sing*(P, p), (*Algorithm* 5.29).
- *Return* E.

[i] factor over k'.

[ii] Equate a_i to 0, that is a linear equation with respect to the variables in V, and solve it. If the degree of $p > 1$, then see Notation 5.31.

Notation 5.31. *If the degree of* p *is greater than* 1*, then* a_i*, a factor of* E *in Algorithm* 5.29*, can be written as a polynomial with respect to Root*(p)*. In this case, we take the coefficients of* a_i *with respect to Root*(p) *instead of* a_i *itself. Therefore, in the output of Algorithm* 5.29*, the list* $[a_i,\, q]$ *becomes*

$$[\{a_{i,z},\, a_{i,z-1}, \ldots, a_{i,0}\}, q]$$

where z *is the degree of* a_i *with respect to Root*(p)*, and* $a_{i,j}$ *is the coefficient of Root*$(p)^j$*.*

Example 5.32. *Let* $P = (c_{1,1} + 2c_{2,1} + c_{2,2} + 1024c_{3,2})x + c_{1,0} - c_{1,2} + c_{2,1} - 2c_{2,2}$ *and* $p = x^2 - 1$*. If we substitute Root*(p) *into* P *and factor it over* $k[x]/(p)$*, we get only one factor which is*

$$a = (c_{1,1} + 2c_{2,1} + c_{2,2} + 1024c_{3,2})\mathit{Root}(p) + c_{1,0} - c_{1,2} + c_{2,1} - 2c_{2,2}.$$

Equating a *to* 0 *is similar to equate the coefficients of Root*(p) *to* 0*. Therefore, instead of taking* a*, we take*

$$\{c_{1,1} + 2c_{2,1} + c_{2,2} + 1024c_{3,2},\, c_{1,0} - c_{1,2} + c_{2,1} - 2c_{2,2}\}.$$

Example 5.33. *If we Apply algorithm* 5.30 *on* $\mathfrak{J}$, (5.17)*, and* **Sing** $= [x,\, x + \frac{5}{2},\, x - \frac{1}{4},\, \infty]$*, we get a set of* 11 *collections of* Q *linear combinations such that each collection has different solution.*

$$S_\infty = [[c_{1,1} + 16384c_{3,1}, 1], [c_{2,1}, 1], [c_{3,1}, 1]],$$

where its solution vanishes the coefficients of x^3*,* x^2*,* x*, so it moves the apparent singularities to* ∞*. The solutions of the other collections move the apparent singularities to* 0*, for example*

$$\begin{aligned} S_1 =& [[3c_{2,1} - 16384c_{3,1} + 4c_{1,0}, 1], [128c_{1,0} + 45c_{1,1} - 573440c_{3,1}, 1], \\ & [c_{1,0} - 20480c_{3,1}, 1]], \\ S_2 =& [[3c_{2,1} - 16384c_{3,1} + 4c_{1,0}, 0], [128c_{1,0} + 45c_{1,1} - 573440c_{3,1}, 0], [c_{1,0}, 1]] \end{aligned}$$

and

$$\begin{aligned} S_3 =& [[3c_{2,1} - 16384c_{3,1} + 4c_{1,0}, 0], [128c_{1,0} + 45c_{1,1} - 573440c_{3,1}, 0], \\ & [c_{1,0} + 8192c_{3,1}, 1]]. \end{aligned}$$

Note that, the first two linear combinations in S_1 *are equal to those in* S_2 *and* S_3*. Therefore, their qualities in* S_2 *and* S_3 *are replaced by* 0*.*

5.6.3 Computing the Best Choices of Linear Combinations of the Unknown Coefficients

Let $L \in \mathbb{C}(x)[\partial]$. Let $\mathfrak{J}$ be as in Section 5.6.1, of degree n with respect to x. For any singular point $p \in \mathbb{C} \cup \{\infty\}$ of L, Algorithm 5.30 finds collections of $\mathbb{Q}$-linear combinations such that the solution of each collection vanishes the coefficients of $(x-p)^0$, $(x-p)^1$, $\ldots$, $(x-p)^{n-1}$, and the coefficients of x^1, x^2, $\ldots$, x^n if $p = \infty$.

Example 5.34. *In Example* 5.33*, the solution of the linear combinations of each collection vanishes the coefficients of* x^0, x, x^2 *(moving the apparent singularities to* 0*) or vanishes the coefficients of* x, x^2, x^3 *(moving the apparent singularities to* ∞*). However, we can not use the same algorithm to find the collections, whose solutions vanish the coefficients of* x^0, x^1, x^3 *or* x^0, x^2, x^3 *(which means moving some of the apparent singularities to* 0 *and the rest of them to* ∞*).*

In this Section, we wrote the following algorithm which treats the cases of moving the apparent singularities to different true singularities.

Algorithm 5.35 (*All_solutions*)**.**

- ***Input:*** *The output of* All_Equations, *Algorithm* 5.30.
- ***Output:*** *All possible solutions of certain quality.*

The algorithm works as the following:

1. Let $S = [S_1, S_2, \ldots, S_j]$ be the output of *All_Equations*$(\mathfrak{J}, \mathbf{Sing})$, Algorithm 5.30, such that
$$S_i = [[a_{i,1}, q_{i,1}], [a_{i,2}, q_{i,2}] ..., [a_{i,l}, q_{1,l}]],$$
where $a_{i,s}$ is a linear combination of $c_1, \ldots, c_m$, and $q_{i,s}$ is its quality.
2. Let $d_i = \sum_{s=1}^{l} q_{i,s}$. Then take $d = \max\{d_1, \ldots, d_j\}$.
3. Take $M = \{S_{1,1}, \ldots, S_{1,l}, \ldots, S_{j,1}, \ldots, S_{j,l}\}$ such that
$$S_{i,s} = [[a_{i,1}, q_{i,1}], [a_{i,2}, q_{i,2}] ..., [a_{i,s}, q_{1,s}]], \text{ for } 1 \le i \le j \text{ and } 1 \le s \le l.$$
4. Take $F = \{\ \}$.
5. For each $h \in M$ do,
   (a) $W = \{a_{h,1}, \ldots, a_{h,s}\}$, and $d_w = \sum_{i=1}^{s} q_{h,i}$.

(b) If $d_w = d$ then $F = F \cup \{W\}$ and go next element in M.

(c) Else for each g in M do,

i. Take $H = W$ and $d_H = d_w$.

ii. Let z be denoted the number of the elements g.

iii. For s from 1 to z do

iv. If $a_{g,s}$ is not an element of H and $d_H < d$, then $H = H \cup \{a_{g,s}\}$ and $d_H = d_H + q_{g,s}$.

v. If $d_H \geq d$, then $F = F \cup \{H\}$ and go to next g.

6. Return the solutions of the collections that are in F.

Example 5.36. *In Example* 5.33, *applying Algorithm* 5.30 *on* $\mathfrak{J}$, (5.17), *and* **Sing** *of* L, *Example* 5.25, *give* 11 *collections. The solutions of* 10 *of the collections move the apparent singularities to* 0, *and then solution of one collection move the apparent singularities to* ∞. *Now, if we apply Algorithm* 5.35 *on the collections in Example* 5.33*, we get* 9 *more collections which move the apparent singularities to* 0 *and* ∞, *for example*

$$S = \{c_{1,0}, c_{1,1} + 16384c_{3,1}, c_{2,1} - 8192c_{3,1}, c_{2,1} - 8192c_{3,1} + c_{1,0}\}.$$

The solution of the linear combinations in S *is*

$$c_{1,0} = 0, \;\; c_{1,1} = -16384c_{3,1}, \;\; c_{2,1} = 8192c_{3,1}, \;\; c_{3,1} = c_{3,1},$$

and substituting this solution in $\mathfrak{J}$, (5.17), *gives that*

$$\mathfrak{J} = 0 \cdot x^3 + -140737488355328c_{3,1}^3 \cdot x^2 + 0 \cdot x + 0 \cdot x^0.$$

Also, substituting that solution in B*,* (5.16), *gives that*

$$L[B] = \partial^3 + \frac{1}{4x}\partial + \frac{3}{8x^2}\partial - \frac{8x+3}{8x^3}.$$

5.6.4 Phase II's Algorithm

Substituting the solutions of the linear combinations that are obtained from *Equations_have_ solutions_*$\mathbb{Q}$ on B gives $\{B_1, ..., B_i\}$, which is a set of gauge transformations. The gauge equivalent differential operators $L[B_1], ..., L[B_i]$ can be computed by using *Maple*'s commands

$$L[B_j] = \text{rightdivision}(\text{LCLM}(L, B_j), B_j)[1]. \tag{5.18}$$

If the output of *All_solutions*, Algorithm 5.35, is an empty list, then the singular points of L are not removable singularities.

Combining all of the algorithms in the previous sections gives the following algorithm:

Algorithm 5.37 (*Phase II*).

Input: $L \in \mathbb{Q}(x)[\partial]$ *of order* n*, and* $j \in \mathbb{N}$.

Output: $[[L[B_1], B_1], [L[B_2], B_2], ..., [L[B_i], B_i]]$*, where* $B_1, ..., B_i$ *are gauge transformations.*

- *Use* normalized_integral_basis *to compute a normalized integral basis* $b_1, b_2, ..., b_n$ *of* L.
- *Compute* B *as in* (5.10).
- *Compute* M *as in* (5.15)*, and compute* $\mathfrak{J}$*, which is the primitive part of the determinant of* M *with respect to the unknown coefficients.*
- *Take* $h = degree(\mathfrak{J}, x)$*, and find* **Sing** *of* L.
- *Let* $A = All\_Equation(J, \boldsymbol{Sing})$*, Algorithm* 5.30.
- *Use Algorithm* 5.35 *to compute the values of the unknown coefficients of quality* $\geq h$*, and substitute them in* B.
- *Return* $[[L[B_1], B_1], [L[B_2], B_2], ..., [L[B_i], B_i]]$.

CHAPTER 6

PHASE III: DIRECT DESCENT

Definition 6.1. *Let $L_2 \in k[\partial]$. (i)+(iii)-simplification of L_2 is a triplet (L_1, r, f), where $L_1 \in k[\partial]$, $r, f \in k$ with $f' \neq 0$, and order(L_1) =order(L_2), such that exp$(\int r) \cdot y(f)$ is a solution of L_2 for any solution $y(x)$ of L_1, (Notation $L_1 \xrightarrow{(i)+(iii)} L_2$). This simplification is called nontrivial if deg$(f) > 1$ (i.e. $\mathbb{C}(f) \neq \mathbb{C}(x)$) (i.e. f is not a Mobius function).*

Goal 6.2. *Our goal in this chapter is to find a nontrivial simplification (L_1, r, f) if it exists. We will compute f first. It suffices to compute $\mathbb{C}(f)$ because any generator $\tilde{f} = (af + b)/(cf + d)$ of $\mathbb{C}(f)$ also belongs to a nontrivial simplification $(\tilde{L}_1, \tilde{r}, \tilde{f})$.*

The definition of the relative invariants and some formulas of computing them are given in this chapter. Also, the relations between relative invariants and the change of variables transformation are studied in this chapter.

6.1 Relative Invariants of Differential Operators

Here, we give important terminologies and relative invariant formulas which are stated in [3, 4, 5]. We will give the definition of the semi-invariants of the first kind and the second kind and the relative invariants for a differential operator over $k[\partial]$ where $k = \mathbb{C}(x)$ and $\partial = \frac{d}{dx}$.

Let

$$c_0^{(0)}, ..., c_{n-1}^{(0)}, c_0^{(1)}, ..., c_{n-1}^{(1)}, ... \tag{6.1}$$

represent algebraic independent variables over the field $\mathbb{Q}$. Let $R_n = \mathbb{Q}[\{c_i^{(j)} | 0 \leq i \leq n-1, j = 0, 1, 2, ...\}]$, and let $'$ be the unique derivation, (Definition 2.1), on R_n such that

$$c_i^{(j)\prime} = c_i^{(j+1)}, \text{ for } 0 \leq i \leq n-1 \text{ and } j \geq 0,$$

and $c_i = c_i^{(0)}$.

Definition 6.3. *The weight of the variable $c_j^{(i)}$ is $i+(n-j)$. The weight of a monomial $c_{j_1}^{(i_1)} c_{j_2}^{(i_2)} ... c_{j_z}^{(i_z)}$ is the sum of the weight of the variables. A differential polynomial $\mathcal{I} \in R_n$ is called **isobaric** of weight m if the weight of the nonzero terms of $\mathcal{I}$ are m.*

Notation 6.4.

- *If* $a_0, a_1, .., a_{n-1} \in k$ *and* $\mathcal{I} \in R_n$, *then* $\mathcal{I}(a_0, a_1, ..., a_{n-1})$ *denotes the element of* k *obtained by replacing* $c_i^{(j)}$ *with* $a_i^{(j)}$.
- *Let* M_n *be the set of* ***monic*** *differential operators of order* n *in* $k[\partial]$.
- *If* $L = \partial^n + a_{n-1}\partial^{n-1} + ... + a_0 \in M_n$, *then* $\mathcal{I}(L)$ *denotes* $\mathcal{I}(a_0, a_1, ..., a_{n-1})$.

Example 6.5. $L = \partial^3 + a_2\partial^2 + a_1\partial + a_0 \in M_n$. *If*

$$\mathcal{I} = c_0c_1 + 2c_2''c_1 + 9c_1'c_1 + 15c_1'c_2' - 7c_2^{(4)} \in R_3,$$

then $\mathcal{I}$ *is an isobaric of weight* 5, *and*

$$\mathcal{I}(L) = a_0a_1 + 2a_2''a_1 + 9a_1'a_1 + 15a_1'a_2' - 7a_2^{(4)}.$$

Definition 6.6 (Definition 1.1 in [5])**.** *Let* $\mathcal{I}$ *as mentioned in Notation* 6.4. *We say that* $\mathcal{I}$ *is a* semi-invariant of the first kind *if* $\mathcal{I}$ *is nonconstant and*

$$\mathcal{I}(L) = \mathcal{I}(L \circledS (\partial - r)), \tag{6.2}$$

for any $L \in M_n$ *and any* $r \in k$. *In other words: invariant under* $\xrightarrow{(iii)}$ *transformation.*

Definition 6.7 (Definition 1.2 in [5])**.** *Let* $\mathcal{I}$ *be a differential polynomial as mentioned in Notation* 6.4. *Then we say that* $\mathcal{I}$ *is a* semi-invariant of the second kind *if* $\mathcal{I}$ *is nonconstant, and there is an integer* s, *which is called the* weight *of* $\mathcal{I}$, *such that*

$$\mathcal{I}(L_f) = (\mathcal{I}(L) \circ f)(f')^s, \tag{6.3}$$

for any $L \in M_n$ *and any nonconstant* $f \in k$, ($L \xrightarrow{(i)} L_f$, *see Notation* 2.23).

Theorem 6.8 (Theorem A.10, Proposition A.15 in [5])**.** *The semi-invariants of the second kind for any differential operators of order* $n \geq 2$ *are isobaric. Moreover, the weight in Definition* 6.3 *is equal to the number* s, *in Definition* 6.7.

Definition 6.9 (Definition 1.3 in [5])**.** *A differential polynomial* $\mathcal{I} \in R_n$ *is a* relative invariant *if it is a semi-invariant of both first and second kind.*

Proposition 6.10 (Proposition A.15 in [5]). *For linear differential operators of order* 1 *and* 2*, there are no relative invariants.*

For the proofs and more details reader can look at [5] pages $140 - 145$.

Definition 6.11. *Let $n \geq 3$. We call $\mathcal{Q} \in Frac(R_n) = \mathbb{Q}(\{c_i^{(j)} | 0 \leq i \leq n-1, j = 0, 1, 2, ...\})$ a* quotient invariant *of weight s if it satisfies* (6.2) *and* (6.3) *for any r, any nonconstant $f \in k$, and any $L \in M_n$ for which the denominator of $\mathcal{Q}$ does not vanish. If $s = 0$, $\mathcal{Q}$ is called an* absolute invariant*. An* absolute invariant *is quotient of two* relative invariants *of the same weight.*

Notation 6.12.

- *We will use $\mathcal{I}_k$ for invariants in R_n, $\mathcal{Q}_k$ for quotient invariants in $Frac(R_n)$, and $\mathcal{A}_k$ for absolute invariants.*
- *Let $\mathcal{A}$ be an absolute invariant. Then*

  $$\mathcal{A}(L_f) = (\mathcal{A}(L) \circ f)(f')^0,$$

  so

  $$\mathcal{A}(L_f) \in \mathbb{C}(f). \tag{6.4}$$

Comment 6.13. *Finding one element in $\mathbb{C}(f)$ is probably not enough to determine $\mathbb{C}(f)$, so for Goal* 6.2 *we will likely need to use* (6.4) *for multiple absolute invariants.*

6.2 Constructing Relative Invariants

In this section, we explained how to construct relative invariants for differential operators of order $n \geq 3$. Section 6.2 gives a construction for relative invariants from [5]. However, there are more relative invariants. Motivated by Comment 6.13 we give an algorithm to compute them in Section 6.2.1.

Constructing Relative Invariants by the Method from [5]. In [5], Chalkley constructed relative invariants of some weights for any monic differential operator in $k[\partial] = \mathbb{C}(x)[\frac{d}{dx}]$ of order $n \geq 3$. Let

$$L = \partial^n + c_{n-1}\partial^{n-1} + ... + c_1\partial + c_0 \in k[\partial],$$

the polynomial

$$a_2 = \frac{1}{\binom{n+1}{3}}(c_{n-2} - \frac{n-1}{2}\, c'_{n-1} - \frac{n-1}{2n}\, c^2_{n-1}) \tag{6.5}$$

is a semi-invariant of the first kind of weight 2, Proposition A.11 [5]. Also, polynomial

$$b_2 = \frac{1}{\binom{n+1}{3}}(c_{n-2} - \frac{n-2}{3}c'_{n-1} - \frac{(3n-1)(n-2)}{6n(n-1)}c^2_{n-1}) \tag{6.6}$$

is a semi-invariant of the second kind of weight 2, Proposition A.13 [5]. By introducing $b_1 = \frac{1}{\binom{n}{2}}c_{n-1}$, $K_{-1,j} = 0$, $K_{0,j} = 1$, and

$$K_{i+1,j} = \sum_{k=j+1}^{n} (K'_{i,k} - \frac{n-1}{2}b_1 K_{i,k} + (n-i-k+1)(1-i-k)a_2 K_{i-1,k}).$$

for any j and $i \geq 0$, Chalkley showed that if

$$L_i = \sum_{j=0}^{i} K_{i-j}c_{n-j}, \text{ for } 0 \leq i \leq n \text{ any } j,$$

then $L_3, ..., L_n$ are semi-invariants of the first kind. By taking

$$h_{m,i} = \frac{-i(n-i)}{(m-i)(m+i-1)}, \text{ and } \lambda_{m,i} = \frac{i(i-1)(n-i)(n-i+1)}{(m-i+1)(m+i-2)},$$

where $2 \leq i \leq m-1$,

$$P_{m,i} = 0, \text{ for } 3 \leq m \leq n, \text{ and } 1 \leq i \leq 2,$$

and

$$P_{m,i+1} = L_{i+1} + h_{m,i}P'_{m,i} + \lambda_{m,i}a_2 P_{m,i-1}, \tag{6.7}$$

for $3 \leq m \leq n$ and $2 \leq i \leq m-1$, the differential polynomials in $\{P_{i,i} |$ for $3 \leq i \leq n\}$ are semi-invariant of the second kind such that $P_{i,i}$ has weight i. Take

$$\mathcal{I}_m = P_{m,m}, \text{ for } 3 \leq m \leq n. \tag{6.8}$$

Theorem 6.14 (Theorems 9.1-C.5 [5])**.** *Let L be a monic differential operator of order $n \geq 3$ over $k[\partial]$. Then, the polynomials $\mathcal{I}_3, \mathcal{I}_4, ..., \mathcal{I}_n$, in* (6.8)*, are relative invariants for L, and the weight of $\mathcal{I}_i$ is i.*

Example 6.15. *Let $L = \partial^3 + a_2\partial^2 + a_1\partial + a_0$. Then, Theorem* 6.14 *gives only one invariant which is*

$$\mathcal{I}_3 = a_0 - \frac{1}{3}a_2a_1 + \frac{2}{27}a_2^3 - \frac{1}{2}a'_1 + \frac{1}{3}a_2a'_2 + \frac{1}{6}a''_2. \tag{6.9}$$

Now, let $L = \partial^4 + a_3\partial^3 + a_2\partial^2 + a_3\partial + a_4$*. Then Theorem* 6.14 *gives two invariants which are*

$$\mathcal{I}_3 = a_1 - \frac{1}{2}a_3a_2 + \frac{1}{8}a_3^3 - a_2' + \frac{3}{4}a_3a_3' + \frac{1}{2}a_3''' \tag{6.10}$$

and

$$\begin{aligned}\mathcal{I}_4 = a_0 - \frac{1}{4}a_3a_1 + \frac{13}{100}a_3^2a_2 - \frac{9}{100}a_2^2 - \frac{1}{20}a_3''' - \frac{33}{200}(a_3')^2 - \frac{3}{20}a_3a_3'' + \frac{1}{5}a_2'' \\ - \frac{39}{200}a_3^2a_3' + \frac{27}{100}a_3'a_2 - \frac{39}{1600}a_3^4 + \frac{1}{4}a_3a_2' - \frac{1}{2}a_1'.\end{aligned} \tag{6.11}$$

6.2.1 Constructing Relative Invariants by Using Computer Algebra

Here we wrote an algorithm which computes relative invariants of any weight m (if it exist) for differential operators of order $n \geq 3$.

Remark 6.16. *Let* $L \in M_n$*. Definition* 6.6 *shows that*

$$\mathcal{I}_i(L) = \mathcal{I}_i(L \circledS (\partial - r)),$$

so

$$\mathcal{I}_i(L) = \mathcal{I}_i(NF(L)),$$

where $NF(L)$ *is the normal form of* L*, Definition* 2.21*. Therefore, it suffices to construct relative invariants of the normal form of* L*.*

Algorithm 6.17 (Invariants)**.**

Input: n, s *are integer number* ≥ 3*.*

Output: *Relative invariants of weight* s*.*

- *Let* $L = \partial^n + c_{n-2}\partial^{n-2} + ... + c_0$*.*
- *Compute* $NF(L_f)$*, see Definition* 2.21 *and Notation* 2.23*, so*

$$NF(L_f) = \partial^n + a_{n-2}(x)\partial^{n-2} + ... + a_0(x)$$

  and $L \xrightarrow{(i)+(iii)} NF(L_f)$*.*
- *The weight of* $c_{n-i}^{(j)}$ *is* $i+j$*. Now, let* $M = \{m_1, m_2, ..., m_N\}$ *be the set of all products of* $c_{n-i}^{(j)}$*'s with weight* s*.*
- *Write the ansatz*

$$\mathcal{I} = C_1m_1 + ... + C_Nm_N$$

- *Substitute $c_i(x) = a_i(x)$ in $\mathcal{I}$ to construct $\mathcal{I}(NF(L_f))$.*
- *Let F be the numerator of $(\mathcal{I}(\mathcal{L}) \circ f)(f')^s - \mathcal{I}(NF(L_f))$.*
- *Let H to be the set of the coefficients of F (viewed as a polynomial with variables $\{f^{(j)}, (c_i(f))^{(j)}\}$). Here H will be a set of $\mathbb{Q}$ linear combinations of $C_1, C_2, ..., C_N$.*
- *Equate the elements of H to zero, find a basis of solutions of H, substitute each in $\mathcal{I}$, and return as output.*

Example 6.18. *If $m = 3$ and $n = 3$, then $L = \partial^3 + c_1\partial + c_0$. The change of variable transformation $x \mapsto f$ sends L to*

$$L_f = \partial^3 - \frac{3f''}{f'}\partial^2 + \frac{c_1(f)(f')^4 - f'''f' + 3(f'')^2}{(f')^2}\partial + (f')^3 c_0(f),$$

and the normal form of L_f is

$$\begin{aligned} NF(L_f) = & \partial^3 + \frac{c_1(f)(f')^4 + 2f'''f' - 3(f'')^2}{(f')^2}\partial \\ & \frac{(f')^6 c_0(f) + c_1(f)(f')^4 f'' - 4f'''f''f' + f^{(4)}(f'')^2 + 3(f'')^3}{(f')^3}. \end{aligned}$$

Now $M = \{c_0, c_1'\}$ is the set of the monomials of weight 3. Write the ansatz $\mathcal{I} = C_1 c_0 + C_2 c_1'$, so $\mathcal{I}_f = C_1 a_0 + C_2 a_1'$ where a_0, a_1 are the coefficient of $NF(L_f)$. Now, compute

$$\begin{aligned} F &= \textit{numerator}((f')^3(\mathcal{I}(L) \circ f) - \mathcal{I}(NF(L_f)) \\ &= -(C_1 + 2C_2)c_1(f)(f')^4 f'' - (C_1 + 2C_2)(f')^2 f^{(4)} + 4(C_1 + 2C_2)f'f''f''' \\ &- 3(C_1 + 2C_2)C_1(f'')^3. \end{aligned}$$

Then, $H = \{-(C_1 + 2C_2), -3(C_1 + 2C_2), 4(C_1 + 2C_2)\}$, which has $(1, \frac{-1}{2})$ as a solution basis, and

$$\mathcal{I}_3 = c_0 - \frac{1}{2}c_1'.$$

Remark 6.19. *Let $L \in M_3$. Then $\mathcal{I}_3(L) = 0$ if and only if there is $L_2 \in k[\partial]$ of order 2 such that $L = L^{\circledS 2}$, see [30]. In this case we use algorithms in [12, 15, 22, 34] for solving second order differential equations and not use any of what follows.*

6.2.2 Differential Operators of Order 3

For any monic differential operator in M_3 of this form $\partial^3 + c_1\partial + c_0$, the algorithm constructs all relative invariants for any weight. Here are all relative invariants of weight ≤ 11,

$$\mathcal{I}_3 = c_0 - \frac{1}{2}c_1', \tag{6.12}$$

which is equal to (6.9) when $a_2 = 0$,

$$\mathcal{I}_6 = (\mathcal{I}_3)^2, \tag{6.13}$$

$$\begin{aligned}\mathcal{I}_8 =& c_1(c_0 - \frac{1}{2}c_1')^2 + \frac{7}{9}(c_0')^2 + \frac{7}{36}(c_1'')^2 - \frac{2}{3}c_0c_0'' + \frac{1}{3}c_0c_1''' \\ &- \frac{7}{9}c_0'c_1'' + \frac{1}{3}c_1'c_0'' - \frac{1}{6}c_1'c_1''',\end{aligned} \tag{6.14}$$

$$\mathcal{I}_9 = (\mathcal{I}_3)^3, \tag{6.15}$$

$$\begin{aligned}\mathcal{I}_{10} =& -72(c_0'')^2 - 18(c_1^{(3)})^2 - 288c_1^2c_0^2 + \frac{185}{2}c_1c_1'c1^{(3)} - \frac{7}{2}c_1''c_0c_1' - 42c_0^{(3)}c_1'' \\ &- 420c_1(c_0')^2 + 72c_0''c_1^{(3)} + 7c_0c_1^{(5)} + 7c_0^{(4)}c_1' - 42c_0'c_1^{(4)} + 420c_1c_0'c_1'' \\ &- 105c_1(c_1'')^2 - 14c_0c_0^{(4)} - \frac{7}{2}c_1'c_1^{(5)} + 21c_1''c_0^2 - \frac{7}{2}c_1^2(c_1')^2 + \frac{35}{2}c_0'(c_1')^2 \\ &- \frac{7}{2}c_1''(c_1')^2 + \frac{35}{2}c_0'(c_1')^2 + 84c_0'c_0^{(3)} + 21c_1''c_1^{(4)} \\ &- 185c_1c_1'c_0'' - 35c_0'c_0c_1' - 185c_1c_0c_1^{(3)} + 370c_1c_0c_0'' - \frac{7}{2}c_1^2c_0c_1',\end{aligned} \tag{6.16}$$

and

$$\mathcal{I}_{11} = \mathcal{I}_3\mathcal{I}_8. \tag{6.17}$$

$\mathcal{I}_6, \mathcal{I}_9, \mathcal{I}_{11}$ are not useful because they are product of prior invariants. The remaining invariants grow quickly when increasing the weight m. To find shorter formula for invariants, we will use quotient invariants as well as the following lemmas.

Lemma 6.20. *We assume $\mathcal{I}_3 \neq 0$, see Remark 6.19. Let $\mathcal{Q}_2 = \mathcal{I}_8/\mathcal{I}_3^2$. Then c_0, c_1 can be written in terms of $\mathcal{I}_3$ and $\mathcal{Q}_2$.*

Proof. From (6.12), it is clear that

$$c_0 = \mathcal{I}_3 + \frac{1}{2}c_1'. \tag{6.18}$$

Substituting (6.18) makes (6.14) much smaller

$$\mathcal{I}_8 = c_1\mathcal{I}_3^2 + \frac{7}{9}(\mathcal{I}_3')^2 - \frac{2}{3}\mathcal{I}_3\mathcal{I}_3''. \tag{6.19}$$

Then,

$$\begin{aligned} c_1 &= \frac{\mathcal{I}_8}{\mathcal{I}_3^2} + \frac{2}{3}\frac{\mathcal{I}_3''}{\mathcal{I}_3} - \frac{7}{9}\frac{(\mathcal{I}_3')^2}{\mathcal{I}_3^2} \\ &= \mathcal{Q}_2 + \frac{2}{3}\frac{\mathcal{I}_3''}{\mathcal{I}_3} - \frac{7}{9}\frac{(\mathcal{I}_3')^2}{\mathcal{I}_3^2} \end{aligned} \tag{6.20}$$

□

Substituting (6.18) and (6.20) often makes invariants much shorter. For instance (6.16) becomes

$$\mathcal{I}_{10} = 21\mathcal{I}_8'' + \frac{224\mathcal{I}_8(\mathcal{I}_3')^2}{\mathcal{I}_3^2} - \frac{119\mathcal{I}_8'\mathcal{I}_3'}{\mathcal{I}_3} - \frac{288\mathcal{I}_8^2}{\mathcal{I}_3^2} - \frac{56\mathcal{I}_8\mathcal{I}_3''}{\mathcal{I}_3} \tag{6.21}$$

Lemma 6.21. *Let $\mathcal{I}_{m_1}, \mathcal{I}_{m_2} \in k_n$ be (quotient) invariants of weight m_1 and m_2 respectively. Then*

$$m_2\mathcal{I}_{m_1}'\mathcal{I}_{m_2} - m_1\mathcal{I}_{m_2}'\mathcal{I}_{m_1}$$

is a (quotient) invariant of weight $m_1 + m_2 + 1$.

Proof. It is a direct proof. □

By using the above lemmas we can find shorter formulas for quotient invariants. For example, (6.21) can be further shortened to

$$\mathcal{I}_{10} = 7\mathcal{Q}_6'\mathcal{I}_3 - 14\mathcal{Q}_6\mathcal{I}_3' - 288\mathcal{Q}_2^2\mathcal{I}_3^2. \tag{6.22}$$

where $\mathcal{Q}_6 = 3\mathcal{Q}_2'\mathcal{I}_3 - 2\mathcal{Q}_2\mathcal{I}_3'$ is an example of Lemma 6.21.

The relative invariants of weight 12, 13, 14 and 15, computed by our algorithm, are

$$\mathcal{I}_{12,1} = 8\mathcal{I}_8\mathcal{I}_3' - 3\mathcal{I}_8'\mathcal{I}_3, \quad \mathcal{I}_{12,2} = \mathcal{I}_3^4, \quad \mathcal{I}_{12,3} = \text{ too large to write here,} \tag{6.23}$$

$$\mathcal{I}_{13} = \mathcal{I}_3\mathcal{I}_{10}, \tag{6.24}$$

$$\mathcal{I}_{14,1} = 10\mathcal{I}_{10}\mathcal{I}_3' - 3\mathcal{I}_{10}'\mathcal{I}_3, \quad \mathcal{I}_{14,2} = \mathcal{I}_3^2\mathcal{I}_8, \quad \mathcal{I}_{14,3} = \text{ too large to write here,} \tag{6.25}$$

$$\mathcal{I}_{15,1} = \mathcal{I}_3^5, \quad \mathcal{I}_{15,2} = \mathcal{I}_3\mathcal{I}_{12,1}, \quad \mathcal{I}_{15,3} = 8\mathcal{I}_8\mathcal{I}_6' - 6\mathcal{I}_8'\mathcal{I}_6, \tag{6.26}$$

and $\mathcal{I}_{15,4} =$ too large to write here.

We can construct many relative invariants and take the quotient of the same weight invariants to obtain absolute invariants. After applying number of absolute invariants to some examples, the absolute invariants that lead to the smallest expression size were:

$$\mathcal{A}_1 = \frac{\mathcal{Q}_2^3}{\mathcal{I}_3^2}, \quad \mathcal{A}_2 = \frac{\mathcal{Q}_6}{\mathcal{I}_3^2}, \quad \mathcal{A}_3 = \frac{\mathcal{I}_{10}\mathcal{I}_3^2}{\mathcal{I}_8^2}, \quad \mathcal{A}_4 = \frac{\mathcal{I}_{12,1}}{\mathcal{I}_{12,2}}. \tag{6.27}$$

Remark 6.22. *For any $L \in M_3$ if a nontrivial simplification (L_1, r, f) exist then (as mentioned in Comment 6.13) it is almost never the case that $\mathbb{C}(A(L))$ equal $\mathbb{C}(f)$, but it is always the case that $\mathbb{C}(A_1(L), A_2(L))$ equal $\mathbb{C}(f)$. Then adding $A_3(L), A_4(L)$ does not make a difference, although we can easily give even more absolute invariants $A_5(L), A_6(L), \ldots$ we have not seen a reason to add so.*

6.2.3 Differential Operators of Order 4

For differential operators in M_4 of this form $\partial^4 + c_2\partial^2 + c_1\partial + c_0$, here are all relative invariants of weight ≤ 8

$$\mathcal{I}_3 = c_1 - c_2', \tag{6.28}$$

$$\mathcal{I}_4 = c_0 - \frac{1}{2}c_1' + \frac{1}{5}c_2'' - \frac{9}{100}c_2^2, \tag{6.29}$$

which are equal to (6.10), (6.11) when $a_3 = 0$,

$$\mathcal{I}_6 = \mathcal{I}_3^2, \tag{6.30}$$

$$\mathcal{I}_7 = \mathcal{I}_3\mathcal{I}_4, \tag{6.31}$$

$$\begin{aligned}\mathcal{I}_{8,1} =& c_2(c_1 - c_2')^2 - \frac{5}{3}c_1c_1'' + \frac{5}{3}c_1c_2''' + \frac{5}{3}c_2'c_1'' - \frac{5}{3}c_2'c_2''' \\ &+ \frac{35}{18}(c_1')^2 - \frac{35}{9}c_1'c_2'' + \frac{35}{18}(c_2'')^2,\end{aligned} \tag{6.32}$$

$\mathcal{I}_{8,2} = 4\mathcal{I}_4\mathcal{I}_3' - 3\mathcal{I}_3\mathcal{I}_4'$, and $\mathcal{I}_{8,3} = \mathcal{I}_4^2$.

Our algorithm can easily computes these, as well as higher weights, but the expressions grow. We will use the following lemma to write the invariants of higher weight in terms of smaller invariants.

Lemma 6.23. *If $\mathcal{I}_3 \neq 0$, then c_0, c_1, c_2 can be written in terms of $\mathcal{I}_3, \mathcal{I}_4$, and $\mathcal{I}_{8,1}$.*

Proof. From (6.28), it is clear that

$$c_1 = \mathcal{I}_3 + c_2', \tag{6.33}$$

and by substituting (6.33) in (6.29) we get

$$c_0 = \mathcal{I}_4 + \frac{9}{100}c_2^2 + \frac{1}{2}\mathcal{I}_3' + \frac{3}{10}c_2''. \tag{6.34}$$

Now, by substituting (6.33) in $\mathcal{I}_{8,1}$, (6.32), it becomes

$$\mathcal{I}_{8,1} = c_2\mathcal{I}_3^2 - \frac{5}{3}\mathcal{I}_3\mathcal{I}_3'' + \frac{35}{18}(\mathcal{I}_3')^2. \tag{6.35}$$

Therefore,

$$
\begin{aligned}
c_2 &= \frac{\mathcal{I}_{8,1}}{\mathcal{I}_3^2} + \frac{5}{3}\frac{\mathcal{I}_3''}{\mathcal{I}_3} - \frac{35}{18}\frac{(\mathcal{I}_3')^2}{\mathcal{I}_3^2} \\
&= \mathcal{Q}_2 + \frac{5}{3}\frac{\mathcal{I}_3''}{\mathcal{I}_3} - \frac{35}{18}\frac{(\mathcal{I}_3')^2}{\mathcal{I}_3^2},
\end{aligned}
\tag{6.36}
$$

where $\mathcal{Q}_2 = \mathcal{I}_{8,1}/\mathcal{I}_3^2$. $\square$

By using the lemma above we can write relative invariants in terms of $\mathcal{I}_3, \mathcal{I}_4$, and $\mathcal{Q}_2$ or $\mathcal{I}_{8,1}$. The relative invariants, computed by our algorithm, of the weight $9, 10, 11$, and 12 are

$$\mathcal{I}_{9,1} = c_2\mathcal{I}_3\mathcal{I}_4 - \frac{35}{64}\mathcal{I}_3\mathcal{I}_4'' + \frac{105}{64}\mathcal{I}_3'\mathcal{I}_4' - \frac{15}{16}\mathcal{I}_4\mathcal{I}_3'',\ \mathcal{I}_{9,2} = \mathcal{I}_3^3, \tag{6.37}$$

$$\mathcal{I}_{10,1} = 3\mathcal{I}_{8,1}'' + 32\mathcal{Q}_2(\mathcal{I}_3')^2 - 8\mathcal{Q}_2\mathcal{I}_3\mathcal{I}_3'' - 17\frac{\mathcal{I}_{8,1}'\mathcal{I}_3'}{\mathcal{I}_3}, \quad \mathcal{I}_{10,3} = \mathcal{I}_4\mathcal{I}_3^2, \tag{6.38}$$

$$\mathcal{I}_{10,2} = \mathcal{I}_4\mathcal{I}_4'' - \frac{4}{5}c_2\mathcal{I}_4^2 - \frac{9}{8}(\mathcal{I}_4')^2, \tag{6.39}$$

$$\mathcal{I}_{10,4} = \frac{113}{5}c_2\mathcal{I}_3\mathcal{I}_4' - \frac{17}{10}c_2'\mathcal{I}_3\mathcal{I}_4 - 29c_3\mathcal{I}_4\mathcal{I}_3' - \frac{7}{2}\mathcal{I}_3\mathcal{I}_4''' - \frac{45}{2}\mathcal{I}_4'\mathcal{I}_3'' + \frac{35}{2}\mathcal{I}_3'\mathcal{I}_4'' + \frac{15}{2}\mathcal{I}_4\mathcal{I}_3'''. \tag{6.40}$$

$$\mathcal{I}_{11,i} = \mathcal{I}_3\mathcal{I}_{8,i},\ \text{for } i = 1, 2, 3, \tag{6.41}$$

$$
\begin{aligned}
\mathcal{I}_{11,4} =& 48c_2^2\mathcal{I}_3\mathcal{I}_4 + \frac{257}{20}c_2'\mathcal{I}_3\mathcal{I}_4' - \frac{263}{5}c_2\mathcal{I}_3\mathcal{I}_4'' - \frac{83}{10}c_2''\mathcal{I}_3\mathcal{I}_{34} - \frac{253}{3}c_2\mathcal{I}_4\mathcal{I}_3''' \\
&+ \frac{4477}{30}c_2\mathcal{I}_3'\mathcal{I}_4' - \frac{33}{10}c_2'\mathcal{I}_4\mathcal{I}_3' + \frac{7}{2}\mathcal{I}_3\mathcal{I}_4^{(4)} + 55\mathcal{I}_3''\mathcal{I}_4'' - \frac{77}{3}\mathcal{I}_3'\mathcal{I}_4''' \\
&- \frac{165}{4}\mathcal{I}_4'\mathcal{I}_3''' + \frac{55}{6}\mathcal{I}_4'\mathcal{I}_3'''',
\end{aligned}
\tag{6.42}
$$

$$\mathcal{I}_{12,1} = 12\mathcal{I}_3\mathcal{I}_3''\mathcal{I}_4 + 27\mathcal{I}_4'\mathcal{I}_3'\mathcal{I}_3 - 9\mathcal{I}_4''\mathcal{I}_3^2 - 32(\mathcal{I}_3')^2\mathcal{I}_4, \tag{6.43}$$

$$\mathcal{I}_{12,i+1} = 8\mathcal{I}_3\mathcal{I}_{8,i} - 3\mathcal{I}_3\mathcal{I}_{8,i}',\ \text{for } i = 1, 2, 3, \tag{6.44}$$

$$\mathcal{I}_{12,i+4} = \mathcal{I}_4\mathcal{I}_{8,i} \text{ for } i = 1, 2, 3, \tag{6.45}$$

$$\mathcal{I}_{12,8} = 4\mathcal{I}_4\mathcal{I}_7' - 7\mathcal{I}_7\mathcal{I}_4',\ \text{and } \mathcal{I}_{12,9} = \mathcal{I}_3^4. \tag{6.46}$$

We applied various absolute invariants to some examples, and we found that if $\mathcal{I}_3 \neq 0$, the absolute invariants that lead to the smallest expression size were:

$$\mathcal{A}_1 = \frac{\mathcal{I}_4^3}{\mathcal{I}_3^4}, \quad \mathcal{A}_2 = \frac{\mathcal{I}_{8,1}^3}{\mathcal{I}_3^8}, \quad \mathcal{A}_3 = \frac{\mathcal{I}_{8,1}}{\mathcal{I}_{8,3}} \quad \mathcal{A}_4 = \frac{\mathcal{Q}_2^2}{\mathcal{I}_4}, \quad \mathcal{A}_5 = \frac{\mathcal{Q}_6}{\mathcal{I}_3^2}, \tag{6.47}$$

where $\mathcal{Q}_6 = 2\mathcal{Q}_2\mathcal{I}_3' - 3\mathcal{I}_3\mathcal{Q}_2'$.

For differential operators $L \in M_4$, if $\mathcal{I}_3 = 0$, all but one of the absolute invariants in (6.47) are undefined, so we need to treat that case separately.

6.2.4 Differential operators of Order 4, Case $\mathcal{I}_3 = 0$

If $\mathcal{I}_3 = 0$, then $c_1 = c_2'$. Then $\mathcal{I}_4$ in (6.29) reduces to:

$$\widetilde{\mathcal{I}}_4 = c_0 - \frac{9}{100}c_2^2 - \frac{3}{10}c_2''. \tag{6.48}$$

From (6.48), it is clear that

$$c_0 = \widetilde{\mathcal{I}}_4 + \frac{9}{100}c_2^2 + \frac{3}{10}c_2''. \tag{6.49}$$

Notation 6.24. *The expression $\widetilde{\mathcal{I}}_4$ in (6.48) does not satisfy Definition 6.9 for every $L \in M_4$. It satisfies (6.2), (6.3) only for $\{L \in M_4 |\ \mathcal{I}_3(L) = 0\}$. We call such $\widetilde{\mathcal{I}}_4$ a $\mathcal{I}_3^0$-invariant and use $\widetilde{\mathcal{I}}_n$ to indicate them.*

Notation 6.25. *Let $L \in M_4$. If $\mathcal{I}_3(L) = \widetilde{\mathcal{I}}_4(L) = 0$, then $L = L_2^{\circledS 3}$, (see e.g. [29]), for some $L_2 \in M_2$. In this case we do not use the remainder of this section, instead, we can use solvers for second order equations [12, 15, 22, 34].*

In the remainder of this section we assume $\widetilde{\mathcal{I}}_4 \neq 0$ and $\mathcal{I}_3 = 0$, (so $c_1 = c_2'$). Equations (6.38) – (6.40) gave 4 relative invariants of weight 10, three of which vanish when $c_1 = c_2'$ and equation (6.39) becomes a constant times:

$$\widetilde{\mathcal{I}}_{10} = c_2\widetilde{\mathcal{I}}_4^2 - \frac{5}{4}\widetilde{\mathcal{I}}_4\widetilde{\mathcal{I}}_4'' + \frac{45}{32}(\widetilde{\mathcal{I}}_4')^2. \tag{6.50}$$

Lemma 6.26. *If $\mathcal{I}_3 = 0$ and $\widetilde{\mathcal{I}}_4 \neq 0$, then c_0, c_1, c_2 can be written in terms of $\widetilde{\mathcal{I}}_4$ and $\widetilde{\mathcal{Q}}_2 = \widetilde{\mathcal{I}}_{10}/\widetilde{\mathcal{I}}_4^2$, see (6.49), $c_1 = c_2'$, and:*

$$c_2 = \widetilde{\mathcal{Q}}_2 + \frac{5}{4}\frac{\widetilde{\mathcal{I}}_4''}{\widetilde{\mathcal{I}}_4} - \frac{45}{32}\frac{(\widetilde{\mathcal{I}}_4')^2}{\widetilde{\mathcal{I}}_4^2}. \tag{6.51}$$

There are two linearly independent $\mathcal{I}_3^0$-invariant of weight 12 and 14, and one $\mathcal{I}_3^0$-invariant of weight 15. They can be written in short forms by substituting (6.49), (6.51) in them. Then they become

$$\widetilde{\mathcal{I}}_{12,1} = \widetilde{\mathcal{I}}_4^3,$$

$$\widetilde{\mathcal{I}}_{12,2} = \widetilde{\mathcal{Q}}_2(\widetilde{\mathcal{I}}_4')^2 + \frac{3}{16}\widetilde{\mathcal{Q}}_2'\widetilde{\mathcal{I}}_4'\widetilde{\mathcal{I}}_4 - \frac{21}{160}\widetilde{\mathcal{Q}}_2(\widetilde{\mathcal{I}}_4')^2 + \frac{3}{40}\widetilde{\mathcal{Q}}_2\widetilde{\mathcal{I}}_4''\widetilde{\mathcal{I}}_4 - \frac{3}{20}\widetilde{\mathcal{Q}}_2''\widetilde{\mathcal{I}}_4^2, \tag{6.52}$$

$$\widetilde{\mathcal{I}}_{14,1} = \widetilde{\mathcal{I}}_4\widetilde{\mathcal{I}}_{10}, \tag{6.53}$$

and

$$
\begin{aligned}
\widetilde{\mathcal{I}}_{14,2} =& 23040 c_2^3 \widetilde{\mathcal{I}}_4^2 + 88816 c_2^2 (\widetilde{\mathcal{I}}_4')^2 + 10640 c_2 c_2' \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{I}}_4' - 81312 c_2^2 \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{I}}_4'' - 10176 c_2 c_2'' \widetilde{\mathcal{I}}_4^2 \\
&+ 2080 (c_2')^2 \widetilde{\mathcal{I}}_4^2 + 12600 c_2 \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{I}}_4'''' - 12285 c_2'' (\widetilde{\mathcal{I}}_4')^2 - 66430 c_2 \widetilde{\mathcal{I}}_4' \widetilde{\mathcal{I}}_4''' \\
&+ 6825 c_2' \widetilde{\mathcal{I}}_4' \widetilde{\mathcal{I}}_4'' - 840 c_2''' \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{I}}_4' - 5740 c_2' \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{I}}_4''' + 54600 c_2 (\widetilde{\mathcal{I}}_4'')^2 + 11760 c_2'' \widetilde{\mathcal{I}}'' \widetilde{\mathcal{I}} \\
&+ 240 c_2^{(4)} \widetilde{\mathcal{I}}_4^2 - 9750 \widetilde{\mathcal{I}}_4'' \widetilde{\mathcal{I}}_4^{(4)} + 2925 \widetilde{\mathcal{I}}_4' \widetilde{\mathcal{I}}_4^{(5)} - 300 \widetilde{\mathcal{I}}_4' \widetilde{\mathcal{I}}_4^{(6)} + 7150 (\widetilde{\mathcal{I}}_4''')^2.
\end{aligned}
\tag{6.54}
$$

$$
\widetilde{\mathcal{I}}_{15} = \widetilde{\mathcal{I}}_4^2 (2 \widetilde{\mathcal{I}}_4 \widetilde{\mathcal{Q}}_2' - \widetilde{\mathcal{Q}}_2 \widetilde{\mathcal{I}}_4'). \tag{6.55}
$$

We applied various $\mathcal{I}_3^0$-absolute invariants to some examples. The $\mathcal{I}_3^0$-absolute invariants that lead to the smallest expression size were:

$$
\widetilde{\mathcal{A}}_1 = \frac{\widetilde{\mathcal{Q}}_2^2}{\widetilde{\mathcal{I}}_4}, \quad \widetilde{\mathcal{A}}_2 = \frac{\widetilde{\mathcal{I}}_{12}}{\widetilde{\mathcal{I}}_4^3}, \quad \widetilde{\mathcal{A}}_3 = \frac{\widetilde{\mathcal{I}}_{15}^2}{\widetilde{\mathcal{I}}_{10}^3}, \quad \widetilde{\mathcal{A}}_4 = \frac{\widetilde{\mathcal{I}}_{14,2}}{\widetilde{\mathcal{I}}_{14,1}}. \tag{6.56}
$$

Comment 6.27. *The same methods, in the above sections, can be applied to differential operators of higher order.*

6.3 Computing Change of Variables Transformation up to a Mobius Transformation

Let $L \in M_n$ where $n \geq 3$. To achieve Goal 6.2, we have to compute $f(x)$ first. It suffices to compute $f(x)$ up to Mobius transformation $(af(x)+b)/(cf(x)+d)$, so it suffices to compute $\mathbb{C}(f(x))$. For this, we extend the method in [7] with the absolute invariants that we constructed in the previous sections.

Notation 6.12 shows that each absolute invariant gives an element $\mathcal{A}(L) \in \mathbb{C}(f(x))$. That means

$$
\mathcal{A}(L) = h \circ f(x),
$$

for some $h \in \mathbb{C}(x)$, i.e. $f(x)$ is a decomposition factor of $\mathcal{A}(L)$. If we have only one absolute invariant $\mathcal{A}$, then to find $f(x)$ we need an algorithm (e.g. [1]) to find all decompositions of $\mathcal{A}(L)$. It is better (see Comment 6.13) to have multiple absolute invariants. Our set of absolute invariants generated $\mathbb{C}(f(x))$ in all examples tested thus far. This allows us to find $f(x)$ without [1].

Definition 6.28. *Let $g(x) \in \mathbb{C}(x)$. Define*

$$
\Delta(g) = \mathit{numerator}(g(x) - g(t)) \in \mathbb{C}[x,t] \tag{6.57}
$$

where t is a new variable.

Lemma 6.29 (Proposition 3.1 in [2])**.** *Let $g(x), f(x) \in \mathbb{C}(x)$. Then $\mathbb{C}(g(x)) \subseteq \mathbb{C}(f(x))$ if and only if $\Delta(g(x))$ is divisible by $\Delta(f(x))$.*

Suppose that $L \in \mathbb{C}(x)[\partial]$ of order $n \geq 3$. To compute $f(x)$, we follow these steps: First, compute[i]

$$\mathcal{A}_1(L), ..., \mathcal{A}_m(L) \in \mathbb{C}(f(x)), \tag{6.58}$$

where $\mathcal{A}_1, ..., \mathcal{A}_m$ as in (6.27), (6.47), or (6.56).

Now, compute

$$g(x,t) = \gcd(\Delta(\mathcal{A}_1(L)), ..., \Delta(\mathcal{A}_m(L))) \in \mathbb{C}[x,t]. \tag{6.59}$$

We will show how to compute $h(x) \in \mathbb{C}(x)$ for which $\Delta(h(x)) = g(x,t)$. Lemma 6.29 shows that

$$\mathbb{C}(\mathcal{A}_1(L), ..., \mathcal{A}_m(L)) = \mathbb{C}(h(x)) \subseteq \mathbb{C}(f(x)). \tag{6.60}$$

Now, by using Definition 6.28

$$\begin{aligned} \Delta(h(x)) &= \text{numerator}(h(x) - h(t)) \\ &= \sum_i G_i(x) t^i \\ &= p(x)q(t) - p(t)q(x), \end{aligned} \tag{6.61}$$

where $p(x)$ and $q(x)$ are relatively prime.

Each polynomial $G_i(x)$ in (6.61) is a linear combination of $p(x)$ and $q(x)$. Since $p(x)$ and $q(x)$ are relatively prime, the quotient $G_i(x)/G_j(x)$ is constant or nonconstant fractional linear combination of $p(x)$ and $q(x)$ such that

$$\frac{G_i(x)}{G_j(x)} = \frac{ap(x) + bq(x)}{cp(x) + dq(x)} = \frac{a\frac{p(x)}{q(x)} + b}{c\frac{p(x)}{q(x)} + d} = \frac{ah(x) + b}{ch(x) + d},$$

where $a, b, c, d \in \mathbb{C}$. Therefore, each nonconstant quotient $G_i(x)/G_j(x)$ is Mobius transformation T_i composed with $h(x)$, so we computed $x \rightarrow T_i(h(x)) = f(x)$.

Algorithm 6.30 (FindDescent)**.**

- ***Input****:* $L = \partial^n + c_{n-2}\partial^{n-2} + ... + c_0 \in M_{n \geq 3}$ *(an operator in normal form.)*
- ***Output****: A candidate for the change of variable transformation, Section* 2.4.

[i] To do thus, we first need to compute $NF(L)$, see Remark 6.16.

- *If* $n = 3$,
  - *If* $\mathcal{I}_3(L) = 0$*, return "*L *can be reduced to an operator of order* 2."
  - *If* $\mathcal{I}_3(L) \neq 0$*, we compute* $\mathcal{A}_i(L)$ *for each absolute invariant* $\mathcal{A}_i$ *in* (6.27).
- *If* $n = 4$,
  - *If* $\mathcal{I}_3(L) = \mathcal{I}_4(L) = 0$*, return "*$L$ *can be reduced to an operator of order* 2."
  - *If* $\mathcal{I}_3(L) = 0$ *and* $\widetilde{\mathcal{I}}_4(L) \neq 0$*, we compute* $\mathcal{A}_i(L)$ *for each absolute invariant* $\mathcal{A}_i$ *in* (6.56).
  - *If* $\mathcal{I}_3(L) \neq 0$*, we compute* $\mathcal{A}_i(L)$ *for each absolute invariant* $\mathcal{A}_i$ *in* (6.47).
- *Compute* $d = gcd(\Delta(\mathcal{A}_1(L)), ..., \Delta(\mathcal{A}_m(L))) \in \mathbb{C}[x, t]$
- *Take* H *to be the set of the coefficients of* d *with respect to* t *and* $g(x)$ *is the leading coefficient of* d *with respect to* t*.*
- *Compute* $S = \{h_i(x)/g(x) |\, h_i(x) \in H\} - \mathbb{C}$.
- *Return a shortest element of* S*.*

6.3.1 Algorithm to Compute Nontrivial Simplification $(L_1, r, f(x))$

Now, we give an algorithm to compute a nontrivial simplification.

Algorithm 6.31 (Dsimplify13)**.**

- ***Input****:* $L = \partial^n + c_{n-1}\partial^{n-1} + ... + c_0 \in M_{n\geq 3}$.
- ***Output****:*
  - *Nontrivial simplification* (L_1, r, f) *as in Definition* 6.1*, or*
  - *"No nontrivial simplification exist", or*
  - *"Fail", which means the algorithm did not find a nontrivial simplification, but also did not prove that there is none. We have not encountered this case so far.*
- *Compute* $NF(L)$ *the normal form of* L*.*
- *Compute* $f(x)$ *with Algorithm* 6.30 *applied to* $NF(L)$*.*
- *If* $f(x)$ *is a Mobius transformation then return "No nontrivial simplification exist".*
- *If* $f(x)$ *is nontrivial, compute the inverse of* $f(x)$*, call it* $f^{-1}(x)$*. This* $f^{-1}(x)$ *is a root of* $f(z) - x$*, so we are computing in a field* $K = \mathbb{C}(x)[z]/(N) \subseteq \overline{\mathbb{C}(x)}$ *where* N *is the numerator of* $f(z) - x$*.*

- *Compute (Notation 2.23)* $L_{f^{-1}(x)} \in K[\partial]$.
- *Let* $L_1 = NF(L_{f^{-1}(x)})$. *If* $L_1 \notin \mathbb{C}(x)[\partial]$, *return "fail".*
- *If* $L_1 \in \mathbb{C}(x)[\partial]$, *compute*

$$r = \frac{\binom{n}{2}\frac{f''(x)}{f'(x)} - c_{n-1}}{n}. \tag{6.62}$$

  Then $L = L_{1_{f(x)}} \circledS (\partial - r)$
- *Return* $(L_1, r, f(x))$.

Example 6.32. *Given*

$$\begin{aligned} L = &\partial^3 - \frac{(27x^6 - 72x^5 + 48x^4 - 18x^3 + 66x^2 - 68x + 20)}{3x(x-1)^2(3x-4)}\partial^2 - \frac{A}{42x^2(x-1)^3(3x-4)^2}\partial \\ &- \frac{x(3x-4)^3}{14(x-1)^4}, \end{aligned} \tag{6.63}$$

where $A = (3321x^8 - 17712x^7 + 34920x^6 - 30144x^5 + 9600x^4 - 168x^3 + 56x^2 + 336x - 224)$.

Applying ***FindDescent*** *on* $NF(L)$ *return:*

$$f(x) = \frac{x^4}{x-1}. \tag{6.64}$$

Compute the inverse of $f(x)$, *denote it by* $f^{-1}(x)$ *which is the congruence class of* z *in* $K = \mathbb{C}(x)[z]/(z^4 - xz + x)$.

Compute $L_{f^{-1}(x)} \in K[\partial]$, *and take its normal form*

$$L_1 = NF(L_{f^{-1}(x)}) = \partial^3 - \frac{126x^2 - 51x - 364}{378x^2}\partial - \frac{756x^3 - 459x^2 - 315x + 9856}{10206x^3}, \tag{6.65}$$

which is in $\mathbb{C}(x)[\partial]$.

To compute r, *we use the formula in* (6.62)

$$r = \frac{\binom{n}{2}\frac{f''(x)}{f'(x)} - c_2}{3} = \frac{(3x-4)(3x^4 - 8x + 8)}{9(x-1)^2 x}, \tag{6.66}$$

where c_2 *is the coefficients of* ∂^2 *in* L *because*

$$L = L_{1_{f(x)}} \circledS (\partial - r).$$

Therefore, if L, (6.63), *is the input of the algorithm* ***Dsimplify13***, *the output is*

$$(L_1 \text{ as in } (6.65), \frac{(3x-4)(3x^4 - 8x + 8)}{9(x-1)^2 x}, \frac{x^4}{x-1}).$$

CHAPTER 7

COMPARISON WITH PRIOR ALGORITHMS

In this chapter, the differential field is the field of rational functions with complex coefficient $k \subseteq \mathbb{C}(x)$, and the derivation ∂ is $\frac{d}{dx}$.

7.1 Second Order Differential Operators

To find the solutions for some kinds of second order differential operators, many algorithms have been developed in the last a few decades [6, 11, 12, 15, 16, 10, 20, 21, 22, 31, 33, 34]. However, there is not a general algorithm that solves all of second order differential operators. If we simplify a differential operator by removing as much as possible of its singularities, we may obtain a small operator (comparing to the original one) that is solvable by other solvers. Therefore, we will show the effects of our simplifiers on computing the solutions of second order differential operators.

Example 7.1. *Let $y = KummerM(\frac{1}{2}; 1; x)$. Then the minimal differential operator for y is*

$$L = \partial^2 - \frac{x-1}{x}\,\partial - \frac{1}{4x}.$$

Applying change of variables transformation followed that by gauge transformation such that $f = \frac{x^2}{x^2+1}$, $r_0 = \frac{1}{x^2}$, and $r_1 = \frac{1}{x}$ gives the the following operator

$$\begin{aligned}\bar{L} =& \partial^2 + \frac{7x^{10} - 5x^8 - 31x^6 - 38x^4 - 22x^2 - 5}{x(x^{10} + x^8 - 3x^6 - 6x^4 - 4x^2 - 1)}\,\partial \\ &+ \frac{8x^{12} - 8x^{10} - 57x^8 - 85x^6 - 58x^4 - 21x^2 - 4}{x^2(x^{10} + x^8 - 3x^6 - 6x^4 - 4x^2 - 1)(x^2+1)}\end{aligned}$$

We tried to find the solutions of $\bar{L}$ by using dsolve*, a command Maple software, and by the algorithm in [11]. Unfortunately, we could not find the solutions of $\bar{L}$. However, if we try to simplify $\bar{L}$ by* Phase II *simplifier, we get* 8 *gauge equivalent operators to $\bar{L}$, and all of them are solvable by* dsolve *command. One of the operators is*

$$L_2 = \partial^2 - \frac{x^4 + 2x^2 - 1}{x(x^2+1)^2}\,\partial - \frac{5}{x^2+1},$$

and its solutions basis is

$$y_1 = (x^2+1)^{\frac{1}{2}}\ KummerM(\frac{1}{4}; 1; \frac{x^2}{x^2+1}),$$

and

$$y_2 = (x^2+1)^{\frac{1}{2}}\ KummerU(\frac{1}{4}; 1; \frac{x^2}{x^2+1}).$$

By using Homomorphisms*, a command in* Maple*,*

$$G = \frac{1}{x(x^2+1)}\partial - \frac{x^2-1}{x^2(x^4+2x^2+1)}$$

is the gauge transformation between L_2 and $\bar{L}$, so

$$Y_i = \frac{1}{x(x^2+1)}\, y_i' - \frac{x^2-1}{x^2(x^4+2x^2+1)}\, y_i,\ \textit{where } i = 1, 2,$$

are elements of a solution basis of $\bar{L}$.

7.2 Third Order Differential Operators

7.2.1 Differential Operators Have Solutions in Terms of Solutions of Second Order Differential Operators

Here, we give a summary of the results in [25, 30]. Let $L \in k[\partial]$ be a third order linear differential operator, and let K be the Picard Vessiot extension of k for L, Definition 2.10. If the Galois group $G(K/k)$, Definition 2.11, is eulerian, Proposition 2.13, then Theorem 4.3 [25] shows that one of the following is hold:

1. L has a has a Liouvillian solution.

2. L is a symmetric square of differential operator of order 2.

3. L is gauge equivalent to symmetric square differential operator.

For the first case, Michael Singer [24] presented an algorithm which computes a basis of $V(L)$ if L has Liouvillian solutions.

The symmetric square of a differential operator of order 2 is an operator of order 3, Lemma 2.19. Let $L_2 = L_1^{Ⓢ2}$ where L_1 is a differential operator of order 2. If $\{y_1, y_2\}$ is a basis of $V(L_1)$, then $\{y_1^2, y_1y_2, y_2^2\}$ is a basis of $V(L_2)$.

Lemma 7.2. *Let $L_1 \in k[\partial]$ of order 3.*

1. *L_1 is a symmetric square of a differential operator L_2 of order 2 with coefficients in some field extension K of k if and only if*

$$L_1 = \partial^3 + c_2\partial^2 + c_1\partial + c_0, \tag{7.1}$$

where

$$c_2'' + 6c_0 - 3c_1' + 2c_2(\frac{2c_2^2}{9} + c_2' - c_1) = 0, \tag{7.2}$$

so

$$L2 := \partial^2 + \frac{a_2}{3}\partial + \frac{1}{4}(a_1 - \frac{a_2'}{3} - \frac{2}{9}a_2^2) \tag{7.3}$$

2. *If $c_2 = 0$, then $L_1 = L_2^{Ⓢ2}$ if and only if $c_0 = \frac{c_1'}{2}$.*

Proof. Let L_1 is a symmetric square of order L_2 of order 2 over differential field extension K of k such that $L_2 = \partial^2 + a_1\partial + a_0$. From (2.4), it can be shown that $c_2 = 3a_1$, $c_1 = 2a_1^2 + a_1' + 4a_0$, and

$$c_0 = 4a_1a_0 + 2a_0'. \tag{7.4}$$

Therefor,

$$a_1 = \frac{c_2}{3} \tag{7.5}$$

$$a_0 = \frac{c_1}{4} - \frac{c_2^2}{18} - \frac{c_2'}{12} \tag{7.6}$$

By substituting the value of a_0 and a_1 in (7.4),

$$\begin{aligned} c_0 &= 4(\frac{c_2}{3}(\frac{c_1}{4} - \frac{c_2^2}{27} - \frac{c_2'}{12})) + 2(\frac{c_1'}{4} - \frac{2c_2c_2'}{18} - \frac{c_2''}{12}) \\ &= \frac{c_1c_2}{3} - \frac{2c_2^3}{27} - \frac{c_2c_2'}{3} + \frac{c_1'}{2} - \frac{c_2''}{6}. \end{aligned}$$

which satisfies (7.2). Also, (7.3) is obtained by inserting the value of a_0 and a_1 in $\partial^2 + a_1\partial + a_0$. □

The left side of equation (7.2) is equal to $I_{3,3}(L)$, in Theorem 6.14, which is a relative invariant of weight 3. For any differential operator of order 3, if the relative invariants of wight 3 is 0, it is a symmetric square.

Lemma 7.3. *Let $L_1, L_2 \in k[\partial]$ of order 3. Let $L_1 \xrightarrow{(i)+(iii)} L_2$, where L_1 is a symmetric square of an operator of order 2. Then L_2 is also a symmetric square of an operator of order 2.*

Proof. By the definition of relative invariant, Definition 6.9,

$$I_{3,3}(L_2) = I_{3,3}(L_1) \restriction_{x \to f} (f')^3.$$

Lemma 7.2 shows that $I_{3,3}(L_1) = 0$ because L_1 is symmetric square. Therefore, $I_{3,3}(L_2) = 0$, and by Lemma 7.2 L_2 is symmetric square. □

If the symmetric square of an operator L of order 3 is an operator of order 5, then L is a symmetric square of an operator $\bar{L}$ of order 2 because $L^{ⓢ2} = \bar{L}^{ⓢ4}$ and by Lemma 2.19 the order of $\bar{L}^{ⓢ4}$ is at most $\begin{pmatrix} 4+2-1 \\ 2-1 \end{pmatrix} = 5$. It would be easy to compute a basis of $V(L)$ if a basis of $V(\bar{L})$ is computed.

Example 7.4. *Let*

$$L_3 = \partial^3 - \frac{3x-1}{x}\partial^2 + \frac{18x^2-30x-1}{9x^2} + \frac{6x+1}{3x^2}.$$

Since $I_{3,3}(L_3) = 0$, it is symmetric square of a differential operator of order 2, L_2. To compute L_2, we use equations (7.5) and (7.6), so

$$L_2 = \partial^2 - \frac{3x-1}{3x} - \frac{1}{2x}.$$

By using dsolve *command in* Maple software*,*

$$\{y_1 = x^{\frac{2}{3}} KummerM(\frac{7}{6}, \frac{5}{3}, x), y_2 = x^{\frac{2}{3}} KummerU(\frac{7}{6}, \frac{5}{3}, x)\}$$

is a basis of the solution space of L_2, so $\{y_1^2, y_1y_2, y_2^2\}$ is a basis of the solution space of L_3.

Let $L_3 \in k[\partial]$ be an operator of order 3. If $L_3^{ⓢ2}$ is an operator of order 6 and have a first order right or lift hand factor L_3, then it is a gauge equivalent to a symmetric square of a differential operator of order 2 [25, 30]. In [30], Van Hoeij implemented an algorithm, *ReduceOrder*, such that if L_3 is a gauge equivalent to $L_2^{ⓢ2}$ where $L2$ is an operator of order 2, the algorithm computes L_2 and the gauge transformation between L_3 and $L_2^{ⓢ2}$.

Example 7.5. *Let*

$$L = \partial^3 + \frac{3(7x+24)}{2x(3x+8)}\partial^2 - \frac{12x^2+65x-8}{4x^2(3x+8)}\partial - \frac{24x^2+179x+216}{8x^3(3x+8)}.$$

The ReduceOrder *algorithm shows that L is a gauge equivalent to $L_1^{ⓢ2}$ where*

$$L_1 = \partial^2 - \frac{1}{2x}\partial - \frac{4x+27}{16x^2}.$$

Applying Phase II *algorithm on L (with 1 in the second input, Phase II$(L,1)$) gives a set of* 8 *operators two of which are symmetric square of a differential operator of order* 2.

7.2.2 Differential Operators that Have Solutions in Terms of Hypergeometric Functions ${}_0F_2$, ${}_1F_2$, ${}_2F_2$, and ${}_3F_2$

In [7], Cheb-Terrab and Roche implemented an algorithm that finds solutions of third order differential equations of this form

$$\exp(\int r\,dx) \cdot \ {}_iF_2(f)$$

where $i = 0, 1, 2, 3$ and $f, r \in \mathbb{C}(x)$. The algorithm in [23] finds solutions of this form

$$\exp(\int r\,dx) \cdot \big(r_0 S(f) + r_1 S'(f) + r_2 S''(f)\big), \tag{7.7}$$

where S is a hypergeometric function ${}_0F_2$, ${}_1F_2$, ${}_2F_2$, or ${}_1F_1^2$, and $f, r, r_0, r_1, r_2 \in \mathbb{C}(x)$. The method, in [23], starts by computing S with its parameters and f. After that, it computes r, r_0, r_1 and r_2. For this kind of differential operators, our algorithms in Chapter 4 and 5 finds a projective equivalent operator, which means it starts by computing r, r_0, r_1 and r_2. Then, use the absolute invariants to compute f, Chapter 6. In the following example, we will explain how our simplification works.

Example 7.6 (L in Chapter 0, [23])**.** *Let*

$$L = \partial^3 + \frac{N_2}{D_2}\,\partial^2 + \frac{N_1}{D_1}\,\partial + \frac{N_0}{D_0}$$

Where

$$N_2 = 45x^8 - 70x^7 + 1756x^6 - 13848x^5 + 44640x^4 - 79520x^3 + 81600x^2 - 44672x + 9984,$$

$$D_2 = 4(x(3x^8 - 7x^7 + 80x^6 - 572x^5 + 1776x^4 - 3040x^3 + 3040x^2 - 1664x + 384)),$$

$$\begin{aligned} N_1 =& 45x^{11} - 54x^{10} + 1976x^9 - 22440x^8 + 100336x^7 - 257888x^6 + 415872x^5 \\ &- 437632x^4 + 318464x^3 - 172032x^2 + 65536x - 12288, \end{aligned}$$

$$D_1 = 8(x^4(3x^8 - 7x^7 + 80x^6 - 572x^5 + 1776x^4 - 3040x^3 + 3040x^2 - 1664x + 384)(x-1)),$$

$$N_0 = 4(2x^8 + 3x^7 - 69x^6 + 244x^5 - 498x^4 + 720x^3 - 688x^2 + 384x - 96),$$

and

$$D_0 = x^5(x-1)(3x^7 - x^6 + 78x^5 - 416x^4 + 944x^3 - 1152x^2 + 736x - 192).$$

The operator L is solvable by Hyp1F2Solutions *algorithm, [23], because it has ${}_1F_2$ solution, and one of its solutions is*

$$ {}_1F_2(\frac{1}{4};\frac{1}{2},\frac{3}{4};\frac{x-1}{x^2}) - \frac{2}{3x}\,{}_1F_2(\frac{5}{4};\frac{3}{2},\frac{7}{4};\frac{x-1}{x^2}) + \frac{4}{3x^2}{}_1F_2(\frac{5}{4};\frac{3}{2},\frac{7}{4};\frac{x-1}{x^2}). $$

If we apply Phase I *algorithm, Chapter* 4*, on L, and apply* Phase II *algorithm, Chapter* 5*, on the result, we get a list of projective equivalent differential operators. One of the operators is a symmetric square of an operator of order* 2*,*

$$ \begin{aligned} \tilde{L}_1 =&\partial^3 + \frac{3(x-4)}{(x-2)x}\partial^2 + \frac{7x^6-72x^5+504x^4-1120x^3+1200x^2-768x+256)}{16(x-2)^2(x-1)^2x^4}\partial \\ &+ \frac{x^5-208x^4+664x^3-800x^2+464x-128}{16(x-2)^2(x-1)^3x^4} \end{aligned} $$

Another operator that is projective equivalent to L is

$$ \begin{aligned} \tilde{L}_2 =&\partial^3 + \frac{3(x-4)}{(x-2)x}\partial^2 + \frac{3x^6-40x^5+408x^4-992x^3+1136x^2-768x+256}{16(x-2)^2(x-1)^2x^4}\partial \\ &- \frac{3x^5-32x^4+328x^3-672x^2+496x-128}{16(x-2)^2(x-1)^2x^4}, \end{aligned} $$

*such that $\tilde{L}_2 \xrightarrow{(ii)+(iii)} L$ where $r = -\frac{x+2}{4(x-1)x}$, $r_0 = -\frac{x^4-8x^3-12x^2+48x-32}{4(x-2)^3}$,
$r_1 = \frac{(3x^4-32x^3+12x^2+48x-32)x}{12(x-2)^3}$, and $r_2 = \frac{(x-1)x^4}{(x-2)^2}$. Applying* FindDescent*, Algorithm* 6.30*, on $\tilde{L}_2$ shows that it is obtained from the following operator*

$$ L_{simp} = \partial^3 + \frac{3x-16}{16x^3} - \frac{3(x-4)}{8x^4} $$

shuch that

$$ L_{simp} \xrightarrow{(i)+(iii)} \tilde{L}_2, $$

where $f = \frac{x^2}{x-1}$ and $r = -\frac{x-3}{(x-1)x}$. The operator L_{simp} is solvable by Maple 18*, and its solutions are*

$$ y_1 = x^2\,{}_1F_2(\frac{-1}{2};\frac{-3}{4},\frac{-1}{4};\frac{1}{x}), \quad y_2 = x^{\frac{3}{4}}\,{}_1F_2(\frac{3}{4};\frac{1}{2},\frac{9}{4};\frac{1}{x}), \quad y_3 = x^{\frac{1}{4}}\,{}_1F_2(\frac{5}{4};\frac{3}{2},\frac{11}{4};\frac{1}{x}). $$

Since we know the solutions of L_{simp} and all of the transformations between L_{simp} and L, we can compute the solutions of L. The solutions of $\tilde{L}_2$ are

$$ Y_i = \frac{(x-1)^2}{x^3}\,y_i(\frac{x^2}{x-1}), \quad \text{for } i = 1,\, 2\,3, $$

and the solution of L are

$$S_i = \frac{x^{\frac{1}{2}}}{(x-1)^{\frac{3}{4}}}\ \big(-\frac{x^4-8x^3-12x^2+48x-32}{4(x-2)^3}Y_i + \frac{(3x^4-32x^3+12x^2+48x-32)x}{12(x-2)^3}Y_i'$$
$$+\frac{(x-1)x^4}{(x-2)^2}Y_i''\big),$$

for $i = 1,\, 2,\, 3$.

For some differential operators that have ${}_0F_2$, we found *Solver* 3 does not find their solutions. Using *Phase I* and *Phase II* to find projective equivalent operators, whose solutions are hypergeometric functions, gives operators that can be solved by other solvers.

Example 7.7. *Let*

$$L = \partial^3 + \frac{N_2}{D_2}\,\partial^2 + \frac{N_1}{D_1}\,\partial + \frac{N_0}{D_0},$$

where

$$\begin{aligned}
N_2 =& 192x^{10} + 8608x^9 - 104368x^8 + 457520x^7 - 975980x^6 + 1006278x^5 - 287529x^4\\
&- 307629x^3 + 240570x^2 - 40095x + 3645,\\
D_2 =& 12x(96x^{10} - 160x^9 - 3736x^8 + 23340x^7 - 63680x^6 + 96679x^5 - 84114x^4\\
&+ 37911x^3 - 4851x^2 - 1890x + 405),\\
N_1 =& -(9024x^{10} - 104896x^9 + 511184x^8 - 1416040x^7 + 2545920x^6 - 3202962x^5\\
&+ 2926776x^4 - 1928829x^3 + 840555x^2 - 172125x - 10935),\\
D_1 =& 12(96x^{10} - 160x^9 - 3736x^8 + 23340x^7 - 63680x^6 + 96679x^5 - 84114x^4\\
&+ 37911x^3 - 4851x^2 - 1890x + 405)(x-1)x,\\
N_0 =& -(384x^{12} - 1792x^{11} - 8960x^10 + 99632x^9 - 376664x^8 + 791496x^7\\
&- 1016844x^6 + 806907x^5 - 390105x^4 + 135243x^3 - 66015x^2 + 34020x - 7290),\\
D_0 =& x(x-1)(x^2-2x+1)(48x^9 - 8x^8 - 1880x^7 + 8850x^6 - 18565x^5\\
&+ 20492x^4 - 11319x^3 + 1977x^2 + 540x - 135).
\end{aligned}$$

We used dsolve*, Maple's command, and* Solver 3 *to get a solution for L. Unfortunately, no solution are found. However, when we apply* Phase II *algorithm on L, it gives two gauge equivalent operators that are solvable by* dsolve*. One of the operators is*

$$\tilde{L} = \partial^3 - \frac{116x^2-96x-63}{12(x-1)(2x-3)x}\,\partial^2 + \frac{324x^3-628x^2-21x+378}{6(x-1)^2(2x-3)^2x}\,\partial$$
$$-\frac{192x^6-1440x^5+4740x^4-7600x^3+5315x^2-625x-588}{6(x-1)^4(2x-3)^2x},$$

where the parameters of the gauge transformation are

$$r_2 = \frac{4x}{(2x-3)^2}, \quad r_1 = -\frac{92x^2 - 108x - 27}{3(2x^2 - 5x + 3)(2x-3)^2},$$

and

$$r_0 = \frac{192x^5 - 1012x^4 + 2172x^3 - 2235x^2 + 958x - 63}{3(2x^3 - 7x^2 + 8x - 3)(x-1)^2(2x-3)^2}.$$

The command dsolve *shows that*

$$y_1 = x(x-1)^2 \, {}_0F_2(0; \frac{5}{4}, \frac{4}{3}; \frac{x^3}{x-1}), \quad y_2 = (x-1)^{\frac{7}{3}} \, {}_0F_2(0; \frac{2}{3}, \frac{11}{12}; \frac{x^3}{x-1}),$$

and

$$y_3 = (x-1)^{\frac{9}{4}} x^{\frac{1}{4}} \, {}_0F_2(0; \frac{3}{4}, \frac{13}{12}; \frac{x^3}{x-1})$$

are solutions of $\tilde{L}$. *Therefore,*

$$Y_i = r_2 \, y_i'' + r_1 \, y_i' + r_0 \, y_i, \quad \textit{for } i = 1, 2, 3,$$

are solutions of L.

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BIOGRAPHICAL SKETCH

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