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SIMPLIFYING DIFFERENCE EQUATIONS

By

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LIST OF SYMBOLS

The following list of symbols are used throughout this thesis.

R	a commutative ring with identity
τ	shift operator
$\mathbb{C}(x)$	field of rational function in x over the complex numbers $\mathbb{C}$
$D, \mathbb{C}(x)[\tau]$	the ring of difference operators over $\mathbb{C}(x)$
$V(L)$	the solution space of L
$ord(L)$	the order of the operator L
M	difference module
v	a valuation map
$v_\varepsilon(a)$	the ε-valuation of a
$V_p(L_\varepsilon)$	the solution space of L_ε at p
$v_{\varepsilon,l}$	the left ε-valuation
$v_{\varepsilon,r}$	the right ε-valuation
$g_{p,\varepsilon}(u)$	the valuation growth of nonzero $u \in V_p(L_\varepsilon)$
$\overline{g}_p(L)$	the set of valuation growth of L at p
$g_{p,min}(L)$	the minimum valuation growth of L at p
$g_{p,max}(L)$	the maximum valuation growth of L at p
$v^a_{p,l}(L)$	the allowed left valuation of L at p
$v^a_{p,r}(L)$	the allowed right valuation of L at p
O_M	the set of integral element of M
$\mathbb{C}((1/x))$	field of Laurent series in $1/x$ over the complex numbers $\mathbb{C}$
$\mathbb{C}[[1/x]]$	ring of formal power series in $1/x$ over the complex numbers $\mathbb{C}$

ABSTRACT

In this thesis, we focus on simplifying difference equations using a method centered around the concept of an integral basis. The primary challenge addressed is how to define "integral" elements, in a way that depends only on the difference module corresponding to a difference equation. We explore how to identify integral elements and give an efficient way to generate an integral basis. We apply this to reduce difference equations to their (near)-simplest form. This approach helps to solve difference equations more efficiently.

CHAPTER 1

INTRODUCTION

A linear homogeneous recurrence equation is an equation of the form:

$$a_n(x)u(x+n) + \cdots + a_0(x)u(x) = 0$$

with $a_i(x) \in \mathbb{C}(x)$. We can write this equation as $L(u) = 0$ where L is a difference operator

$$L = a_n\tau^n + a_{n-1}\tau^{n-1} + \ldots + a_0\tau^0 \in D := \mathbb{C}(x)[\tau].$$

Here τ is the shift operator, it sends $u(x)$ to $u(x+1)$.

The goal of this thesis is to simplify a difference equation to its simplest form. To illustrate this goal, consider the algorithm from [2] (see more such algorithms see [12, 11]). It can decide if a third order difference operator L_3 can be solved in terms of an operator of order two, say L_2. However, such L_2 is not unique and there is no reason to assume that [2] will find the "simplest" L_2. As an example, take the minimal operator for sequence A295371 from the OEIS (Online Encyclopedia for Integer Sequences):

$$L_3 := (2x+3)(x+4)^2\,\tau^3 - (2x+3)(7x^2+52x+97)\,\tau^2 - 3(2x+7)(7x^2+18x+12)\,\tau + 27(2x+7)(x+1)^2.$$

Algorithm `ReduceOrder` in [2] finds that solutions of L_3 can be written in terms of solutions of

$$L_2 := \tau^2 + \tau - \frac{3\,(x+1)\,(5x^2+7x+3)\,(x+2)\,(5x^2+27x+37)}{4\,(5x^3+27x^2+46x+27)\,(5x^3+12x^2+7x+3)}. \tag{1.1}$$

Given such L_2 as starting point, our goal is to find the "simplest" operator $\tilde{L}$ such that solutions of L_2 (and hence solutions of L_3) can be written in terms of solutions of $\tilde{L}$. Our implementation reduces L_2 to

$$\tilde{L} = \tau^2 + (2x+1)\,\tau - 3x^2. \tag{1.2}$$

Note that the order did not change, just the size of the coefficients $a_i \in \mathbb{C}(x)$. The key step for such reduction is to compute a so-called *integral basis*, and this will be the main topic in this thesis.

To explain its relevance, let M be a difference module, i.e. a D-module that is finite dimensional as a $\mathbb{C}(x)$-vector space. A cyclic vector (a generator) of M gives, by taking its annihilator, an operator L for which $M \cong D/DL$. But unless one can find the "best" cyclic vector in M, one cannot expect the corresponding L to have minimal size. This is the reason why the above mentioned Algorithm `ReduceOrder` finds an operator L_2 that is much larger than optimal. To do better, we have to search the D-module $M \cong D/DL_2$ to find an element with a small annihilator. The idea is to define, and then compute, the "integral" elements in a D-module (Chapters 3 and 4), and then select an integral element of minimal degree (Chapter 5).

For this to work, we need to define integral elements of a module M in a way that depends only on the module M, and not on the operator L that was used to define M. We do this in Chapter 3 where we define left and right-integral by comparing valuations with *allowed left and right-valuations*. Theorem 3.4 will show an equivalence to a shorter criterion. The definition and computation of these valuations is quite technical, so Section 3.2 will give examples to clarify this.

The goal in Chapter 4 is to produce an integral basis. Theorem 4.14 combined with Equation (3.9) provides an efficient algorithm for calculating such a basis. Section 4.3 gives an alternative way to describe an integral basis, in terms of the matrix that represents the action of τ on a basis. This description avoids many of the prior technicalities because it can be stated without the valuations that are central until this point. It also allows us to directly define an integral basis in a difference module, without having to represent that module as D/DL for some operator L.

1.1 Related and Future Work

1.1.1 Differential Equations

Linear homogeneous differential equations with coefficients in $\mathbb{C}(x)$ play an important role in several scientific areas such as mathematics, statistics, and physics. There are many algorithms to find several types of closed form solutions of differential equations. In [10], there is an algorithm `hypergeometricsols` that can find solutions for second order differential equations that can be written in terms of hypergeometric functions. A key step in this algorithm is to reduce a differential equation to its simplest form. We expect therefore that simplification for difference equations will also facilitate the development of algorithms to find closed form solutions.

In [1] there are three simplification algorithms for differential equations. A primary motivation was to find closed form solutions, but simplification can also improve numerical computations.

1.1.2 Related Work in Difference Equations

There is also an definition of integral in [4]. However, this definition depends on the operator. So if $M \cong D/DL_1 \cong D/DL_2$ then this does not provide an intrinsic definition of integral in M, as the definition depends on a choice (L_1 or L_2). This makes it more difficult to search for an element of M with a (near)-optimal annihilator.

1.1.3 Future Work

Our next goal is to use integral basis techniques to develop algorithms for finding closed form solutions of difference equations.

CHAPTER 2

PRELIMINARIES

2.1 Linear Recurrence Relation

A *linear recurrence relation* is an equation defining a term of a sequence as a linear combination of previous terms, i.e.

$$u(x+n) = r_{n-1}(x)u(x+n-1) + \cdots + r_0(x)u(x) \tag{2.1}$$

where $r_i(x) \in \mathbb{C}(x)$. We can multiply by denominators to rewrite (2.1) as

$$a_n(x)u(x+n) + \cdots + a_0(x)u(x) = 0 \tag{2.2}$$

with $a_i(x) \in \mathbb{C}[x]$. We write (2.2) as $L(u) = 0$ where

$$L = a_n\tau^n + a_{n-1}\tau^{n-1} + \ldots + a_0\tau^0, \tag{2.3}$$

and τ sends $u(x)$ to $u(x+1)$ and where a_i is short for $a_i(x)$.

The purpose of a recurrence relation is, when given the value of $u(x)$ at n consecutive points $x \in \{q, q+1, \ldots, q+n-1\}$, to give the value of $u(x)$ at the next points $x = q+n, q+n+1, \ldots$, or at the previous points $x = q-1, q-2, \ldots$. This is possible with finitely exceptions: the *problem points* are points where computing $u(x)$ triggers a division by zero error, which we illustrate here:

Example 2.1. *Let*

$$u(x+2) = \frac{1}{x-\frac{1}{2}}u(x+1) + \frac{1}{x+2}u(x) \tag{2.4}$$

be recurrence relation. For almost every $q \in \mathbb{C}$, if the values of $u(q)$ and $u(q+1)$ are given, then we can use Equation (2.4) to compute the value of $u(q+2)$, as well as the value of $u(q-1)$. However, if $q = -2$, if we have $u(-2)$ and $u(-1)$, then trying to compute $u(0)$ by substituting $x = -2$ in Equation (2.4) leads to a division by zero. Likewise, we can not compute $u(\frac{5}{2})$ from $u(\frac{1}{2})$ and $u(\frac{3}{2})$ because substituting $x = \frac{1}{2}$ in Equation 2.4 also results in a division by zero. The same is also true if we try to compute $u(\frac{1}{2})$ from $u(\frac{3}{2})$ and $u(\frac{5}{2})$.

Definition 2.2. *[6] Let L be as in Equation (2.2). A point $q \in \mathbb{C}$ is called a* problem point *of L if q is a root of the polynomial $a_0(x)a_n(x-n)$. This is equivalent to saying that (due to division by zero errors) the recurrence relation does not determine the value of $u(q)$ from the values of $u(q-1), \ldots, u(q-n)$, or does not determine $u(q)$ from $u(q+1), \ldots, u(q+n)$.*

Definition 2.3. *Let L be a recurrence relation and $q \in \mathbb{C}$. The sets*

- $q - \mathbb{N} = \{q, q-1, q-2, ...\}$.
- $q + \mathbb{N} = \{q, q+1, q+2, ...\}$.
- $q + \mathbb{Z} = \{\ldots, q-1, q, q+1, ...\}$

are called a left set, right set, *and* two sided set *respectively if they contain no problem points. The set $q + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$ is called a* finite singularity *if it does contain problem points.*

Remark 2.4. *If $q + \mathbb{Z}$ is a finite singularity, then it will still contain a left set and a right set*[1] *because the number of problem points is finite. Relating solutions on left sets and right sets will be investigated in Section 2.5.*

Definition 2.5. *Let L be a recurrence operator. A* left solution *of L is a function defined on some left-set $q - \mathbb{N}$ that satisfies the recurrence given by L. We denote the set of* left solutions *on $q - \mathbb{N}$ as*

$$V_{q-\mathbb{N}}(L) := \{u : q - \mathbb{N} \to \mathbb{C} \mid u \text{ satisfies } L\}.$$

Likewise we can define sets of right solutions *and* two-sided solutions*.*

The definition is motivated by the following lemma.

Lemma 2.6. *If $q - \mathbb{N}$ is a left set of L and $a+1, ..., a+n \in q - \mathbb{N}$ and $c_1, ..., c_n \in \mathbb{C}$, then $\exists!\, u \in V_{q-\mathbb{N}}(L)$ such that $u(a+i) = c_i$ where $i = 1, ..., n$. Hence $dim(V_{q-\mathbb{N}}(L)) = n$. Similarly for right sets and two-sided sets.*

Proof. It is trivial because given $u(a+1), \ldots, u(a+n)$, we can compute $u(a)$ since a is not a problem point (see Definition 2.3), as well as $u(a+n+1)$ if $a+n+1$ is still in $q - \mathbb{N}$, so $u(a+1), \ldots, u(a+n)$ give $u(x)$ for all $x \in q - \mathbb{N}$. □

[1] denoted $q_l - 1 - \mathbb{N}$ and $q_r + 1 + \mathbb{N}$ in 2.3

Example 2.7. *Let $L = \tau - x$ be a difference operator. The Gamma function $\Gamma(x) = (x-1)!$ satisfies L. Here we will illustrate the definitions viewed from this solution.*

The Gamma function has poles of order 1 at non-positive integers and no root or pole at positive integers. This shows $\Gamma(x)$ is a right solution on the right set $1 + \mathbb{N} = \{1, 2, 3, \ldots\}$, and it is also a two sided solution on $q + \mathbb{Z}$ for any $q \notin \mathbb{Z}$.

Now $0 - \mathbb{N} = \{0, -1, -2, \ldots\}$ is a left-set, but $1 - \mathbb{N} = \{1, 0, -1, \ldots\}$ is not, because if we try to compute $u(0)$ from $u(1)$ then we get a division by zero error. To obtain a solution on $-\mathbb{N}$, we can choose $u(0) = 1$. Then the recurrence gives $u(-1) = -1$, $u(-2) = \frac{1}{2}$ and $u(-3) = \frac{-1}{6}$ and so.

If we try to extend the function $u : 0 - \mathbb{N} \to \mathbb{C}$ to positive integers, then we find that it is zero there. Conversely, if we try to extend the Gamma function from $1 + \mathbb{N}$ to $0 - \mathbb{N}$ then we get poles.

To relate solutions on left sets to solutions on right sets, we will introduce ε-method in Section 2.5 which is a method to bypass the problem points. This will then also allow us to measure difference between the *valuations* (orders of roots or poles, see Definition 2.23) on right-sets versus left-sets. The above example is then characterized by saying that it has valuation-growth 1 on $\mathbb{Z}$ and valuation-growth 0 on $q + \mathbb{Z}$ for every $q \notin \mathbb{Z}$.

2.2 Difference Ring and Difference Operators

Definition 2.8. *A* difference ring *is a commutative ring R together with an automorphism $\tau : R \to R$. If R is a field then we say R is a difference field. The constants of a difference ring R are the elements $c \in R$ satisfying $\tau(c) = c$.*

In this thesis R is usually the difference field $\mathbb{C}(x)$ with τ being the *shift operator* that sends $a(x)$ to $a(x+1)$, or an extension of $\mathbb{C}(x)$ such as Definition 2.11 below.

Definition 2.9. *Let $D = \mathbb{C}(x)[\tau, \tau^{-1}] = \{\sum_{i=n}^{m} a_i \tau^i \mid a_i \in \mathbb{C}(x),\ n, m \in \mathbb{Z},\ n \leq m\}$. A* difference operator *is an element $L \in D$. Here $L = \sum_{i=n}^{m} a_i \tau^i$ acts on $u(x)$ as*

$$L(u)(x) = \sum_{i=n}^{m} a_i u(x+i). \tag{2.5}$$

A difference operator L corresponds to a recurrence relation $L(u) = 0$. The set D is a non-commutative ring, where multiplication is defined as composition of operators.

Example 2.10. *Let $a(x), b(x) \in \mathbb{C}$. Then $(\tau + a(x)) \cdot (\tau + b(x)) = \tau^2 + (a(x) + b(x+1))\tau + a(x)b(x)$. If we abbreviate $a(x), b(x)$ as a, b then we would write this as $(\tau + a)(\tau + b) = \tau^2 + (a + \tau(b))\tau + ab$.*

Definition 2.11. *([14, Example 1.3]) Let the ring* $S = \mathbb{C}^{\mathbb{N}}/\sim$ *where* $u_1 \sim u_2$ *if there exist* $N \in \mathbb{N}$ *such that, for all* $x > N$*,* $u_1(x) = u_2(x)$*.*

Let τ act on S by $\tau(u(x)) = u(x+1)$. This is an automorphism of S due to the equivalence $\sim$. We can multiply elements of S with elements of $\mathbb{C}(x)$ (the finitely many poles of a a rational function can be ignored due to $\sim$). Hence S is a D-module.

Definition 2.12. *Let*

$$V_S(L) = \{u \in S | L(u) = 0\}$$

be the set of solutions of L *in* S*. We also write* $V(L)$ *instead of* $V_S(L)$*.*

Lemma 2.13. *([13, Theorem 8.2.1])* $V(L)$ *is a* $\mathbb{C}$*-vector space of dimension ord(L), defined below.*

Definition 2.14. *Let* $L = \sum_{i=n}^{m} a_i \tau^i$ *where* $a_m, a_n \neq 0$*. The order of* L *denote by* $ord(L) = m - n$*.*

Definition 2.15. *([14, Chapter 1]) Let* $k = \mathbb{C}(x)$*. A* k*-algebra* V *is called a* universal extension *of* k *if the following three conditions hold*

- $\tau\colon V \to V$ *is an automorphism that extends* $\tau\colon k \to k$*.*
- *For every* $L \in D$ *the kernel of* $L : V \to V$ *is an ord(L)-dimensional* $\mathbb{C}$*-vector space.*
- *For every* $u \in V$ *there exists a non-zero* $L \in D$ *such that* $L(u) = 0$*.*

The set S from definition 2.11 is almost a universal extension because it meets the first and second conditions. If we delete elements from S that do not satisfy any non-zero $L \in D$ then what remains is a universal extension.

2.3 Difference Module

Definition 2.16. *A* difference module *is a* D*-module that is finite dimensional as a* $\mathbb{C}(x)$*-vector space. Equivalently, it is a finite-dimensional* $\mathbb{C}(x)$*-vector space* M *equipped with a bijection* $\tau : M \to M$ *that satisfies*

- $\tau(m_1 + m_2) = \tau(m_1) + \tau(m_2)$ *for all* $m_1, m_2 \in M$
- $\tau(fm) = \tau(f)\tau(m)$ *for all* $f \in \mathbb{C}(x)$ *and* $m \in M$*.*

Theorem 2.17. *[14] (cyclic vector theorem). Every difference module is isomorphic to a module of the form* D/DL *for some* $L \in D$*.*

Remark 2.18. *Let $L \in D$ have order n. Then $\{\tau^0, \dots, \tau^{n-1}\}$ is a $\mathbb{C}(x)$-vector space basis of D/DL. By taking the remainder modulo L, we can write $\tau^n, \tau^{n+1}, \dots$ as $\mathbb{C}(x)$-linear combinations of this basis. We can do the same for $\tau^{-1}, \tau^{-2}, \dots$. This way, any b in D/DL can be written in* standard representation*, meaning, as a $\mathbb{C}(x)$-linear combination of $\{\tau^0, \dots, \tau^{n-1}\}$.*

Definition 2.19. *Let M be a D-module and $b \in M$. The* minimal operator *of b is the monic generator of the left ideal $\{L \in D \mid L(b) = 0\}$ of D and it is denoted by MinOp(b,D).*

A D-module M is called irreducible if it has no non-trivial sub-modules. This is equivalent to saying that the minimal operator of any non-zero element is an irreducible operator of order $\dim(M)$. All of these operators are *gauge equivalent* (Definition 2.20 below) to each other.

2.4 Transformations

Our goal is, given an operator L, to transform it to an equivalent operator in simplest form. There are two (listed in subsections 2.4.1 and 2.4.2 below) types of transformations that we can use for this purpose, that (i) preserve the order, and (ii) allow the solutions of each to be written in terms of the solutions of the other.

2.4.1 Gauge Transformations

Definition 2.20. *[14, Section 2.1] Let L_1 and L_2 two difference operators. L_1 is* gauge equivalent *to L_2 (notation: $L_1 \sim_g L_2$) when D/DL_1 and D/DL_2 are isomorphic as D-modules.*

Lemma 2.21. *[14, Section 2.3] Let L_1 and L_2 two difference operators then the following are equivalent:*

- *L_1 is* gauge equivalent *to L_2*
- *$\exists_{G \in D}$ such that $G(V(L_1)) = V(L_2)$ and L_1, L_2 have the same order.*

This G gives a bijection $V(L_1) \to V(L_2)$ called a gauge transformation.

2.4.2 Term Transformations

Definition 2.22. *[8] Let L_1, L_2 be difference operators, then the* symmetric product *of L_1 and L_2 is an operator, denoted by $L_1 \otimes L_2$, whose solution space is*

$$\text{SPAN}\{u_1 \cdot u_2 \mid L_1(u_1) = 0,\ L_2(u_2) = 0\}. \tag{2.6}$$

If we restrict L_2 to order 1, say $L_2 = \tau - r$, then the transformation $L_1 \mapsto L_1 \otimes (\tau - r)$ is order preserving. This is called a *term transformation* [3] because it multiplies the solutions of L_1 by a solution of $\tau - r$, and solutions of first order operators are called *hypergeometric terms*.

Our goal is to find transformations that reduce an operator to its simplest form. Since a term transformation is determined by a single rational function r, it relatively easy to find a suitable choice for r, see [3, Section 2.3.2]. Thus we will focus on finding gauge-transformations.

2.5 Valuation Growth

The concept of valuation growth was introduced in [9], as well as an algorithm to compute $\overline{g}_p(L)$, the set of valuation growths of L at $p \in \mathbb{C}/\mathbb{Z}$ (see also Definition 2.31). For a finite singularity we discussed left and right solutions of the difference equation in Section 2.1. The key to computing valuations, valuation growths, and relating right solutions to left solutions, is to deform the operator by substituting $x \to x + \varepsilon$, see Definition 2.26. After this deformation, the solutions are defined over an extended field of constants, $\mathbb{C}(\varepsilon)$. By analyzing the difference in ε-valuation between these left and right solutions, we obtain $\overline{g}_p(L)$, the set of valuation growths.

Definition 2.23. *A* valuation on an ring R *is a map* $v : R \to \mathbb{Z} \cup \{\infty\}$ *such that*

- $v(f) = \infty$ *if and only if* $f = 0$.
- $v(fg) = v(f) + v(g)$ *for all* $f, g \in R \setminus \{0\}$.
- $v(f + g) \geq min(v(f), v(g))$ *with equality if* $v(f) \neq v(g)$.

Example 2.24. *Let* $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *be a meromorphic function and* $p \in \mathbb{C}$. *Let*

$$v_p(f) = \begin{cases} n & \text{if } f \text{ has a root at } p \text{ of order } n \\ \text{-}n & \text{if } f \text{ has a pole at } p \text{ of order } n \\ 0 & \text{if } f \text{ has no root nor pole at } p \\ \infty & \text{if } f = 0 \end{cases}$$

Then $v_p : R \to \mathbb{Z} \cup \{\infty\}$ *is a valuation where* R *is the field of meromorphic function.*

Example 2.25. *From Example 2.7,* $\Gamma(x) = (x-1)!$ *is a solution of* $L = \tau - x$. *Let* $q \in \mathbb{C}$, *then* $v_q(\Gamma)$ *is -1 if* $q \in \{0, -1, -2, \ldots\}$ *and 0 otherwise. By* valuation growth *at* $q + \mathbb{Z}$ *we mean the valuation on the right minus the valuation on the left. As seen in Figure 2.1 below, the valuation growth of* $\Gamma(x)$ *is* $0 - (-1) = 1$ *at* $\mathbb{Z}$. *It is* 0 *at* $q + \mathbb{Z}$ *when* $q \notin \mathbb{Z}$.

x	-4	-3	-2	-1	0	1	2	3	4	5
$\Gamma(x)$	$\cdots$	*pole*	*pole*	*pole*	*pole*	1	1	2	6	$\cdots$
valuation of $\Gamma(x)$	$\cdots$	-1	-1	-1	-1	0	0	0	0	$\cdots$
$u(x)$	$\cdots$	$\frac{-1}{6}$	$\frac{1}{2}$	-1	1	0	0	0	0	$\cdots$
valuation of $u(x)$	$\cdots$	0	0	0	0	1	1	1	1	$\cdots$

Figure 2.1: Comparing the Valuation of Right and Left Solutions.

The table also lists $u(x)$ from Example 2.7 except that we used the recurrence to extend its domain from $-\mathbb{N}$ to $\mathbb{Z}$. Observe that $u(x)$ has roots at $x = 1, 2, 3, \ldots$. It is not a priori obvious how to define the valuation of $u(x)$ at $x = 1, 2, 3, \ldots$ or that this valuation should be 1. This will be explained in Example 2.28. The valuation growth *of $u(x)$ is $1 - 0 = 1$, the same as for $\Gamma(x)$.*

Definition 2.26. *Let $L = \sum_{i=n}^{m} a_i(x)\tau^i, a_i \in \mathbb{C}(x)$ with $a_n \neq 0, a_m \neq 0$ be a difference operator. We define $L_\varepsilon = \sum_{i=n}^{m} a_i(x+\varepsilon)\tau^i$, i.e. substituting $x \mapsto x + \varepsilon$ in L. The map $L \mapsto L_\varepsilon$ defines an embedding (as non-commutative rings) of $\mathbb{C}(x)[\tau, \tau^{-1}]$ in $\mathbb{C}(x, \varepsilon)[\tau, \tau^{-1}]$.*

The ε-method moves a singularity $p \in \mathbb{C}/\mathbb{Z}$ to $p - \varepsilon \in \mathbb{C}(\epsilon)/\mathbb{Z}$. This makes p non-singular, as it replaces divisions by zero with divisions by ε. The field of constants increases from $\mathbb{C}$ to $\mathbb{C}(\varepsilon)$.

Definition 2.27. *Let $a \in \mathbb{C}(\varepsilon)$. The* valuation *$v_\varepsilon(a) \in \mathbb{Z} \cup \{\infty\}$ of a at $\varepsilon = 0$ is*

$$v_\varepsilon(a) = \sup\{m \in \mathbb{Z} \mid \frac{a}{\varepsilon^m} \in \mathbb{C}[[\varepsilon]]\}.$$

Example 2.28. *Continuing example 2.7, the new recurrence relation is L_ε: $u(x+1) = (x+\varepsilon)\cdot u(x)$. If we choose $u(0) = 1$, we can use L_ε to compute the previous terms $u(-1) = \frac{1}{(-1+\varepsilon)}$, $u(-2) = \frac{1}{(-1+\varepsilon)(-2+\varepsilon)}, \ldots$ as well as the next terms $u(1) = \varepsilon$, $u(2) = \varepsilon \cdot (1+\varepsilon), \ldots$. Their valuations at $\varepsilon = 0$ are the valuations listed in the last row of Figure 2.1. Substituting $\varepsilon = 0$ in $u(x)$ produces the $u(x)$ from Figure 2.1.*

Let $p \in \mathbb{C}/\mathbb{Z}$ and $L = a_n(x)\tau^n + \cdots + a_0(x)\tau^0 \in \mathbb{C}(x)[\tau]$. Due to the ε shift, p will automatically non-singular and hence be a *two sided set* (Definition 2.3) for L_ε. As in Definition 2.5 we denote the set of two-sided solutions of L_ε at p as

$$V_p(L_\varepsilon) = \{u : p \to \mathbb{C}(\varepsilon) \mid L_\varepsilon(u) = 0\}.$$

Definition 2.29. *Let* $L = a_n(x)\tau^n + \cdots + a_0(x)\tau^0 \in \mathbb{C}(x)[\tau]$. *Let* q_l *be the smallest root of* $a_0(x)a_n(x-n)$ *in* p, *i.e. the smallest* problem point *in* p *(Definition 2.2). Likewise, let* q_r *be the largest root of* $a_0(x)a_n(x-n)$ *in* p. *(If* p *is not singular, if there are no problem points in* p, *then choose two arbitrary elements* $q_l, q_r \in p$*). We define two bases* $u_1, \ldots, u_n$ *and* $v_1, \ldots, v_n$ *of* $V_p(L_\varepsilon)$ *by requiring*

$$u_i(q_l - j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad v_i(q_r + j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Since there are no problem points in $\{q_l - 1, q_l - 2, \ldots\}$, substituting $\varepsilon = 0$ in u_i produces a basis of $V_{q_l-1-\mathbb{N}}(L)$, and likewise, substituting $\varepsilon = 0$ in v_i produces a basis of $V_{q_r+1+\mathbb{N}}(L)$. See Example 3.10.

Definition 2.30. *For non-zero* $u \in V_p(L_\varepsilon)$. *The* left ε-valuation *is*

$$v_{\varepsilon,l}(u) = min\{v_\varepsilon(u(m)) \mid m \in q_l - 1, \ldots, q_l - n\} = \liminf_{m\in p,\ m\to-\infty} v_\varepsilon(u(m))$$

(for the latter equality see [9, Section 4.1], or Corollary 2.34). Likewise the right ε-valuation *is*

$$v_{\varepsilon,r}(u) = min\{v_\varepsilon(u(m)) \mid m \in q_r + 1, \ldots, q_r + n\} = \liminf_{m\in p,\ m\to\infty} v_\varepsilon(u(m)).$$

Definition 2.31. *([9, Definition 13]) Define the* valuation growth *of* $u \in V_p(L_\varepsilon) - \{0\}$ *as*

$$g_{p,\varepsilon}(u) = v_{\varepsilon,r}(u) - v_{\varepsilon,l}(u) \in \mathbb{Z}.$$

Also define the the set of valuation growths *of* L *at* p *as*

$$\overline{g}_p(L) = \{g_{p,\varepsilon}(u) \mid u \in V_p(L_\varepsilon) - \{0\}\} \subset \mathbb{Z}.$$

The minimum valuation growth *of* L *at* p *is*

$$g_{p,\min}(L) = \min(\overline{g}_p(L))$$

The maximum valuation growth *of* L *at* p *is*

$$g_{p,\max}(L) = \max(\overline{g}_p(L)).$$

The set of valuation growths can be computed with the following lemma.

Lemma 2.32. *([5, Definition 10]) Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ as in Definition 2.29. Then*

$$g_{p,\min}(L) = \min\{v_\varepsilon(u_*(q)) \mid q \in p,\ q - q_r > 0\}$$

$$g_{p,\max}(L) = -\min\{v_\varepsilon(v_*(q)) \mid q \in p,\ q - q_l < 0\}$$

$$\overline{g}_p(L) = \{m \in \mathbb{Z} \mid g_{p,\max}(L) \leq m \leq g_{p,\min}(L)\}$$

where we use the abbreviation

$$v_\varepsilon(u_*(q)) := \min\{v_\varepsilon(u_i(q)) \mid i = 1, \ldots, n\}$$

and likewise for $v_\varepsilon(v_*(q))$.

The following lemma and its corollary show that the minima in the first two formulas in lemma 2.32 can be computed by taking the minimum valuation of at n consecutive q's.

Lemma 2.33. *Let $L \in \mathbb{C}[x][\tau]$. If $p = q + \mathbb{Z}$, $u \in V_p(L_\varepsilon)$, and q, $q+n$ are not problem points (see Definition 2.2), then*

$$\min(v_\varepsilon(u(q+i)), i = 0, \ldots, n-1) = \min(v_\varepsilon(u(q+i)), i = 1, \ldots, n)$$

Proof. Let m_1 be the left-hand-side and let m_2 be the right-hand-side. It suffices to prove that $v_\varepsilon(u(q+n)) \geq m_1$ and $v_\varepsilon(u(q)) \geq m_2$.

Write $L = \sum_{i=0}^{n} a_i(x)\tau^i$ with $a_i \in \mathbb{C}[x]$, then $L_\varepsilon = \sum_{i=0}^{n} a_i(x+\varepsilon)\tau^i$ and thus

$$\sum_{i=0}^{n} a_i(q+\varepsilon)u(q+i) = 0 \tag{2.7}$$

$$u(q+n) = -\frac{\sum_{i=0}^{n-1} a_i(q+\varepsilon)u(q+i)}{a_n(q+\varepsilon)} \tag{2.8}$$

Now for $i = 0, \ldots, n-1$, $v_\varepsilon(a_i(q+\varepsilon)) \geq 0$, and $v_\varepsilon(u(q+i)) \geq m_1$. Then, by the third property of valuation from Definition 2.23 for $i = 0, \ldots, n-1$

$$v_\varepsilon(a_i(q+\varepsilon)u(q+i)) \geq m_1$$

Furthermore, $v_\varepsilon(a_n(q+\varepsilon)) = 0$ because $q+n$ is not a problem point (i.e. q is not a root of a_n from Definition 2.2). Thus equation (2.8) implies $v_\varepsilon(u(q+n)) \geq m_1$. Similarly, $v_\varepsilon(u(q)) \geq m_2$ due to the fact that q is not a problem point (equivalent to $v_\varepsilon(a_0(q+\varepsilon)) = 0$). □

Corollary 2.34. *Let $p \in \mathbb{C}/\mathbb{Z}$. Let $q_l, q_r \in p$ were defined in such away that there are no problem points in $q_r + 1 + \mathbb{N}$ and $q_l - 1 - \mathbb{N}$. Let $u \in V_p(L_\varepsilon)$. Let*

$$m = \liminf_{i \to \infty, i \in \mathbb{N}} v_\varepsilon(u(q_r + 1 + i))$$

from Definition 2.30. Then m is the minimum valuation of u at any n consecutive points in $q_r + 1 + \mathbb{N}$, i.e.

$$m = \min \{v_\varepsilon(u(q_r + k + 1)), \ldots, v_\varepsilon(u(q_r + k + n))\} \quad \text{for any } k \in \mathbb{N}.$$

So this m is the number $v_{\varepsilon,r}(u)$ from Definition 2.30. A similar property holds for $v_{\varepsilon,l}(u)$.

Remark 2.35. *Let $m = v_{\varepsilon,r}(u)$. Then $(u/\varepsilon^m)^{\varepsilon=0} \in V_{q_r+1+\mathbb{N}}(L)$ is a nonzero right solution of L. Similarly, dividing u by $\varepsilon^{v_{\varepsilon,l}(u)}$ and substituting $\varepsilon = 0$ gives a nonzero element of $V_{q_l-1-\mathbb{N}}(L)$.*

CHAPTER 3

INTEGRAL ELEMENTS OF DIFFERENCE MODULES

3.1 How to Define Integral Elements

Definition 3.1. *[4, Section 5.2] Let $L \in \mathbb{C}(x)[\tau]$ be a difference operator, and let $b \in D/DL$. Define the* valuation *at $z \in \mathbb{C}$ as*

$$val_z(b) = \min_{u \in Sol(L)} \left[v_\epsilon(b(u)(z)) - \liminf_{n \to \infty} v_\epsilon(u(z-n)) \right].$$

Then b is called integral *at z if $val_z(b) \geq 0$. Note that we will not use this definition in this thesis, and will give a different definition instead.*

A problem with the above definition is that b could be non-integral at infinitely many $z \in \mathbb{C}$. To define integral elements of $M = D/DL$, the paper [4] limits z to a finite range. However, this range depends on L. This means that if $D/DL_1 \cong D/DL_2$ (if L_1, L_2 are gauge-equivalent, Definition 2.20), then the sets of integral elements in these modules can differ with the above definition from [4], despite the modules being isomorphic. For this reason we will give a different definition in this thesis, in Definition 3.3 below, with an equivalent but shorter condition given in Theorem 3.4. The idea behind our definition comes from the following lemma.

Lemma 3.2. *Let $p \in \mathbb{C}/\mathbb{Z}$. Let $q_l, q_r \in p$ and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ as in Definition 2.29. Then*

$$v_\varepsilon(u_i(q))) \geq \begin{cases} 0 & \forall q \in q_l - 1 - \mathbb{N} \\ g_{p,\min} & \forall q \in q_r + 1 + \mathbb{N} \end{cases} \qquad v_\varepsilon(v_i(q))) \geq \begin{cases} -g_{p,\max} & \forall q \in q_l - 1 - \mathbb{N} \\ 0 & \forall q \in q_r + 1 + \mathbb{N} \end{cases}$$

Proof. From the description of $g_{p,\min}$ in Lemma 2.32 and the fact that $q_r + 1 + \mathbb{N}$ is a right-set (it contains no problem points), Corollary 2.34 then gives $v_\varepsilon(u_i(q))) \geq g_{p,\min}$ for all $q \in q_r + 1 + \mathbb{N}$. In a similar way Corollary 2.34 gives $v_\varepsilon(u_i(q))) \geq 0$ for all q in the left-set $q_l - 1 - \mathbb{N}$. The argument for the v_i's is similar. □

The lemma motivates the following definition, because it shows that if we take $b = 1$, so that $b(u_i)$ is just u_i, then b will satisfy the following definition of integral at all but finitely many $q \in \mathbb{C}$.

Definition 3.3. *Let $q \in \mathbb{C}$, $p = q + \mathbb{Z}$ and $L \in D$. Then $b \in D/DL$ is* left-integral at q *when*

$$v_\varepsilon(b(u_*)(q)) \;\geq\; v^a_{q,l} := \begin{cases} 0 & Re(q) < 1 \\ g_{p,\min} & Re(q) \geq 1 \end{cases} \tag{3.1}$$

and right-integral at q *when*

$$v_\varepsilon(b(v_*)(q)) \;\geq\; v^a_{q,r} := \begin{cases} -g_{p,\max} & Re(q) < 1 \\ 0 & Re(q) \geq 1 \end{cases} \tag{3.2}$$

where $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are as in Definition 2.29, and, similar to Lemma 2.32, we abbreviate

$$v_\varepsilon(b(u_*)(q)) := \min\{v_\varepsilon(b(u_i)(q)) \mid i = 1, \ldots, n\}$$

and likewise $b(v_*)$ refers to $b(v_i)$ for all i. If b is left and right integral at q then we say that b is integral at q. *We call these numbers $v^a_{q,l}$ and $v^a_{q,r}$ the* allowed left and right valuations. *We denote the set of all integral elements of $M = D/DL$ as O_M.*

Theorem 3.4. *Let $p \in \mathbb{C}/\mathbb{Z}$ and $b \in D/DL$. Then b is integral at every $q \in p \iff$ the following conditions hold for all $q \in p$.*

$$v_\varepsilon(b(u_*))(q) \geq 0 \quad \textit{if } \mathrm{Re}(q) < 1 \tag{3.3}$$

$$v_\varepsilon(b(v_*))(q) \geq 0 \quad \textit{if } \mathrm{Re}(q) \geq 1 \tag{3.4}$$

Proof. ($\Rightarrow$) Follows immediately from Definition 3.3 because we simply deleted the inequalities involving $g_{p,\min}$ and $g_{p,\max}$.

($\Leftarrow$) Assume (3.3) and (3.4) for all $q \in p$. We have to prove the deleted inequalities:

$$v_\varepsilon(b(u_i))(q) \geq g_{p,\min} \quad \text{if } \mathrm{Re}(q) \geq 1 \tag{3.5}$$

$$v_\varepsilon(b(v_i))(q) \geq -g_{p,\max} \quad \text{if } \mathrm{Re}(q) < 1 \tag{3.6}$$

for $i = 1, \ldots, n$. Write $u_i = a_1 v_1 + \cdots + a_n v_n$ where $a_i \in \mathbb{C}(\varepsilon)$. Lemma 2.32 shows (3.5), but only for $b = 1$ and $q \in \{q_r + 1, q_r + 2 \ldots\}$. That implies $v_\varepsilon(a_j) \geq g_{p,\min}$ for every j because $a_j = u_i(q_r + j)$, see Definition 2.29. Now (3.5) follows from $b(u_i) = a_1 b(v_1) + \cdots + a_n b(v_n)$ and (3.4). The proof for (3.6) is similar. □

Corollary 3.5. *Let*

$$c = \prod_{q \in \mathbb{C}} (x - q)^{\begin{cases} -v_\varepsilon(b(u_*)(q)) & Re(q) < 1 \\ -v_\varepsilon(b(v_*)(q)) & Re(q) \geq 1 \end{cases}} \tag{3.7}$$

and $r \in \mathbb{C}(x)$. *Then*

$$r \cdot b \in O_M \Longleftrightarrow r \in c \cdot \mathbb{C}[x]. \tag{3.8}$$

Let

$$c_i = \prod_{q \in \mathbb{C}} (x-q)^{\begin{cases} -v_\varepsilon(u_*(q+i)) & Re(q) < 1 \\ -v_\varepsilon(v_*(q+i)) & Re(q) \geq 1. \end{cases}} \tag{3.9}$$

Then $r \cdot \tau^i \in O_M \Longleftrightarrow r \in c_i \cdot \mathbb{C}[x]$. *To obtain this formula for* c_i, *note that substituting* $b = \tau^i$ *in (3.7) shifts* q *to* $q + i$.

Notation 3.6. *(Standard representative) If* $p \in \mathbb{C}/\mathbb{Z}$ *then* $\exists!\, q \in p$ *with* $Re(q) \in [0, 1)$.

Lemma 3.7. *Let* $p_1, \ldots, p_l \in \mathbb{C}/\mathbb{Z}$ *be the singularities of* L. *Write* $p_i = q_i + \mathbb{Z}$ *with* $q_i \in \mathbb{C}$ *and* $Re(q_i) \in [0, 1)$. *Let* m_i, M_i *be the* $g_{p_i,\min}, g_{p_i,\max}$ *from Definition 2.31. If* $b \in D/DL$ *is integral, then so are:* $P \cdot \tau(b)$ *and* $Q \cdot \tau^{-1}(b)$ *where*

$$P = \prod_{i=1}^{n} (x - q_i)^{-m_i} \tag{3.10}$$

$$Q = \prod_{i=1}^{n} (x - q_i - 1)^{M_i} \tag{3.11}$$

Proof. In this proof we only cover the singularity $p_i = \mathbb{Z}$, i.e. $q_i = 0$. The proof for the other singularities in $\mathbb{C}/\mathbb{Z}$ is similar.

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be as before. Then from Equations (3.1) and (3.2), we have

$$v_\varepsilon(\tau(b)(u_*)(q)) = v_\varepsilon(b(u_*)(q+1)) \geq 0, \text{ if } Re(q+1) < 1 \tag{3.12}$$

$$v_\varepsilon(\tau(b)(v_*)(q)) = v_\varepsilon(b(v_*)(q+1)) \geq 0, \text{ if } Re(q+1) \geq 1 \tag{3.13}$$

Equations (3.12) and (3.13) imply $\tau(b)$ is integral at all $q \in \mathbb{Z}$ except possibly at $q = 0$. To make $\tau(b)$ integral at $q = 0$ $(= q_i)$ we multiply by $(x - q_i)^{-m_i}$.

Next we examine $\tau^{-1}(b)$. Applying τ^{-1} to b in Equations (3.1) and (3.2) gives

$$v_\varepsilon(\tau^{-1}(b)(u_*)(q)) = v_\varepsilon(b(u_*)(q-1)) \geq 0, \text{ if } Re(q-1) < 1 \tag{3.14}$$

$$v_\varepsilon(\tau^{-1}(b)(v_*)(q)) = v_\varepsilon(b(v_*)(q-1)) \geq 0, \text{ if } Re(q-1) \geq 1 \tag{3.15}$$

At $q = 1$ we have $v_\varepsilon(b) \geq 0$ while $v_\varepsilon(\tau^{-1}(b)) \geq -M_i$. Consequently, $\tau^{-1}(b)$ is integral at all $q \in \mathbb{Z}$ except possibly at $q = 1$ $(= q_i + 1)$. To remedy this we multiply by $(x - q_i - 1)^{M_i}$. □

Remark 3.8. *Let* $b = c_i \tau^i \in O_M$ *with* c_i *as in Equation (3.9) and let* P *be as in Lemma 3.7. Then* $P \cdot \tau(c_i) \in c_{i+1} \cdot \mathbb{C}[x]$.

Proof. $P \cdot \tau(c_i)\tau^{i+1} = P \cdot \tau(b) \in O_M$ by Lemma 3.7. The result then follows from Corollary 3.5. □

3.2 Example Illustrating the Definitions

In the remainder of this section, we provide examples to illustrate Definition 3.3. In the next examples we will have $p = \mathbb{Z}$.

3.2.1 Example of Order 2

Example 3.9.

$$L = (x-5)\tau^2 + \tau + (x-2) \tag{3.16}$$

Let u_1, u_2 be the "left basis" of $V_{\mathbb{Z}}(L_\varepsilon)$ as defined in Definition 2.29. As mentioned there, substituting $\varepsilon = 0$ into u_1, u_2 yields a basis $u_1^{\varepsilon=0}, u_2^{\varepsilon=0}$ of $V_{1-\mathbb{N}}(L)$. This substitution does not necessarily produce elements $V_{\mathbb{Z}}(L)$ because of possible poles at $\epsilon = 0$ as illustrated in the next figure.

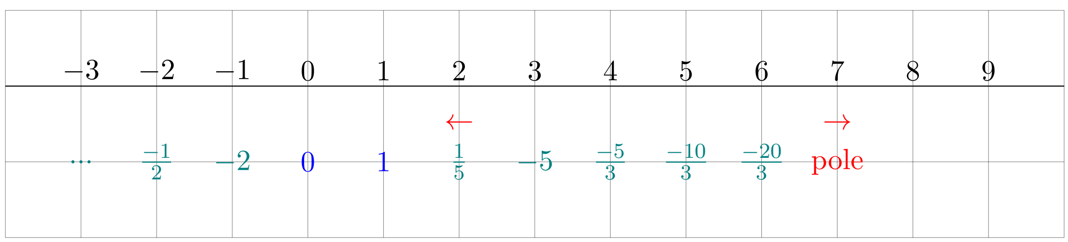

Figure 3.1: Attempting to Compute $u_1^{\varepsilon=0}$ in $V_{\mathbb{Z}}(L)$

In figure 3.2 below, the blue numbers are the initial values from Definition 2.29 and the green numbers are computed from the recurrence relation (3.16). We drew cut off lines *L_1, L_2 to indicate q_l, q_r from Definition 2.29. In principle, $u_1^{\varepsilon=0}, u_2^{\varepsilon=0}$ in figure 3.2 could be extended past L_1 to $x = 2, 3, 4, 5, 6$ (like in Figure 3.1). However, we do not go beyond L_1 because Corollary 2.34 no longer holds there.*

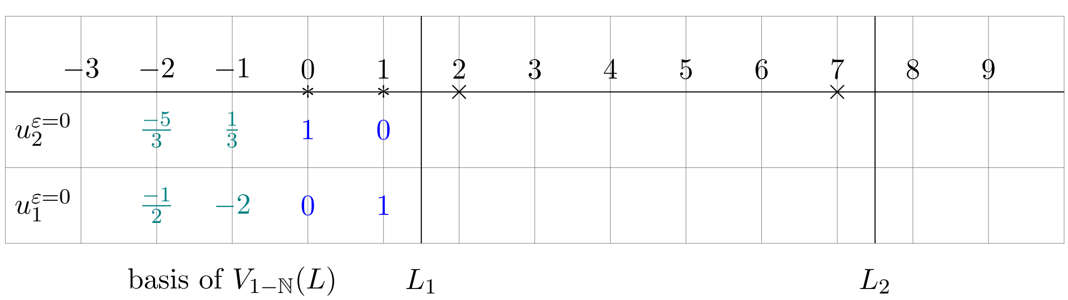

Figure 3.2: $u_1^{\varepsilon=0}, u_2^{\varepsilon=0} \in V_{1-\mathbb{N}}(L)$

Now, we will do the same for the "right basis" v_1, v_2 from Definition 2.29.

	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	
					$\times$					$\times$					
											1	0	-2	$\frac{1}{2}$	$v_1^{\varepsilon=0}$
											0	1	$\frac{1}{3}$	$\frac{11}{6}$	$v_2^{\varepsilon=0}$

L_1 (between 1 and 2) $\quad L_2$ (between 7 and 8) $\quad$ basis of $V_{8+\mathbb{N}}(L)$

Figure 3.3: $v_1^{\varepsilon=0}, v_2^{\varepsilon=0} \in V_{8+\mathbb{N}}(L)$

Any problem points are between lines L_1 and L_2. Outside these two lines are the regions where Corollary 2.34 applies. Remark 2.35 explains how "left solutions" (elements of $V_{1-\mathbb{N}}(L)$, see Figure 3.2) can be mapped to "right solutions" (elements of $V_{8+\mathbb{N}}(L)$, see Figure 3.3). This cannot be done with the recurrence (3.16) as that would lead to division by zero errors at problem points.

To illustrate Remark 2.35, observe in the next figure that $(u_1/\varepsilon^{-1})^{\varepsilon=0}$ is in $V_{8+\mathbb{N}}(L)$. Note that $(u_2/\varepsilon^{-1})^{\varepsilon=0}$ is a scalar multiple of $(u_1/\varepsilon^{-1})^{\varepsilon=0}$. So the map from $V_{1-\mathbb{N}}(L)$ to $V_{8+\mathbb{N}}(L)$ in Remark 2.35 is not a bijection.

	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
				$*$	$*$	$\times$					$\times$			
u_2		$\frac{1}{12}+\varepsilon$	$\frac{1}{3}+\varepsilon$	1	0	$\frac{-2}{5}+\varepsilon$	$\frac{-1}{10}+\varepsilon$	$\frac{-1}{30}+\varepsilon$	$\frac{-1}{15}+\varepsilon$	$\frac{-2}{15}+\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{-1}{3}\varepsilon^{-1}$	$\frac{-2}{3}\varepsilon^{-1}$	$\frac{8}{9}\varepsilon^{-1}$
u_1		$\frac{-1}{2}-\varepsilon$	$-2-\varepsilon$	0	1	$\frac{1}{5}+\varepsilon$	$\frac{-1}{5}+\varepsilon$	$\frac{-1}{15}+\varepsilon$	$\frac{-2}{15}+\varepsilon$	$\frac{-4}{15}+\varepsilon$	$\frac{2}{3}\varepsilon^{-1}$	$\frac{-2}{3}\varepsilon^{-1}$	$\frac{-4}{3}\varepsilon^{-1}$	$\frac{16}{9}\varepsilon^{-1}$
$v_\varepsilon(u_2)$		0	0	0	$+\infty$	0	0	0	0	0	-1	-1	-1	-1
$v_\varepsilon(u_1)$		0	0	$+\infty$	0	0	0	0	0	0	-1	-1	-1	-1

Figure 3.4: Values (Truncated) and Valuations of u_1, u_2

Note that $g_p(u_1) = g_p(u_2) = -1 - 0 = -1$ (valuation on the right minus valuation on the left).

	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1

Figure 3.5: $v_\varepsilon(u_*)$ from Lemma 2.32 (Also in Definition 3.3)

Next we do the same process, but this time starting with basis v_1, v_2.

	−3	−2	−1	0	1	2 ×	3	4	5	6	7 ×	8 *	9 *	10	11
v_1		$\frac{-17}{9}\varepsilon^{-1}$	$\frac{16}{9}\varepsilon^{-1}$	$\frac{4}{3}\varepsilon^{-1}$	$\frac{-2}{3}\varepsilon^{-1}$	$\frac{-2}{3}\varepsilon^{-1}$	$\frac{4}{15}-\varepsilon$	$\frac{-2}{15}+\varepsilon$	$\frac{1}{15}+\varepsilon$	$\frac{-1}{5}-\varepsilon$	$\frac{-1}{5}+\varepsilon$	1	0	$-2+\varepsilon$	$\frac{1}{2}-\varepsilon$
v_2		$\frac{17}{18}\varepsilon^{-1}$	$\frac{-8}{9}\varepsilon^{-1}$	$\frac{-2}{3}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{-2}{15}+\varepsilon$	$\frac{1}{15}-\varepsilon$	$\frac{-1}{30}+\varepsilon$	$\frac{1}{10}+\varepsilon$	$\frac{-2}{5}+\varepsilon$	0	1	$\frac{-1}{3}+\varepsilon$	$\frac{-5}{3}+\varepsilon$
$v_\varepsilon(v_1)$		−1	−1	−1	−1	−1	0	0	0	0	0	0	$+\infty$	0	0
$v_\varepsilon(v_2)$		−1	−1	−1	−1	−1	0	0	0	0	0	$+\infty$	0	0	0
$v_\varepsilon(v_*)$		−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0

Figure 3.6: Values (Truncated) and Valuations of v_1, v_2

It is observed that $g_p(v_1) = g_p(v_2) = 0 - (-1) = 1$. Now, we will combine the minimum ε-valuation for left and right solution in one diagram and we will draw a cut off line L_3 to separate $Re(q) \leq 1$ from $Re(q) > 1$ see Definition 3.3.

	−2	−1	0	L_3	1	2	3	4	5	6	7	8	9	10	11
$v_\varepsilon(u_*)$	0	0	0		0	0	0	0	0	0	−1	−1	−1	−1	−1
$v_\varepsilon(v_*)$	−1	−1	−1		−1	−1	0	0	0	0	0	0	0	0	0

Figure 3.7: $v_\varepsilon(u_*)$ and $v_\varepsilon(v_*)$

	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
$v^a_{q,l}$	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$v^a_{q,r}$	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0

L_3

Figure 3.8: Valuation Allowed from Definition 3.3

Figures 3.7 and 3.8 show that $b = 1 \in D/DL$ is left-integral (Definition 3.3) at every $q \in \mathbb{Z}$, i.e. we have $v_\epsilon(u_*) \geq v^a_{q,l}$ at every $q \in \mathbb{Z}$. But $b = 1$ is not right-integral (and hence not integral) at $q = 1$ and $q = 2$ as indicated in red in Figures 3.7 and 3.8.

Alternatively, we can the criterion given in Theorem 3.4 to conclude that $b = 1$ is not integral. Corollary 3.5 shows how to remedy this, it shows how to compute $c \in \mathbb{C}(x)$ for which $c \cdot b$ becomes integral. We can see from Figures 3.7 and 3.8 that such c needs to have roots at the points $q = 1$ and $q = 2$.

3.2.2 Example from Prior Work

The previous section showed how to check if $b = \tau^0$ is integral or not. In this section, we will do the same for $b = \tau^i$, and also illustrate Equation (3.9) which produces the minimal degree $c_i \in \mathbb{Q}(x)$ minimal such that $c_i\tau^i$ is integral.

The following example comes from [4]. This section shows how to compute an integral basis with our definition 3.3.

Example 3.10.

$$L = (x+2)\tau^3 + x\tau^2 + (x+2)^2$$

$$(x+2)u(x+3) + xu(x+2) + (x+2)^2u(x) = 0 \tag{3.17}$$

Let u_1, u_2, u_3 be the "left basis" of $V_{\mathbb{Z}}(L_\varepsilon)$ from Definition 2.29. From the recurrence relation 3.17, it is clear that the problem points is at $q_l = -2$ and $q_r = 1$ which are the smallest and largest problem point respectively see Definition 2.29. We noticed that $u_1^{\varepsilon=0}, u_2^{\varepsilon=0}, u_3^{\varepsilon=0}$ are defined on $V_{-3-\mathbb{N}}(L)$. Now, we will pick up the the basis of left solutions $V_{-3-\mathbb{N}}(L)$ from Definition 2.5.

$$u_3^{\varepsilon=0}(-5) = 1,\ u_3^{\varepsilon=0}(-4) = 0,\ u_3^{\varepsilon=0}(-3) = 0$$

$$u_2^{\varepsilon=0}(-5) = 0,\ u_2^{\varepsilon=0}(-4) = 1,\ u_2^{\varepsilon=0}(-3) = 0$$

$$u_1^{\varepsilon=0}(-5) = 0,\ u_1^{\varepsilon=0}(-4) = 0,\ u_1^{\varepsilon=0}(-3) = 1$$

In the next figure, we will pick up the basis for left solution u_1, u_2, u_3 from Definition 2.29, and we will take minimum ε-valuation from Definition 3.3. Then, we will extend the left and right solutions as we did in Example 3.9.

	-8	-7	-6	-5	-4	-3	-2 (q_l)	-1	0	1 (q_r)	2	3	4	5	6	7
u_3	$\frac{1}{6}+\varepsilon$	$\frac{7}{25}+\varepsilon$	0	1	0	0	$3-\varepsilon$	$-6+\varepsilon$	$18+\varepsilon$	$36\varepsilon^{-1}$	$36\varepsilon^{-1}$	$-54+\varepsilon$	$-108\varepsilon^{-1}$	$-90\varepsilon^{-1}$	$54\varepsilon^{-1}$	$612\varepsilon^{-1}$
u_2	$\frac{1}{12}+\varepsilon$	$\frac{1}{5}+\varepsilon$	$\frac{3}{8}+\varepsilon$	0	1	0	0	$2-\varepsilon$	$-6-\varepsilon$	$-12\varepsilon^{-1}$	$-12\varepsilon^{-1}$	$18-\varepsilon$	$36\varepsilon^{-1}$	$30\varepsilon^{-1}$	$-18\varepsilon^{-1}$	$-204\varepsilon^{-1}$
u_1	$\frac{1}{18}+\varepsilon$	0	$\frac{1}{4}+\varepsilon$	0	0	1	$\frac{-5}{3}+\varepsilon$	$\frac{10}{3}+\varepsilon$	$-9+\varepsilon$	$-18\varepsilon^{-1}$	$-18\varepsilon^{-1}$	$27+\varepsilon$	$54\varepsilon^{-1}$	$45\varepsilon^{-1}$	$-27\varepsilon^{-1}$	$-306\varepsilon^{-1}$
$v_\varepsilon(u_3)$	0	0	$+\infty$	0	$+\infty$	$+\infty$	0	0	0	-1	-1	0	-1	-1	-1	-1
$v_\varepsilon(u_2)$	0	0	0	$+\infty$	0	$+\infty$	$+\infty$	0	0	-1	-1	0	-1	-1	-1	-1
$v_\varepsilon(u_1)$	0	$+\infty$	0	$+\infty$	$+\infty$	0	0	0	0	-1	-1	0	-1	-1	-1	-1
$v_\varepsilon(u_*)$	0	0	0	0	0	0	0	0	0	-1	-1	0	-1	-1	-1	-1

L_1 (between -3 and -2) L_2 (between 1 and 2)

Figure 3.9: Values (Truncated) and Valuations of u_1, u_2, u_3

Lemma 2.33 states that the minimum value over any three consecutive points beyond q_r is the same. This minimum value, denoted $g_{p,min}$ (Definition 2.31), is equal to -1. Now to separate $Re(q) \leq 1$ from $Re(q) > 1$, we will draw a cut off line L_3.

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$v^a_{q,l}$	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1

L_3 (between 0 and 1)

Figure 3.10: Valuation Allowed for Left Solutions

To do the same process for the right integral, we require the right solutions v_1, v_2, v_3 as given in Definition 2.30. These are illustrated in the following figure.

	-8	-7	-6	-5	-4	-3	-2 ×	-1	0	1 ×	2	3	4	5	6	7
v_1	$\frac{-5}{72}\varepsilon^{-1}$	$\frac{-11}{75}\varepsilon^{-1}$	$\frac{-3}{16}\varepsilon^{-1}$	$\frac{-1}{6}\varepsilon^{-1}$	$\frac{-1}{2}\varepsilon^{-1}$	$-3-\varepsilon$	$\frac{-1}{2}\varepsilon^{-1}$	$-1+\varepsilon$	$\frac{-1}{4}\varepsilon$	0	1	0	0	$-4-\varepsilon$	$\frac{12}{5}+\varepsilon$	$\frac{-8}{5}-\varepsilon$
v_2	$\frac{-5}{36}\varepsilon^{-2}$	$\frac{-22}{75}\varepsilon^{-2}$	$\frac{-3}{8}\varepsilon^{-2}$	$\frac{-1}{3}\varepsilon^{-2}$	$-\varepsilon^{-2}$	$\frac{-5}{6}+\varepsilon$	$-\varepsilon^{-2}$	$\frac{-1}{9}+\varepsilon$	$\frac{-1}{2}+\varepsilon$	$-\frac{-1}{9}-\varepsilon$	0	1	0	0	$-5-\varepsilon$	$\frac{10}{3}+\varepsilon$
v_3	$\frac{5}{108}\varepsilon^{-1}$	$\frac{22}{225}\varepsilon^{-1}$	$\frac{1}{8}\varepsilon^{-1}$	$-\frac{1}{9}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$	$-1+\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{-1}{3}+\varepsilon$	0	$\frac{-1}{3}+\varepsilon$	0	0	1	$\frac{-1}{2}-\varepsilon$	$\frac{3}{10}+\varepsilon$	$\frac{-31}{5}-\varepsilon$
$v_\varepsilon(v_1)$	-1	-1	-1	-1	-1	0	-1	0	1	0	0	$+\infty$	$+\infty$	0	0	0
$v_\varepsilon(v_2)$	-2	-2	-2	-2	-2	0	-2	0	0	0	$+\infty$	0	$+\infty$	$+\infty$	0	0
$v_\varepsilon(v_3)$	-1	-1	-1	-1	-1	0	-1	0	$+\infty$	0	$+\infty$	$+\infty$	0	0	0	0
$v_\varepsilon(v_*)$	-2	-2	-2	-2	-2	0	-2	0	0	0	0	0	0	0	0	0

Figure 3.11: Values (Truncated) and Valuations of v_1, v_2, v_3

This figure shows that $g_{p,max}$ from Definition 2.31 is 2.

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$v_\varepsilon(u_*)$	0	0	0	0	0	0	0	0	0	-1	-1	0	-1	-1	-1	-1
$v_\varepsilon(v_*)$	-2	-2	-2	-2	-2	0	-2	0	0	0	0	0	0	0	0	0

L_3 (between 0 and 1)

Figure 3.12: $v_\varepsilon(u_*)$ and $v_\varepsilon(v_*)$

Observe that τ^0 is integral because the valuations in Figure 3.12 satisfies the two condition in Theorem 3.4.

Example 3.11 (Example 3.10 continued)**.** *Next we check τ^1. Since τ^1 shifts by 1, all we have to do to produce the next two figures is to shift figure 3.12 to the left by 1.*

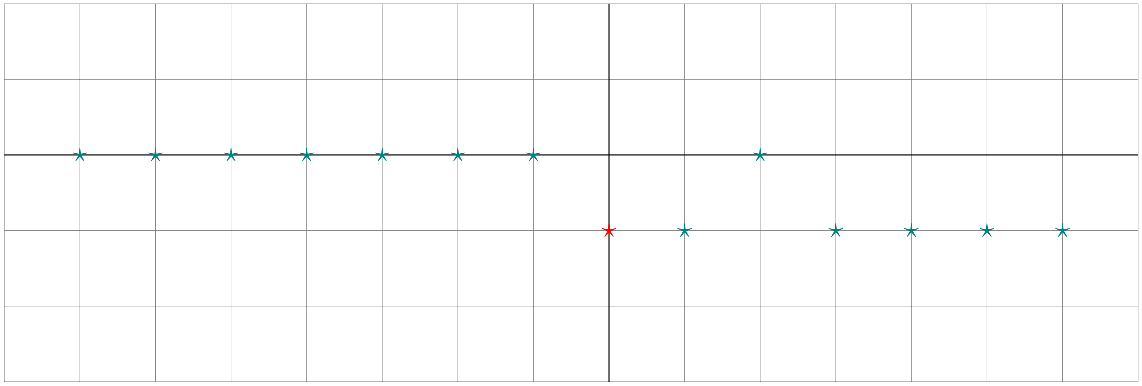

Figure 3.13: $v_\varepsilon(u_*)$

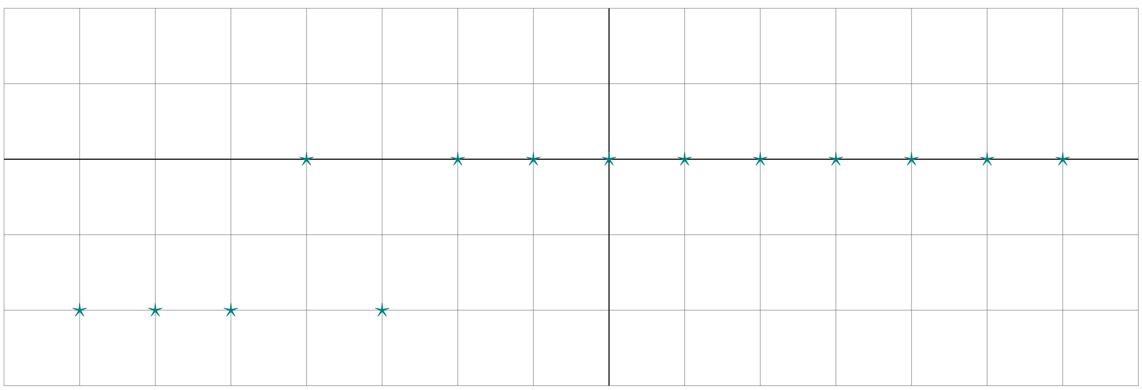

Figure 3.14: $v_\varepsilon(v_*)$

Figures (3.13) and (3.14) show that τ^1 *is not integral because if* $v_\varepsilon(u_*)$ *is not* ≥ 0 *at every* q *with* $Re(q) < 1$*. Multiplying* τ^1 *by* $c_1 = x$ *from Equation (3.9) fixes this. For* τ^2*, shift figure 3.13 by two. Then we can see that* τ^2 *is not integral at* $x = 0, 1$*. Multiplying* τ^2 *with* $c_2 = x(x+1)$ *from Equation (3.9) fixes this.*

Doing the same process for other τ^i *produces the following integral elements:*

τ^0, $x\tau^1$, $x(x+1)\tau^2$, $(x+1)(x+2)\tau^3$, $x(x+2)(x+3)\tau^4$, $x(x+1)(x+3)(x+4)\tau^5, \ldots$.

$\tau^{-1}, \tau^{-2}, (x-1)^2\tau^{-3}, (x-2)^2\tau^{-4}, (x-1)^2(x-3)^2\tau^{-5}, (x-1)^2(x-2)^2(x-4)^2\tau^{-6}, \ldots$.

The question if this will eventually span all integral elements will be answered in Chapter 4.

3.3 Properties of Integral Elements

In this section, $p \in \mathbb{C}/\mathbb{Z}$ and $q \in p$.

Definition 3.12. *For $L \in D$ and $M = D/DL$ we define $O_{(q)} = \{b \in M \mid b \text{ is integral at } q\}$, which is an $R_{(q)}$-module where $R_{(q)} = \{f \in \mathbb{C}(x) \mid v_q(f(x)) \geq 0\} = \{\frac{A}{B} \mid A, B \in \mathbb{C}[x], B(q) \neq 0\}$.*

Definition 3.13. *Let $O_M = \bigcap_{q \in \mathbb{C}} O_{(q)}$ be the $\mathbb{C}[x]$-module of all integral elements in M.*

Definition 3.14. *We say that $b_1, \ldots, b_n$ is* local integral basis at q *if $R_{(q)}b_1 + \cdots + R_{(q)}b_n = O_{(q)}$.*

Definition 3.15. *We say that $b_1, \ldots, b_n$ is* (global) integral basis *if it is local integral basis at all $q \in \mathbb{C}$. Equivalently, if it is $\mathbb{C}[x]$-basis of O_M.*

CHAPTER 4

INTEGRAL BASIS

In [1], an integral basis is used to simplify differential equations. We adopt the same approach to define an integral basis for difference equations.

Let $L \in D$ have order n and let O_M be the $\mathbb{C}[x]$-module of integral elements of $M := D/DL$. Equation (3.9) gives $c_i \in \mathbb{C}(x)$ of minimal degree with $c_i \tau^i \in O_M$. Then $\{c_0\tau^0, \ldots, c_{n-1}\tau^{n-1}\}$ spans a sub-module of O_M of maximal rank. Our goal is to obtain a basis of O_M.

4.1 Main Theorem

Let $M_k = SPAN_{\mathbb{C}[x]}\{c_i\tau^i \mid i = -k, \ldots, n-1+k\}$, so $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq O_M$. In Theorem 4.14, we will show that

$$M_k = M_{k+1} \implies M_k = O_M. \tag{4.1}$$

Combined with Equation (3.9) this provides an efficient algorithm for calculating a basis of O_M. Since $\mathbb{C}[x]$ is a PID, any $\mathbb{C}[x]$-module $M \subseteq D/DL$ is free if and only if it is finitely generated. We will only consider free $\mathbb{C}[x]$-modules of full rank n. In our algorithms such modules are represented by triangular bases:

Definition 4.1. *$\{b_1, \ldots, b_n\} \subset D/DL$ is a* triangular set *when each b_i is represented by an element of $D = \mathbb{C}(x)[\tau]$ of degree $i-1$ in τ.*

Remark 4.2. *If $b \in M$, and $b_1, \ldots, b_n$ is a triangular set, then the* standard reduction *of b* mod $b_1, \ldots, b_n$ *is defined as follows. First, write $b = a_1b_1 + \cdots + a_nb_n$ for some $a_i \in \mathbb{C}(x)$. Now write $a_i = p_i + \tilde{a}_i$ with $p_i \in \mathbb{C}[x]$ and degree$(\tilde{a}_i) < 0$, where degree(A/B) denotes degree$(A) -$ degree(B). Then $\tilde{a}_1b_1 + \cdots + \tilde{a}_nb_n$ is the standard reduction of $b \mod \{b_1 \ldots b_n\}$. Note that this is zero if and only if $b \in SPAN_{\mathbb{C}[x]}(b_1, \ldots, b_n)$.*

We can make a triangular basis nearly-unique (unique up to constant factors) by requiring that b_{i+1} minus its τ^i term is equal to its standard reduction mod $\{b_1 \ldots b_i\}$.

Algorithm 4.3. *Algorithm* `AddToTriangularSet`

- *Input: a triangular set* $\{b_1 \dots b_n\} \subset M$*, and one more element* $b \in M$*.*
- *Output: a triangular basis of* $SPAN_{\mathbb{C}[x]}(b_1, \dots, b_n, b)$*.*

Step 1. Let b' *= standard reduction of* $b \mod \{b_1 \dots b_n\}$*. If* $b' = 0$*, then return* $\{b_1 \dots b_n\}$*.*

Step 2. Let d *be the largest integer for which the* τ^{d-1} *term in* b' *is not 0.*

Step 3. Return `AddToTriangularSet`*(*$\{b_1, \dots, b_{d-1}, b', b_{d+1}, \dots b_{n-1}, b_n\}$*,* b_d*).*

Lemma 4.4. *Algorithm* `AddToTriangularSet` *terminates after finitely many steps.*

Proof. A triangular set $\{b_1 \dots b_n\}$ can be written as $b_i = \sum_{j=0}^{i-1} a_{i,j}\tau^j$ with $a_{i,j} \in \mathbb{C}(x)$. Let's call

$$\sum_{i=1}^{n} deg(a_{i,i-1}) \tag{4.2}$$

the *total degree* of our triangular set with degree as in Remark 4.2. Observe that the triangular set in Step 3 has lower total degree than the triangular set in the input. This implies that the algorithm terminates (denominators during the algorithm are bounded by the maximal denominator in $SPAN_{\mathbb{C}[x]}(b_1, \dots, b_n, b)$, which gives a lower bound for the total degree of any triangular subset in this SPAN). □

Algorithm 4.5. *Algorithm* `IB`

- *Input:* $L \in D$*.*
- *Output: A basis of* O_M *where* $M = D/DL$*.*

Step 1. Let $n = ord(L)$*. Let* $k = 0$*. Let* $B_0 := \{c_0\tau^0, \dots, c_{n-1}\tau^{n-1}\}$ *with* c_i *as in Equation (3.9).*

Step 2. Compute $\tau^{-(k+1)}$ *and* τ^{n+k} *modulo* L *as explained in Remark 2.18. Then let*
$B_{k+1} :=$ `AddToTriangularSet`$($`AddToTriangularSet`$(B_k, c_{-(k+1)}\tau^{-(k+1)}), c_{n+k}\tau^{n+k})$.

Step 3. If $B_k = B_{k+1}$ *then return* B_k*. Otherwise increase* k *by 1 and go back to Step 2.*

The set B_k in the algorithm is a basis of M_k. Equation (4.1) shows that the output in Step 3 is a basis of O_M. Section 5.3 will explain why this algorithm terminates.

Example 4.6 (Example 3.10 continued)**.**

$$L = (x+2)\tau^3 + x\tau^2 + (x+2)^2$$

In example 3.10, we calculated the following basis for M_0

$$B_0 = \{c_0\tau^0, c_1\tau^1, c_2\tau^2\} = \{\tau^0, x\tau^1, x(x+1)\tau^2\}.$$

Equation (4.1) tells us that we can check if M_0 equals O_M by checking if M_1 equals M_0. We have to check if $c_{-1}\tau^{-1}$ and $c_3\tau^3$ are in $M_0 = \mathrm{SPAN}(B_0)$ or not. We will write $c_{-1}\tau^{-1}$ and $c_3\tau^3$ in terms of τ^0, τ^1, τ^2 as in a standard form see Remark 2.18. Then we reduce (see Remark 4.2) $c_{-1}\tau^{-1}$ and $c_3\tau^3$ mod B_0.

$$\begin{aligned} c_3\tau^3 &= (x+2)(x+1)x\tau^3 \\ &\equiv (x+2)(x+1)x\left(\frac{-x\tau^2}{x+2} - x - 2\right) \quad \bmod DL \\ &\equiv 0 \quad \bmod B_0 \end{aligned}$$

and

$$\begin{aligned} c_{-1}\tau^{-1} &= \tau^{-1} \\ &\equiv \frac{-1}{x+1}\tau^2 - \frac{x-1}{(x+1)^2}\tau \quad \bmod DL \\ &\equiv \frac{-1}{x+1}\tau^2 - \frac{x-1}{(x+1)^2}\tau \quad \bmod B_0 \end{aligned}$$

We see that $c_3\tau^3$ vanishes mod B_0, which means that AddToTriangularSet(B_0, $c_3\tau^3$) returns B_0 in Step 1. Now $c_{-1}\tau^{-1}$ does not vanish mod B_0, and AddToTriangularSet(B_0, $c_{-1}\tau^{-1}$) returns

$$B_1 = \{\tau^0, x\tau^1, \frac{-1}{x+1}\tau^2 - \frac{x-1}{(x+1)^2}\tau\} = \{b_1, b_2, b_3^{\mathrm{new}}\}.$$

Going from b_3 to b_3^{new}, the degree in x decreased from 2 to -1.

Now $c_4\tau^4$ will be in M_2. (We can show this with a computation, or, by combining the fact that $c_3\tau^3$ vanished mod B_0 with a slight generalization of Theorem 4.14). To check if $M_2 = M_1$, it remains to compute:

$$\begin{aligned} c_{-2}\tau^{-2} &= \tau^{-2} \\ &\equiv \frac{-1}{x}\tau - \frac{x-2}{x^2} \quad \bmod DL \\ &\equiv \frac{-1}{x}\tau - \frac{x-2}{x^2} \quad \bmod B_1 \end{aligned}$$

So $c_{-2}\tau^{-2} \notin M_1$. AddToTriangularSet($B_1$, $c_{-2}\tau^{-2}$) returns the following basis of M_2:

$$B_2 = \{\tau^0, \frac{-1}{x}\tau - \frac{x-2}{x^2}, \frac{-1}{x+1}\tau^2 - \frac{x-1}{(x+1)^2}\tau\}.$$

Next, $c_5\tau^5$ and $c_{-3}\tau^{-3}$ are M_2, so $M_2 = M_3$. Then Theorem 4.14 says that B_2 is an integral basis. The basis in [4] has the same SPAN. (This will not always be the case because the definitions differ.)

4.2 Matrix Expressing the D-module Structure

Definition 4.7. *We say that a matrix A* expresses the action *of τ on a basis*[1] $b_1, \ldots, b_n$ *if*

$$\begin{bmatrix} \tau(b_1) \\ \tau(b_2) \\ \vdots \\ \tau(b_n) \end{bmatrix} = A \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{4.3}$$

If $\tau(v) = Av$ and if $w = Pv$ where P is invertible matrix, then

$$\begin{aligned} \tau(w) &= \tau(P)\tau(v) \\ &= \tau(P)A\,v \\ &= \tau(P)AP^{-1}\,w \end{aligned} \tag{4.4}$$

This shows that if A expresses the action of τ on some basis of D/DL, then so does $\tau(P)AP^{-1}$, where P is the *change of basis matrix* between the two bases.

Example 4.8. *Let $L = \sum_{i=0}^{n} a_i\tau^i$, $a_i \in \mathbb{C}(x)$ with $a_0 \neq 0$ and $a_n = 1$. The following matrix expresses the action of τ on the basis $\tau^0, \ldots, \tau^{n-1}$ of D/DL.*

$$A_L := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}$$

This matrix is called the companion matrix *of L.*

The recurrence relation of order n

$$u(x+n) + a_{n-1}(x)u(x+n-1) + \cdots + a_0(x)u(x) = 0 \tag{4.5}$$

where $a_i(x) \in \mathbb{C}(x)$ can be written as a first-order recurrence relation $\tau(v) = A_L \cdot v$ where

$$v = \begin{bmatrix} u(x) \\ u(x+1) \\ \vdots \\ u(x+n-1) \end{bmatrix}.$$

[1] basis of D/DL as $\mathbb{C}(x)$-vector space

4.3 Characterizing Integral Bases in Terms of the Matrix that Expresses the D-module Structure

Definition 4.9. *Let A be an n by n matrix and let P, Q, p_i, q_i, m_i and M_i be as in Lemma 3.7. Then A meet the* strict valuation requirement *if*

1. *Every pole $q \in \mathbb{C}$ of A or A^{-1} has $Re(q) \in [0, 1)$. (Recall that $Re(q_i) \in [0, 1)$.)*
2. *$v_{q_i}(A) \geq m_i$ for each i, in other words $PA \in \mathrm{MAT}_{n\times n}(\mathbb{C}[x])$.*
3. *$v_{q_i}(A^{-1}) \geq -M_i$ for each i, in other words $\tau(Q)A^{-1} \in \mathrm{MAT}_{n\times n}(\mathbb{C}[x])$.*

The matrix A meets the weak valuation requirement *if condition* 1 *holds.*

Lemma 4.10. *If A expresses the action of τ on an integral basis $b_1, \ldots, b_n$, then A meets the* strict valuation requirement.

Proof. Let P, Q, m_i and M_i be as in Lemma 3.7. If $b_1, \ldots, b_n$ are *integral* then $P\tau(b_1), \ldots, P\tau(b_n)$ are integral by Lemma 3.7. Since $b_1, \ldots, b_n$ is an integral *basis*, this implies that PA has polynomial entries. Hence condition 2 holds. Multiplying Equation (4.3) by A^{-1} and applying τ^{-1} gives

$$\begin{bmatrix} \tau^{-1}(b_1) \\ \tau^{-1}(b_2) \\ \vdots \\ \tau^{-1}(b_n) \end{bmatrix} = \tau^{-1}(A^{-1}) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{4.6}$$

Lemma 3.7 now gives condition 3. Roots of $P, \tau(Q)$ are in $\{q_1, \ldots, q_l\}$ which gives condition 1. □

Lemma 4.11. *If $b_1, \ldots, b_n \in M$ are $\mathbb{C}(x)$-linearly independent and the matrix A that expresses the action of τ meets the weak valuation requirement, then $b_1, \ldots, b_n$ are integral.*

Proof. Assume $\exists_i$ such that b_i is not locally integral at q. Let q_1 (respectively, q_2) denote the smallest (respectively, largest) integer at which some b_i is not integral.

- case 1: $q_1 \leq 0$, and q_1 is minimal for which there is a b_i that is not integral at q_i. Then $\tau(b_i)$ is not integral at $q_1 - 1$ since τ shifts by 1. Now $\tau(b_i) = A_{i1}b_1 + \cdots + A_{in}b_n$ since A expresses the action of τ on $b_1, \ldots, b_n$. We have $v_{q_1-1}(A_{ij}) \geq 0$ because A meets the weak valuation requirement and $Re(q_1 - 1) \notin [0, 1)$. Since $\tau(b_i)$ is not integral at $q_1 - 1$, the same must be true for at least one of its terms, and hence for at least one of $b_1, \ldots, b_n$. That contradicts the minimality of q_1.
- case 2: $q_2 > 0$. This is similar to case 1 except we apply τ^{-1} and use $v_{q_2+1}(A^{-1}) \geq 0$, which leads to a contradiction with the maximality of q_2. □

Lemma 4.12. *If A meets the weak valuation requirement, then $b_1, \ldots, b_n$ is an integral basis.*

Proof. Lemma 4.11 says that $b_1, \ldots, b_n$ are integral. To show that it is an integral *basis*, we need to show that if $a_1 b_1 + \cdots + a_n b_n \in O_M$ then $\forall i, a_i \in \mathbb{C}[x]$. We proceed by contradiction. Assume there is some $a_1 b_1 + \cdots + a_n b_n \in O_M$ for which some a_i has a pole. Assume also that this pole is in $\mathbb{Z}$. (Other elements of $\mathbb{C}/\mathbb{Z}$ are handled similarly.)

Let q_1 resp. q_2 be the smallest resp. largest integer for which $\exists\ b := a_1 b_1 + \cdots + a_n b_n \in O_M$ with some a_i having a pole at that integer.

- case 1: $q_1 \leq 0$. Since $b \in O_M$, $\tilde{b} := P\tau(b) \in O_M$ with P as in Lemma 3.7. Write

$$\tilde{b} = P \cdot (\tau(a_1)\tau(b_1) + \cdots + \tau(a_n)\tau(b_n)) = \tilde{a}_1 b_1 + \cdots + \tilde{a}_n b_n \tag{4.7}$$

  for some $\tilde{a}_i$ that can be computed from P and $\tau(a_1), \ldots, \tau(a_n)$ and the matrix A that expresses $\tau(b_1), \ldots, \tau(b_n)$ in terms of $b_1, \ldots, b_n$. Since a_i has a pole at q_1, $\tau(a_i)$ has a pole at $q := q_1 - 1 \leq -1$. Then $q \notin \{q_1, \ldots, q_l\}$ and so $v_q(P) = 0$ and $v_q(A) = v_q(A^{-1}) = 0$ since A satisfies the weak valuation requirement. Then $\tilde{a}_j$ must have a pole at q for some j.

  That means $b_1, \ldots, b_n$ is not a local integral *basis* at q contradicting the minimality of q_1.

- case 2: $q_2 > 0$. A similar computation with $Q\tau^{-1}(b)$ from Lemma 3.7 contradicts the maximality of q_2. □

The following theorem characterizes integral bases of $M = D/DL$, and can serve as a simpler definition of integral basis that can be used for any D-module M.

Theorem 4.13. *Let A express the action of τ on a $\mathbb{C}(x)$-basis $b_1, \ldots, b_n$ of M. Then the following are equivalent:*

1. *$b_1, \ldots, b_n$ is an integral basis.*
2. *A meets the* strict *valuation requirement.*
3. *A meets the* weak *valuation requirement.*

In particular, specifying "weak" or "strict" is no longer necessary.

Proof. (1) ⇒ (2) is Lemma 4.10, (2) ⇒ (3) is straightforward, and (3) ⇒ (1) is Lemma 4.12. □

Theorem 4.14. *Let A express the action of τ on a basis $b_1, \ldots, b_n$ of M_k from Section 4.1. If $M_k = M_{k+1}$, then A meets the (strict) valuation requirement and hence $M_k = O_M$.*

Proof. Let P, Q, m_i and M_i be as in Lemma 3.7. Since $M_k = M_{k+1}$, we have

$$M_{k+1} = M_k = SPAN_{\mathbb{C}[x]}\{b_1, \ldots, b_n\}. \tag{4.8}$$

Recall that M_k and M_{k+1} are defined as

$$M_k = SPAN_{\mathbb{C}[x]}\{c_{-k}\tau^{-k}, \ldots, c_{n-1+k}\tau^{n-1+k}\} \tag{4.9}$$

$$M_{k+1} = SPAN_{\mathbb{C}[x]}\{c_{-(k+1)}\tau^{-(k+1)}, \ldots, c_{n+k}\tau^{n+k}\}. \tag{4.10}$$

Applying τ to Equations (4.9) and (4.8) gives

$$\tau(M_k) = SPAN_{\mathbb{C}[x]}\{\tau(c_{-k})\tau^{-k+1}, \ldots, \tau(c_{n-1+k})\tau^{n+k}\} = SPAN_{\mathbb{C}[x]}\{\tau(b_1), \ldots, \tau(b_n)\}. \tag{4.11}$$

We need to show that each $\tau(b_i)$ can be written as a linear combination of $\{b_1, \ldots, b_n\}$, where the denominators of the coefficients divide P (equivalently, the coefficients have poles at $q_1, \ldots, q_l$ of order at most $m_1, \ldots, m_l$).

The generators $\tau(c_i)\tau^{i+1}$ ($i \in \{-k, \ldots, n-1+k\}$) of $\tau(M_k)$ can be written by Remark 3.8 as

$$\tau(c_i)\tau^{i+1} = \frac{1}{P} \cdot f_i(x) \cdot c_{i+1}\tau^{i+1} \in \frac{1}{P}M_{k+1} \tag{4.12}$$

for some $f_i(x) \in \mathbb{C}[x]$. Therefore $\tau(M_k) \subseteq \frac{1}{P} \cdot M_{k+1}$. Then from Equations (4.11) and (4.8) we get

$$SPAN_{\mathbb{C}[x]}\{\tau(b_1), \ldots, \tau(b_n)\} = \tau(M_k) \subseteq \frac{1}{P}M_{k+1} = \frac{1}{P}SPAN_{\mathbb{C}[x]}\{b_1, \ldots, b_n\} \tag{4.13}$$

so $A \in \frac{1}{P} \cdot MAT_{n\times n}(\mathbb{C}[x])$. For A^{-1} the argument is similar, we apply τ^{-1} on M_k and we get $A^{-1} \in \frac{1}{\tau(Q)} \cdot MAT_{n\times n}(\mathbb{C}[x])$. Therefore, the strict valuation condition is satisfied. □

Example 4.15 (Example 3.10 continued)**.**

$$L = (x+2)\tau^3 + x\tau^2 + (x+2)^2 \tag{4.14}$$

The set M_0 defined in Section 4.1 has basis $B_0 := \{\tau^0, x\tau^1, x(x+1)\tau^2\}$. (The coefficient c_i in front of τ^i is computed with Corollary 3.5.) The matrix that expresses the action of τ on B_0 is:

$$A = \begin{bmatrix} 0 & \frac{1}{x} & 0 \\ 0 & 0 & \frac{1}{x} \\ -(x+1)(x+2)^2 & 0 & -1 \end{bmatrix}$$

Its inverse is

$$A^{-1} = \begin{bmatrix} 0 & \frac{-x}{(x+1)(x+2)^2} & \frac{-1}{(x+1)(x+2)^2} \\ x & 0 & 0 \\ 0 & x & 0 \end{bmatrix}$$

The matrix A has a pole at $x = 0$, but the matrix A^{-1} has poles at $x = -1$ and -2 so the valuation requirement from Definition 4.9 does not hold. That implies (see Theorem 4.13) that B_0 is not an integral basis, so $M_0 \subsetneq O_M$. According to Equation (4.1) this is equivalent $M_0 \subsetneq M_1$. Example 4.6 confirms that $M_0 \subsetneq M_1$.

To further test Theorem 4.13 we computed the matrix that represents the action of τ on the integral basis $\{\tau^0, \frac{-1}{x}\tau - \frac{x-2}{x^2}, \frac{-1}{x+1}\tau^2 - \frac{x-1}{(x+1)^2}\tau\}$ from Example 4.6. This matrix is

$$\begin{bmatrix} \frac{-(x-2)}{x} & -x & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{x}\begin{bmatrix} -(x-2) & -x^2 & 0 \\ 0 & 0 & x \\ x & 0 & 0 \end{bmatrix}$$

and its inverse is

$$\begin{bmatrix} 0 & 0 & 1 \\ \frac{-1}{x} & 0 & \frac{-(x-2)}{x^2} \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{x^2}\begin{bmatrix} 0 & 0 & x^2 \\ -x & 0 & -(x-2) \\ 0 & x^2 & 0 \end{bmatrix}.$$

Both matrices only have poles at $x = 0$, so the (weak) valuation requirement holds.

Example 4.16. *We test Theorem 4.13 on one more example*

$$L = (x+2)\tau^3 + x\tau^2 + (x+2)^2(x+1)^3.$$

Our implementation [7] gives the following integral basis

$$\{1, \frac{-1}{x}\tau - \frac{(x-2)}{x^2}, \frac{-1}{x^3(x+1)}\tau^2 - \frac{(x-1)}{x^3(x+1)^2}\tau\}.$$

Here is the matrix, and its inverse, for the action of τ on this basis

$$A = \begin{bmatrix} \frac{-(x-2)}{x} & -x & 0 \\ 0 & 0 & x^3 \\ 1 & 0 & 0 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{-1}{x} & 0 & \frac{-(x-2)}{x^2} \\ 0 & \frac{1}{x^3} & 0 \end{bmatrix}$$

and one can see that they satisfy the (weak) valuation requirement.

CHAPTER 5

REDUCED INTEGRAL BASES

5.1 Reduced Basis

Our goal is to simplify a difference operator with transformations from Section 2.4. As mentioned there, our focus will be gauge-transformations. Finding the best gauge-transformation is equivalent to finding a generator of $M = D/DL$ whose annihilator (Definition 2.19) is as small as possible.

Random elements of M are likely to have annihilators with coefficients of high degree. To obtain low degrees, the idea is to look for an element of O_M with minimal *degree* (Definition 5.1 below).

To simplify notations we will work with difference operators of order $n = 2$ in this chapter. Let $b_1, b_2 \in M$ be an integral basis. Then $b_1 \wedge b_2$ is a basis of $\bigwedge^2 O_M$. If $b \in O_M$ then $P \cdot \tau(b) \in O_M$ and hence $b \wedge \tau(b) \in \frac{1}{P}\mathbb{C}[x] \cdot b_1 \wedge b_2$, with P as in lemma 3.7. Likewise $P\tau(P) \cdot \tau^2(b) \in O_M$.

Definition 5.1. *Let $b \in O_M$ and let b_1, b_2 be an integral basis. Define $P_b \in \frac{1}{P}\mathbb{C}[x]$, $Q_b \in \frac{1}{P\tau(P)}\mathbb{C}[x]$, and $R \in \frac{1}{P^2}\mathbb{C}[x]$ by the equations*

$$b \wedge \tau(b) = P_b \, (b_1 \wedge b_2)$$

$$b \wedge \tau^2(b) = Q_b \, (b_1 \wedge b_2)$$

$$\tau(b_1) \wedge \tau(b_2) = R \, (b_1 \wedge b_2).$$

The degree *of b is defined as the degree of P_b.*

Our goal is to minimize this degree since this minimizes degrees in the annihilator of b:

Lemma 5.2. *The annihilator $a_2\tau^2 + a_1\tau + a_0$ of b is given by*

$$a_2 = P_b \quad a_1 = -Q_b \quad a_0 = \tau(P_b)\, R.$$

Proof. If $a_2\tau^2 + a_1\tau + a_0$ annihilates b, then $0 = b \wedge 0 = b \wedge (a_2\tau^2(b) + a_1\tau(b) + a_0 b)$ which gives $0 = a_2 \cdot Q_b + a_1 \cdot P_b + a_0 \cdot 0$. Taking wedge products with $\tau(b)$ and $\tau^2(b)$ produces similar equations and it is easy to check that the values of a_2, a_1, a_0 in the lemma satisfy these equations. □

Definition 5.3. *We say that b_1, b_2 is a* reduced basis *if*

- *There is no polynomial* $p \in \mathbb{C}[x]$ *such that* $\deg(b_1 - p \cdot b_2) < \deg(b_1)$.

- *There is no polynomial* $p \in \mathbb{C}[x]$ *such that* $\deg(b_2 - p \cdot b_1) < \deg(b_2)$.

In the next example, to see if a basis is reduced or not, we compute the polynomial(s) p for which $b_1 - p\, b_2$ and has minimal degree, and then do the same for $b_2 - p\, b_1$.

Example 5.4. *Let M be a two-dimensional $\mathbb{C}(x)$-vector space, and let b_1, b_2 be a basis of M. To turn M into a D-module we need to specify how τ acts on this basis. Lets choose $A = \begin{bmatrix} x^2 & x \\ x & x+1 \end{bmatrix}$ to be the action of τ on b_1, b_2. which is equivalent to choosing:*

$$\tau(b_1) := x^2 \cdot b_1 + x \cdot b_2 \qquad \tau(b_2) := x \cdot b_1 + (x+1) \cdot b_2.$$

From $A^{-1} = \frac{1}{x^3} \begin{bmatrix} x+1 & -x \\ -x & x^2 \end{bmatrix}$ we see that A meets the (weak) valuation requirement (Definition 4.9). So b_1, b_2 is an integral basis (Theorem 4.13). Let $p \in \mathbb{C}[x]$ and let $b_3 := b_1 - p \cdot b_2 \in O_M$. Then

$$\begin{aligned} b_3 \wedge \tau(b_3) &= (b_1 - p \cdot b_2) \wedge (\tau(b_1) - \tau(p) \cdot \tau(b_2)) \\ &= (b_1 \wedge \tau(b_1)) - p \cdot (b_2 \wedge \tau(b_1)) - \tau(p) \cdot (b_1 \wedge \tau(b_2)) + p \cdot \tau(p) \cdot (b_2 \wedge \tau(b_2)) \\ &= x \cdot (b_1 \wedge b_2) + x^2 \cdot p \cdot (b_1 \wedge b_2) - \tau(p) \cdot (x+1) \cdot (b_1 \wedge b_2) - p \cdot \tau(p) \cdot x \cdot (b_1 \wedge b_2) \\ &= P_b\, (b_1 \wedge b_2) \end{aligned}$$

where

$$P_b = x + x^2\, p - (x+1)\, \tau(p) - x\, p\, \tau(p). \tag{5.1}$$

Our goal is finding $p \in \mathbb{C}[x]$ that minimizes the degree of P_b. Let $p = c_n x^n + \cdots + c_0 x^0$ where we assume that either $p = 0$, or $c_n \neq 0$ so that deg(p) $= n$. Now $\tau(p)$ has the same degree n and the same leading coefficient c_n.

Equation (5.1) implies an upper bound for n if we want to minimize the degree of P_b. To see this, consider the degrees of the four terms:

- *x has degree 1.*

- *$x^2 p$ has degree $n+2$.*

- *$(x+1)\tau(p)$ has degree $n+1$.*

- *$x\; p\; \tau(p)$ has degree $2n+1$.*

If $n \geq 2$, then the degree of the last term is strictly higher than the degree of the other terms; the highest degree term cannot cancel. So the degree of P_b can only be minimized if $n < 2$.

1. *If $n = 0$ then p is a constant. Then the dominant term is $x^2 p$ and thus $p = 0$ minimizes the degree of P_b.*

2. *If $n = 1$ then $p = c_1 x + c_0$ with $c_1 \neq 0$. Now there are two terms with maximal degree namely $x^2 p$ and $x\, p\, \tau(p)$ with degree $n+2 = 2n+1 = 3$. Their leading coefficients must cancel because we want to minimize the degree of P_b. This gives $c_1 - c_1^2 = 0$. But $c_1 \neq 0$, so $c_1 = 1$, so $p = x + c_0$ for some constant c_0. If we substitute this p in P_b, the result is a polynomial with leading term $c_0 + 2$. Hence $c_0 = -2$ and $p = x - 2$.*

The computation produces two polynomials, $p = 0$ and $p = x - 2$, that minimize the degree of P_b. In both cases the degree of P_b is equal to the degree of b_1, so the first requirement in Definition 5.3 is met.

To examine the second requirement we will compute all polynomials p that minimize the degree of $b_0 := b_2 - p\, b_1$. We get

$$P_b = -x - p\,(x+1) + \tau(p)\, x^2 + p\, \tau(p)\, x \tag{5.2}$$

We find that its degree is minimized when $p = 0$ or $p = -x + 1$. Hence b_1, b_2 is a reduced basis by Definition 5.3.

The computation in Example 5.4 not only showed that b_1, b_2 is a reduced basis, it also found other reduced bases b_0, b_1 and b_2, b_3 where $b_3 = b_1 - (x-2)\, b_2$ and $b_0 = b_2 - (-x+1)\, b_1$. We can repeat this and find a sequence

$$\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, b_3, b_4, \ldots$$

in which b_i, b_{i+1} is a reduced basis for any $i \in \mathbb{Z}$. This scenario is called the *$\mathbb{Z}$-sequence case* in our implementation [7] which computes 10 terms in this sequence and then selects the one with the smallest annihilator.

Lemma 5.5. *Let M be a 2-dimensional D-module with integral basis b_1, b_2 and let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ describe the action of τ on this basis. Let d_{ij} be the degree of a_{ij} and let d_p be the largest integer for which the following inequality holds:*

$$2d_p + d_{21} \leq \max\big(d_{12},\ d_p + d_{11},\ d_p + d_{22}\big). \tag{5.3}$$

Then any polynomial $p \in \mathbb{C}[x]$ that minimizes the degree of $b := b_1 - p\, b_2$ has degree $\leq d_p$.

Proof. $b \wedge \tau(b) = P_b \cdot (b_1 \wedge b_2)$ where

$$P_b = a_{12} + p \cdot a_{11} - \tau(p) \cdot a_{22} - p \cdot \tau(p) \cdot a_{21} \tag{5.4}$$

If $\deg(p) > d_p$ then the last term has strictly higher degree than the other terms, implying that the degree of P_b cannot be minimal in this case. □

Algorithm 5.6. *Algorithm* `ReduceBasisElement`

- *Input:* b_1, b_2.
- *Output: a polynomial* p *such that* $b_1 - p\, b_2$ *has lower degree than* b_1 *if such* p *exists, otherwise* 0.

Step 1: Write $p = \sum_{i=0}^{d_p} c_i x^i$ *where the* c_i *are unknowns and where* d_p *is as in Lemma 5.5*

Step 2: Let $b := b_1 - pb_2$ *and compute* P_b *from Equation 5.4.*

Step 3: Let $S = \{\text{coefficient}(P_b, x^i) \in \mathbb{C}[c_0, \ldots, c_{d_p}] \mid \deg(b_1) \leq i \leq \deg(P_b)\}$

Step 4: Solve S *with respect to the unknowns* c_i*. If there is a solution, then pick one and substitute it into* p*, and return that. Otherwise return* 0*.*

Algorithm 5.7. *Algorithm* `ReduceBasis`

- *Input:* b_1, b_2.
- *Output: a reduced basis.*

* *Let* $p =$ `ReduceBasisElement`(b_1, b_2).

* *If* $p \neq 0$*, then return* `ReduceBasis`$(b_2, b_1 - p \cdot b_2)$ *and stop.*

* *Let* $p =$ `ReduceBasisElement`(b_2, b_1).

* *If* $p = 0$*, then return* (b_1, b_2)*, otherwise return* `ReduceBasis`$(b_1, b_2 - p \cdot b_1)$.

Definition 5.8. *Let* b_1, b_2 *be a reduced basis. We say that* b_2, b_3 *is a* neighbor basis *if* b_3 *is not equal to* b_1 *and* $\deg(b_3) = \deg(b_1)$*.*

In Example 5.4 (b_2, b_3) is a neighbor basis of (b_1, b_2).

Algorithm 5.9. *Algorithm* `NeighborBasis`

- *Input:* b_1, b_2.
- *Output: a neighbor basis* b_2, b_3 *if it exists, otherwise "failed".*

Steps 1,2,3. Same as in Algorithm `ReduceBasisElement` *except we replace* $\deg(b_1) \leq i$ *in Step 3 with* $\deg(b_1) < i$.
Step 4. Solve S, if there is no non-zero solution then return "failed", otherwise, substitute a non-zero solution in p and return $(b_2, b_1 - p \cdot b_2)$.

The situation mentioned after Example 5.4, where each reduced basis has a unique neighbor basis on each side, leading to a sequence $b_i, i \in \mathbb{Z}$, is quite common in experiments. To explain this observation, the next section will study when this situation is likely to happen.

5.2 Power Series and Generalized Exponents

Generalized exponents of difference equation have been defined in [3]. Let $K = \mathbb{C}((1/x)) = \mathbb{C}((t))$ where $t = \frac{1}{x}$. The algebraic closure of K according to [15, Theorem 3.1] is

$$\overline{K} = \bigcup_{n \in \mathbb{N}} \mathbb{C}((t^{1/n}))$$

where $\mathbb{C}((t^{1/n}))$ is the field of Laurent series in $t^{1/n}$. The action of τ on K is

$$\tau(t) = \tau(\frac{1}{x}) = \frac{1}{x+1} = \frac{t}{1+t} = t - t^2 + t^3 - \cdots$$

which extends to an action on $\overline{K}$

$$\tau(t^{1/n}) = t^{1/n}\ (1+t)^{-1/n} = t^{1/n}\ (1 - \frac{1}{n}t + \frac{(\frac{-1}{n})(\frac{-1}{n} - 1)}{2}\ t^2 - \cdots).$$

The definition of *generalized exponents* in [3, Definition 3.2.13] is technical, but their main property can be described in a relatively short way as follows: Any operator $L \in K[\tau]$ can be written as a product of first order factors in $\overline{K}[\tau]$. If $\tau - a$ with $a \in \overline{K}$ is a *right-factor* of L, and we write a in the following form:

$$a = c\, t^v\ (1 + a_1 t^{1/n} + a_2 t^{2/n} + \cdots) \tag{5.5}$$

where $c \in \mathbb{C}^*, v \in \frac{1}{n}\mathbb{Z}$, then the corresponding *generalized exponent* consists of the terms from t^v to t^{v+1}, i.e.

$$e = c\, t^v\ (1 + a_1 t^{1/n} + a_2 t^{2/n} + \cdots a_n t^1). \tag{5.6}$$

Example 5.10. *With the matrix A in Example 5.4, we can compute the minimal operator for* b_2*. We found* $L = \tau^2 + (-x^2 - 2x - 2)\,\tau + x^2(x+1)$*. The implementation GeneralizedExponents in Maple returns*

$$e_1 = x\ (1 + 0 \cdot \frac{1}{x}), \qquad e_2 = x^2\ (1 + (-1) \cdot \frac{1}{x}).$$

This tells us that L has two right-factors in $K[\tau]$, namely

$$\tau - x\,(1 + 0.\frac{1}{x} + \cdots \frac{1}{x^2} + \cdots \frac{1}{x^3} + \cdots) \quad \text{and} \quad \tau - x^2\,(1 + (-1)\frac{1}{x} + \cdots \frac{1}{x^2} + \cdots \frac{1}{x^3} + \cdots).$$

(We could compute more terms but those will not be needed.)

If $L \in K[\tau]$ has order 2, then there are 4 possible cases for the $K[\tau]$-module $K[\tau]/K[\tau]L$.

1. Irreducible case. No 1-dimensional submodules.
   (There will be two 1-dimensional submodules if we extend K to $\mathbb{C}((1/\sqrt{x}))$.)
2. Non-semi-simple case. Exactly one 1-dimensional sub-module.
3. Semi-simple split case. Exactly two 1-dimensional submodules, say Ku_1 and Ku_2.
4. Fully reducible case. Infinitely many 1-dimensional submodules.

The generalized exponents e_1, e_2 in Example 5.10 tell us that this example is in case 3, which appears to be the most common case. One can choose u_1, u_2 in such a way that $\tau(u_1) = e_1 u_1$ and $\tau(u_2) = e_2 u_2$.

Since $\mathbb{C}(x) \subset K$, the D-module $M \cong D/DL$ from Example 5.4 can be viewed as a subset of $K[\tau]/K[\tau]L = Ku_1 + Ku_2$. This way any $b \in M$ can be written as

$$b = a_1\,u_1 + a_2\,u_2$$

for some $a_1, a_2 \in K$. Then

$$\begin{aligned} b \wedge \tau(b) &= (a_1\;u_1 + a_2\;u_2) \wedge (\tau(a_1)\,\tau(u_1) + \tau(a_2)\,\tau(u_2)) \\ &= (a_1\,\tau(a_2)\,e_2 - a_2\,\tau(a_1)\,e_1)\,(u_1 \wedge u_2) \end{aligned}$$

The leading term in $a_1\,\tau(a_2)\,e_2 - a_2\,\tau(a_1)\,e_1$ cannot cancel. This implies that the degree of b is a constant (that depends only on e_1, e_2, u_1, u_2 but not on b) plus $\deg(a_1) + \deg(a_2)$ where $\deg(a) = -v_t(a)$ (i.e. -1 times the t-adic valuation, recall that the a_i are Laurent series in $t = 1/x$.)

Next we write the basis b_1, b_2 of M in terms of u_1, u_2.

$$b_1 = c_{11}\,u_1 + c_{12}\,u_2 \qquad b_2 = c_{21}\,u_1 + c_{22}\,u_2 \tag{5.7}$$

with $c_{11}, c_{12}, c_{21}, c_{22} \in K$. The degree of b_i equals $\deg(c_{i1}) + \deg(c_{i2})$ plus the previously mentioned constant. The degree of $b_2 - p\,b_1$ is that constant plus $\deg(c_{21} - p\,c_{11}) + \deg(c_{22} - p\,c_{12})$. The only way a polynomial p could minimize this sum is when

$$p \in \left\{ round\left(\frac{c_{21}}{c_{11}}\right), round\left(\frac{c_{22}}{c_{12}}\right) \right\} \tag{5.8}$$

where $round(a) \in \mathbb{C}[x]$ denotes the polynomial part of Laurent series $a \in \mathbb{C}((1/x))$. If b_1, b_2 is a reduced basis, then 0 is an element of the set in (5.8), leaving exactly one non-zero element p in this set, and therefore we find exactly one neighbor basis. This argument repeats itself, in both directions, leading to a $\mathbb{Z}$-sequence b_i, $i \in \mathbb{Z}$.

5.3 Termination of Algorithm 4.5

Definition 5.11. *[16] Let $K = \mathbb{C}((t))$ as in Section 5.2. For a Laurent series*

$$f = ct^n(1 + d_1 t + d_2 t^2 + \cdots) \in K^*$$

let $slc(f)$ denote d_1 ("second leading coefficient"). Recall that $t = 1/x$, so for a monic polynomial

$$A = x^n + c_{n-1}x^{n-1} + \cdots + c_0 x^0$$

we have $slc(f) = c_{n-1}$.

Lemma 5.12. *Let $M_0 \subseteq M_1 \subseteq \ldots$ be as in Section 4.1. Let $d_i = det(A_i)$ where A_i is the matrix of the action of τ on a basis of M_i then*

$$\dim_{\mathbb{C}}(M_i/M_0) = slc(d_i) - slc(d_0)$$

Proof. Let P be the change of basis matrix between bases of M_0 and M_1. Because $M_0 \subseteq M_1$, matrix P has entries in $\mathbb{C}[x]$ and $\dim_{\mathbb{C}}(M_1/M_0) = \deg(p)$ where $p := \det(P) \in \mathbb{C}[x]$.

From Equation (4.4) we have $A_1 = \tau(P)A_0P^{-1}$, so $d_1 = d_0\tau(p)/p$. This implies $slc(d_1) = slc(d_0) + \deg(p)$. The lemma now follows by induction. □

Lemma 5.13. *The sequence $M_0 \subseteq M_1 \subseteq \ldots$ in Section 4.1 terminates, and hence, Algorithm 4.5 terminates as well.*

Proof. If $M_k \neq M_{k+1}$ then increasing k by 1 increases $slc(det(A_k))$ by a positive integer, namely $\dim(M_{k+1}/M_k)$. The strict valuation requirement in Definition 4.9 implies an a prior upper bound for how large the number $slc(det(A_k))$ can get. Hence $M_k \neq M_{k+1}$ can only be true finitely many times, and as soon as $M_k = M_{k+1}$, we have found O_M by Equation (4.1). □

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BIOGRAPHICAL SKETCH

Safiah Bawazeer was born in Jeddah, Saudi Arabia. She earned her Bachelor's degree in Mathematics at King AbdulAziz University, Saudi Arabia, in 2010. In the next year, she was awarded the general diploma in education from the same university. After that, Safiah started her Master's degree while working as a teaching assistant at King AbdulAziz University. After completing her master's, she continue in her role as a teaching assistant. In 2021, she started her Ph.D starting under the advisement of Professor Mark van Hoeij at Florida State University.