Hotel Infinity  (Harm Derksen’s version of a story by David Hilbert)

Once upon a time, there was a hotel called Holiday Infinity with an infinite number of rooms, numbered 1, 2, 3, . . .
One day, when all rooms were already occupied, a new guest arrived and asked for a room. The receptionist came up with a clever solution: All current guests were asked to move up 1 room. So guests in room \( n \) moved to room \( n + 1 \) like this:

\[
1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto \cdots
\]

This freed up room number 1 which was given to the new guest.

The next day, Comfort Infinity, the hotel next door, had a fire. It too had infinitely many rooms. All of its infinitely many guests survived but they all needed a room in Holiday Infinity. Once again, the receptionist of Holiday Infinity found a clever solution: Guests in room \( n \) were asked to move to room \( 2n \) like this:

\[
1 \mapsto 2, \ 2 \mapsto 4, \ 3 \mapsto 6, \ 4 \mapsto 8, \ \cdots
\]

This freed up the rooms with numbers 1, 3, 5, 7, . . . in Holiday Infinity, enough to house all guests from the now burnt Comfort Infinity.

Once people started to realize that one Infinite Hotel can hold all guests from two Infinite Hotels, a price war arose. This led to the sudden bankruptcy of a hotel chain called Days Infinity. This hotel chain had infinitely many hotels (1, 2, 3, . . .) each of which had infinitely many rooms (1, 2, 3, . . .).

After the bankruptcy, all guests of Days Infinity hotel chain suddenly needed a room in Holiday Infinity. Unfortunately, the receptionist quit in despair. Nevertheless, there would have been a solution! Can you find it? (use Theorem 1 on page 22 in the book).