Test 1 Answers, Feb 6 2019, Intro Advanced Math.

1. Let \( A, B, C \) be sets. If \( A \subseteq B \cup C \) then show \( A - B \subseteq C \).

To prove: \( x \in A - B \implies x \in C \). Read WP#1 (Writing Proofs item 1):
Let \( x \in A - B \). To prove \( x \in C \).
\( x \in A - B \) means \( x \in A \) and \( x \notin B \).
\( x \in A \) and \( A \subseteq B \cup C \) so \( x \in B \cup C \) but since \( x \notin B \) we have \( x \in C \).

2. Let \( p, q \) be statements. Which of the following statements are logically equivalent, if any? Which are tautologies, if any?
\( S_1 \) : \( p \lor q \)
\( S_2 \) : \( (\neg q) \lor (p \implies q) \)
\( S_3 \) : \( (\neg q) \implies p \).

\( S_2 \) is a tautology. \( S_1 \) is logically equivalent to \( S_3 \).

3. Give the definitions of:
   (a) A function \( f : A \to B \) is one-to-one (injective) when:
   \[ \forall_{a_1, a_2 \in A} f(a_1) = f(a_2) \implies a_1 = a_2. \]
   (b) The function \( f^{-1} : P(B) \to P(A) \) is defined as follows: If \( T \in P(B) \) then \( x \in f^{-1}(T) \) if and only if: \( f(x) \in T \).
   (c) If \( S \subseteq L \) where \( L \) is a p.o.set with ordering \( \leq \) then \( u \) is a greatest lower bound for \( S \) when:
   (1): \( u \) is a lower bound for \( S \): \( \forall_{s \in S} u \leq s \)
   (2): any other lower bound for \( S \) is \( \leq u \).
   (d) If \( u \) is a bottom element of \( S \), must \( u \) then also be a greatest lower bound for \( S \)?
   Yes. Bottom element means \( u \) is a lower bound and \( u \in S \). If \( v \) is any other lower bound, then \( v \leq s \) for all \( s \in S \). But \( u \in S \), so \( v \leq u \).

4. Let \( f : A \to B \) and \( g : B \to C \). If the composition \( g \circ f : A \to C \) is onto then show that \( g \) is onto.

Given: (G) \( \forall_{c \in C} \exists_{a \in A} g(f(a)) = c \).
To prove: \( \forall_{c \in C} \exists_{b \in B} g(b) = c \). Read WP#5:
Let \( c \in C \).
To prove: (*) \( \exists_{b \in B} g(b) = c \).
(G) says: \( \exists_{a \in A} g(f(a)) = c \). Read WP#6 on how to prove (*).
Proof of (*): Take \( b = f(a) \).

5. Suppose \( L \) is a chain and that \( S \subseteq L \) has no top element.
To prove: \( \forall_{a \in S} \exists_{b \in S} b > a \).
If \( a \in S \) then \( a \) is a top element if \( \forall_{b \in S} b \leq a \).
So \( S \) has a top element if: \( \exists_{a \in S} \forall_{b \in S} b \leq a \).
So \( S \) has no top element if: \( \forall_{a \in S} \exists_{b \in S} \neg(b \leq a) \)
(Read "Quantifiers and Negation" in Handouts.html !)
Note that \( \neg(b \leq a) \) is the same \( b > a \) since \( L \) is a chain.
Writing Proofs.

1. **Direct proof for** \( p \implies q \).
   Assume: \( p \). To prove: \( q \).

2. **Proving** \( p \implies q \) **by contrapositive**.
   Assume: \( \neg q \). To prove: \( \neg p \).

3. **Proving** \( S \) **by contradiction**.
   Assume: \( \neg S \). To prove: a contradiction.

4. **Proving** \( p \implies q \) **by contradiction**.
   Assume: \( p \) and \( \neg q \). To prove: a contradiction.

5. **Direct proof for a** \( \forall x \in A \) \( P(x) \) **statement**.
   To ensure you prove \( P(x) \) for all (rather than for some) \( x \) in \( A \), do this:
   Start your proof with: Let \( x \in A \). To prove: \( P(x) \).

6. **Direct proof for** \( \exists x \in A \) \( P(x) \) **statement**.
   Take \( x := [ \text{write down an expression that is in } A, \text{ and satisfies } P(x) ] \).

7. **Proving** \( \forall x \in A \) \( P(x) \) **by contradiction**.
   Assume: \( x \in A \) and \( \neg P(x) \). To prove: a contradiction.

8. **Proving** \( \exists x \in A \) \( P(x) \) **by contradiction**.
   Assume: \( \neg P(x) \) for every \( x \in A \). To prove: a contradiction.

9. **Proving** \( S \) **by cases**.
   Suppose for example a statement \( p \) can help to prove \( S \). Write two proofs:
   Case 1: Assume \( p \). To prove: \( S \).
   Case 2: Assume \( \neg p \). To prove \( S \).

10. **Proving** \( p \land q \)
    Write two separate proofs: To prove: \( p \). To prove: \( q \).

11. **Proving** \( p \iff q \)
    Write two proofs. To prove: \( p \implies q \) To prove: \( q \implies p \).

12. **Proving** \( p \lor q \)
    Method (1): Assume \( \neg p \). To prove: \( q \).
    Method (2): Assume \( \neg q \). To prove: \( p \).
    Method (3): Assume \( \neg p \) and \( \neg q \). To prove: a contradiction.

13. **Using** \( p \lor q \) **to prove another statement** \( r \).
    Write two proofs:
    Assume \( p \). To prove \( r \).
    Assume \( q \). To prove \( r \).

14. **How to use a for-all statement** \( \forall x \in A \) \( P(x) \).
    You need to produce an element of \( A \), then use \( P \) for that element.