Desingularization and p-Curvature of Recurrence Operators

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Recurrence operators with rational function coefficients

Let $a_i(x) \in \mathbb{Q}(x)$ be rational functions in x.

Recurrence relation:

$$a_n(x)u(x+n) + \cdots + a_1(x)u(x+1) + a_0(x)u(x) = 0.$$

Solutions $u(x)$ are functions on subsets of $\mathbb{C}$.
For subset $\{0, 1, 2, \ldots\}$, solution is a sequence $u(0), u(1), u(2), \ldots$

Recurrence operator: write the recurrence as $L(u) = 0$ where

$$L = a_n\tau^n + \cdots + a_0\tau^0 \in \mathbb{Q}(x)[\tau]$$

Here τ is the shift operator. It sends $u(x)$ to $u(x+1)$.

Clearing denominators $\rightsquigarrow$ $a_i \in \mathbb{Q}[x]$.
Then $\max\{\text{degree } a_i\}$ is the degree and n is the order.

Recurrence relations come from many sources: creative telescoping, walks, QFT computations, OEIS, etc.

Example: Entry A025184 in OEIS

$$
\begin{aligned}
L(u) = {} & 33x(3x-1)(3x-2)u(x) \\
& +11(2047x^3 - 10725x^2 + 17192x - 8520)u(x-1) \\
& -9(4397x^3 + 10169x^2 - 110500x + 145368)u(x-2) \\
& -54(2x-5)(5353x^2 - 33313x + 53904)u(x-3) \\
& -115668(x-4)(2x-5)(2x-7)u(x-4) = 0.
\end{aligned}
$$

$L \in \mathbb{Q}(x)[\tau^{-1}]$ has order 4 and **degree 3**.

MinimalRecurrence in Maple 2021 finds

$$
\begin{aligned}
& 3x(3x-1)(3x-2)(221x^2 - 723x + 574)u(x) \\
& -2(2x-1)(7735x^4 - 33040x^3 + 48239x^2 - 27998x + 5280)u(x-1) \\
& -36(x-2)(2x-1)(2x-3)(221x^2 - 281x + 72)u(x-2)
\end{aligned}
$$

Price to pay: lower order but **higher degree** (order-degree curve).

Minimal Order Recurrence has 3 true and 2 removable singularities.

Desingularization

Given a recurrence relation:

$$\begin{aligned}
&3x(3x-1)(3x-2)(221x^2-723x+574)u(x)\\
-&2(2x-1)(7735x^4-33040x^3+48239x^2-27998x+5280)u(x-1)\\
-&36(x-2)(2x-1)(2x-3)(221x^2-281x+72)u(x-2)
\end{aligned}$$

Desingularization: find out which are true singularities and which ones can be removed if one allows the order to increase.

$$\begin{aligned}
&3{\color{blue}x(3x-1)(3x-2)}{\color{red}(221x^2-723x+574)}u(x)\\
-&2(2x-1)(7735x^4-33040x^3+48239x^2-27998x+5280)u(x-1)\\
-&36{\color{blue}(x-2)(2x-1)(2x-3)}{\color{red}(221x^2-281x+72)}u(x-2)
\end{aligned}$$

Applications desingularization

Many algorithms (closed form solutions, RightFactors in Maple 2021) only need the true singularities to work correctly. Discarding removable singularities reduces the amount of work.

p-curvature: a common tool to study differential or recurrence operators (appears in at least one talk and a poster at this conference).

The Norm of the leading coefficient (leading singularities) plays a key role in the algorithm for computing the p-curvature (Bostan, Caruso, Schost, ISSAC'2014). It is used as denominator-bound.

We expected: discarding removable singularities $\rightsquigarrow$ sharper bound $\rightsquigarrow$ faster computation.
To our surprise we actually get the exact denominator:

Main result: Norm(true singularities) $=$ denominator(p-curvature).

Gauss' lemma does not hold for $\mathbb{Q}[x][\tau] \subset \mathbb{Q}(x)[\tau]$

As illustrated:
L may have a right-factor with lower order but higher degree (after clearing denominators).

Conversely, a multiple of L can have lower degree.
We call L Gaussian if that does not happen.

We first prove the main result for Gaussian operators.
To prove the result in general, we use the fact that any operator has a Gaussian multiple.

First some notation.

Recurrence Operators

F: a field

$F(x)$: the field of rational functions

τ: the shift operator

$\mathcal{D} := F(x)[\tau]$: ring of recurrence operators over $F(x)$

$\mathcal{P} := F[x][\tau]$: recurrence operators with polynomial coefficients

Addition: same as polynomials

Multiplication: $\tau \cdot r(x) = r(x+1) \cdot \tau$

Recurrence Relation and Singularities

A recurrence operator $L = \sum_{i=0}^{n} a_i \tau^i \in \mathcal{P}$ corresponds to the equation

$$a_n(x)u(x+n) + \cdots + a_0(x)u(x+0) = 0.$$

Generally, $u(x+n)$ is determined by $u(x), \cdots, u(x+n-1)$ and L. Except at roots of $a_n(x)$.

Definition

Roots or factors of $a_n(x-n)$ are called (leading) *singularities* of L.

Singularities

$L = \sum_{i=0}^{n} a_i \tau^i$

Some notations:

$\mathrm{lc}(L) := a_n$ the *leading coefficient*

$\mathrm{lc}^*(L) := a_n(x - n)$ the *adjusted leading coefficient*

$\mathcal{P} = F[x][\tau]$
$\mathcal{D} = F(x)[\tau]$

If $A \in \mathcal{P}$ then $\mathrm{lc}^*(L) \mid \mathrm{lc}^*(AL)$ because

$$\mathrm{lc}^*(AL) = \mathrm{lc}^*(L) \cdot \text{a shift of } \mathrm{lc}^*(A)$$

However, we will allow $A \in \mathcal{D}$ as long as $AL \in \mathcal{P}$.

Then $\mathrm{lc}^*(AL)$ could be smaller than $\mathrm{lc}^*(L)$ even if L is primitive!
(i.e. Gauss' lemma does not hold for $\mathcal{P} \subset \mathcal{D}$)

Desingularization

Definition

The essential part of $\mathrm{lc}^*(L)$, denoted $\mathrm{LC}^*(L)$,

- divides $\mathrm{lc}^*(AL)$ for any $AL \in \mathcal{P} \quad (A \in \mathcal{D})$
- it is maximal satisfying the first condition.

True singularities: factors/roots of $\mathrm{LC}^*(L)$

Removable singularities: remaining factors/roots of $\mathrm{lc}^*(L)$, these can disappear in some AL

Desingularization: Find out which singularities are removable.

Gaussian Operators

Definition

$L \in \mathcal{P}$ is *Gaussian* if $AL \in \mathcal{P}$ implies $A \in \mathcal{P}$ (for any $A \in \mathcal{D}$).

Lemma

$L \in \mathcal{P}$ is Gaussian if $\mathrm{LC}^*(L) = \mathrm{lc}^*(L)$ *(no removable singularities)*

Recurrence Operators in Characteristic p

F a field of characteristic p, where p is a prime.
Let $Z = x^p - x = \prod_{i=0}^{p-1} \tau^i(x) = x(x+1)\cdots(x+p-1)$.
Then $F(x)/F(Z)$ has Galois group $<\tau> \cong Z_p$.

Let

$$\mathcal{N} : F(x) \to F(Z)$$

$$f(x) \mapsto f(x)f(x+1)\cdots f(x+p-1)$$

be the norm map.

Recurrence Operators in Characteristic p

The center of $\mathcal{D} = F(x)[\tau]$ is $F(Z)[\tau^p]$.
A $\mathcal{D}$-module is also an $F(x)[\tau^p]$-module, which is an $F(x)$-v.s. equipped with a linear transformation induced by τ^p.

Definition

- Suppose M is a finitely dimensional $\mathcal{D}$-module. The *p-curvature* of M is the linear transformation induced by τ^p.
- An operator $L \in \mathcal{D}$ is naturally associated with the $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}L$. Define the p-curvature to be that of the module.
- Let $\chi(L) \in F(x)[T]$ be its characteristic polynomial.

p-Curvature: Properties

1. $\chi(L) \in F(Z)[T] \subset F(x)[T]$
2. Identify T with τ^p. Then $\chi(L) \in F(Z)[\tau^p] = \mathrm{center}(\mathcal{D})$.
   By Cayley-Hamilton, it is the zero map on $\mathcal{D}/\mathcal{D}L$.
   As a result, $\chi(L) \in \mathcal{D}L$.
3. χ is multiplicative. $\chi(L_1 L_2) = \chi(L_1)\chi(L_2)$.
4. Denote $\tilde{\chi}(L) = \mathcal{N}(\mathrm{lc}(L)) \cdot \chi(L)$.
   When $L \in \mathcal{P}$, $\tilde{\chi}(L) \in F[Z][T]$.
5. For $L \in \mathrm{center}(\mathcal{D})$, $\tilde{\chi}(L) = L^p$.

Item 4 says that $\mathcal{N}(\mathrm{lc}(L))$ is a denominator bound for $\chi(L)$, this is used in the algorithm for computing $\chi(L)$:

Bostan, Caruso, Schost, ISSAC'2014. A fast algorithm for computing the characteristic polynomial of the p-curvature

Main Theorem

Theorem

Let $L \in \mathcal{P}$. *Then* $\mathrm{denom}(\chi(L)) = \mathcal{N}(\mathrm{LC}^*(L))$.

The leading coefficient of L contains the

- removable singularities, and the
- true singularities $\mathrm{LC}^*(L)$

The theorem implies that all removable singularities disappear in the p-curvature, but more surprisingly:

All true singularities (their norm) do appear in denom(p-curvature), there is no cancellation!

This is unexpected because it implies corollaries in characteristic p that do not hold in characteristic 0.

Proof: Step 1

$\mathrm{denom}(\chi(L)) \mid \mathcal{N}(\mathrm{LC}(L))$

Proof: $\chi(AL) = \chi(A)\chi(L)$. Since characteristic polynomials are always monic,

$$\mathrm{denom}(\chi(AL)) = \mathrm{denom}(\chi(A)) \cdot \mathrm{denom}(\chi(L))$$

There exists A such that $\mathrm{lc}^*(AL) = \mathrm{LC}^*(L)$. Hence

$$\mathrm{denom}(\chi(L)) \mid \mathrm{denom}(\chi(AL)) \mid \mathcal{N}(\mathrm{lc}(AL)) = \mathcal{N}(\mathrm{LC}(L))$$

Note: $\mathrm{lc}^*(AL)$ and $\mathrm{lc}(AL)$ have the same Norm.

Proof: Step 2, Key Step

$\mathrm{denom}(\chi(L)) = \mathcal{N}(\mathrm{LC}^*(L))$ when L is Gaussian.

To prove: $\tilde{\chi}(L) = \mathcal{N}(\mathrm{lc}(L)) \cdot \chi(L) \in F[Z][T]$ is primitive.
We know

$$\mathrm{Prim}(\chi(L)) = QL$$

for some $Q \in \mathcal{D}$.
Since L is Gaussian, $Q \in \mathcal{P}$.
Apply $\tilde{\chi}$ on $\mathrm{Prim}(\chi(L))$:

$$\tilde{\chi}(\mathrm{Prim}(\chi(L))) = \tilde{\chi}(Q)\tilde{\chi}(L).$$

The LHS is $(\mathrm{Prim}(\chi(L)))^p$, which is primitive; the two factors of the RHS are in $F[Z][T]$. By Gauss's Lemma, $\tilde{\chi}(L)$ is primitive.

Proof: Step 3, Sketch

Idea of the proof:
In general, for $L \in \mathcal{P}$, there exists A such that $\mathrm{lc}^*(AL) = \mathrm{LC}^*(L)$, which implies AL is Gaussian. Then

$$\mathrm{denom}(\chi(AL)) = \mathcal{N}(\mathrm{LC}^*(L)) = \mathrm{denom}(\chi(A)) \cdot \mathrm{denom}(\chi(L))$$

Attempt to prove: $\mathrm{denom}(\chi(A)) = 1$.

We can show that using a technical computation based on what we know about A from this paper:

Chen, Kauers, Singer, 2016. Desingularization of Ore operators.

p-curvature complexity?

- Say d is the degree of $\mathrm{lc}(L)$ and d^* the degree of $\mathrm{LC}^*(L)$. Fast (partial) desingularization replaces d by $\approx d^*$.
- Random operators often have no removable singularities. Then $d^* = d$, no complexity improvement!
- Does this matter? After all, I learned from Joris' invited talk that I'm a complexity extremist.
- Factoring random operators is not useful, they are irreducible.
- LCLM(two random operators of order n and degree n). Experimentally $d^* = 2n = O(\sqrt{d})$.
- So in theory, desingularization does not speed up the p-curvature, and in practice it does.
- Worst case complexity? Average case complexity? What matters is: Actual case complexity.