Second Order Differential Equations with Hypergeometric Solutions of Degree Three

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Introduction

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- We will focus on differential operators with regular singularities only. Such operators are also called 'Fuchsian' operators.
- $\partial = \frac{d}{dx}$.
- We want the solutions in terms of ${}_2F_1(a, b; c \mid f)$ where f is a rational function of degree 3.

An Example

- Consider the following differential operator:

$$L := \partial^2 + \frac{\left(27\,x^7 - 39\,x^5 + 17\,x^3 - 5\,x - 9\,x^4 - 3\right)}{3x\,(x^2-1)(x^3-x-1)(3\,x^2-1)}\partial - \frac{5\left(3\,x^2-1\right)^2}{36x\,(x^3-x-1)(x^2-1)}.$$

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- Our algorithm finds the following solution of L:

$$Sol(L) = -\frac{{}_2F_1\left(\frac{5}{6},\frac{7}{6};1\mid\frac{-1}{x^3-x-1}\right)}{(x^3-x-1)^{5/6}} + \frac{35}{36}\,\frac{x\left(x^2-1\right)\cdot{}_2F_1\left(\frac{11}{6},\frac{13}{6};2\mid\frac{-1}{x^3-x-1}\right)}{(x^3-x-1)^{11/6}}.$$

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- This algorithm solves many equations, and helps reduce the tabulation work for other algorithms as well.

Notation

We use the following notation:

- $H_{c,x}^{a,b}$: Gauss hypergeometric differential operator.
  $H_{c,x}^{a,b} = x(1-x)\partial^2 + (c-(a+b+1)x)\partial - ab.$

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- $(e_0, e_1, e_\infty) = (1-c, c-a-b, b-a)$.
- $f \in \mathbb{C}(x)$: a rational function of degree 3.

Singularities

Theorem (Wang, Guo)

If $x = p$ is a regular singularity or a regular point of L, then there exist the following independent solutions of L at $x = p$:

$$y_1 = (x-p)^{e_1} \sum_{i=0}^{\infty} a_i (x-p)^i, \;\; a_0 \neq 0 \text{ and}$$

$$y_2 = (x-p)^{e_2} \sum_{i=0}^{\infty} b_i (x-p)^i + c y_1 \log(x-p), \;\; b_0 \neq 0$$

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- e_1, e_2 are called the exponents of L at $x = p$.
- Exponent difference of L at $x = p$ is defined as $\Delta_p(L) := \pm(e_1 - e_2)$.

Transformations

- We define the following transformations on a second order differential operator:

  (i) Change of variables: $y(x) \mapsto y(f)$
  (ii) Gauge transformation: $y \mapsto r_0 y + r_1 y'$
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- These transformations are denoted as $\xrightarrow{f}_C$, $\xrightarrow{r_0, r_1}_G$ and $\xrightarrow{r}_E$ respectively.
- We use the notation $L_1 \longrightarrow L_2$ if L_1 can be transformed to L_2 with any combination of these transformations.

Problem Statement

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such that $L_{inp}(y) = 0$,
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- The expression (1) is called a ${}_2F_1$-type solution.

Problem Discussion

Lemma (Debeerst)

Let $L_1, L_2 \in \mathbb{C}(x)[\partial]$ such that $L_1 \longrightarrow L_2$. Then there exists an operator $M \in \mathbb{C}(x)[\partial]$ such that:

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- There are algorithms to compute $\xrightarrow{r_0,r_1}_G \xrightarrow{r}_E$. Hence the crucial part is to compute f and a, b, c.

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- Singularities are generated under $\xrightarrow{f}_C$ from the roots of f, $1 - f$ and poles of f.

Effect of $\xrightarrow{f}_C$ on Singularity Structure

$H_{1,f}^{\frac{1}{12},\frac{5}{12}}$:

p	37	$\sqrt{-3}$	$-\sqrt{-3}$	∞
Δ_p	0	0	0	$\frac{1}{2}$

↑

$$f = \frac{27(x-37)(x^2+3)}{(3x-13)^3} \qquad 1 - f = \frac{8\,(9\,x+10)^2}{(3\,x-13)^3}$$

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  (ii) $min(d) = 2$ when branching pattern is $[1, 2], [1, 2], [3]$ and $(e_0, e_1, e_\infty) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$ (that has Liouvillian solutions!).

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- Hence $\boxed{4 \leq d \leq 9}$.

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$\frac{*}{2}$: an element of $\frac{1}{2} + \mathbb{Z}$.
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d	Case	Exponent difference at 0, 1, ∞ resp.	Branching pattern above 0, 1, ∞ resp.
5	case5.1	$\neq \frac{*}{3}, \neq \frac{*}{3}, E$	[3], [3], [1,1,1]
	Liouv	$\frac{*}{2}, \frac{*}{2}, E$	[1,2], [1,2], [1,1,1]
	case5.2	$\neq \frac{*}{2}, \frac{*}{3}, E$	[1,2], [3], [1,1,1]
	case5.3	$\frac{*}{2}, E, \neq \frac{*}{3}$	[1,2], [1,1,1], [3]
	case5.4	$\neq \frac{*}{2}, \neq \frac{*}{2}, \frac{*}{2}$	[1,2], [1,2], [1,2]
	case5.5	$\neq \frac{*}{3}, \neq \frac{*}{2}, \neq \frac{*}{2}$	[3], [1,2], [1,2]

One Case

$(e_0, e_1, e_\infty) = (\frac{*}{2}, E, \neq \frac{*}{3})$, Branching pattern: [1,2], [1,1,1], [3]

Let $L_{inp} \in C(x)[\partial]$ where $C \subseteq \mathbb{C}$.

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Algorithm[5.3]: Compute $f \in C(x)$ of degree 3 and exponent differences (e_0, e_1, e_∞) for $H_{c,x}^{a,b}$ corresponding to 'case5.3'.

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Output: A set of lists $[f, (e_0, e_1, e_\infty)]$ compatible to 'case5.3'.

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- **STEP 3:** Find the elements of degree 1 in $Sing(L_{inp})$ with exponent-difference in $\frac{1}{2} + \mathbb{Z}$. $(x - a_1)$ loops over these, and e_0 is the exponent-difference at $x = a_1$.

One Case

$(e_0, e_1, e_\infty) = (\frac{*}{2}, E, \neq \frac{*}{3})$, Branching pattern: [1,2], [1,1,1], [3]

- **STEP 4:**
  (i) Loop $(x - b)$ over the elements of degree 1 in $Sing(L_{inp})$, skipping $(x - a_1)$, such that the remaining three singularities have matching exponent-difference mod $\mathbb{Z}$.

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  (ii) Let e be the exponent-difference at $x = b$. Now loop e_∞ over $\frac{e}{3}, \frac{(e-1)}{3}, \frac{(e+1)}{3}$. (This fixes exponent difference mod $\mathbb{Z}$)

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- **STEP 5:**
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  (ii) We could choose any of these exponent differences for e_1.

One Case

- **STEP 6:** Take the numerator of $1 - f$ and compute its remainder mod P. That produces equations in k_1, a_2. Compute the solutions $k_1 \in C$ and $a_2 \in C \bigcup \{\infty\}$. If any solution exists, then $[f, (e_0, e_1, e_\infty)]$ is one of the candidates.

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- **STEP 7:** Repeating the procedure with all other possibilities returns the complete list of candidates.

Main Algorithm

Algorithm: Solve an irreducible second order linear differential operator $L_{inp} \in C(x)[\partial]$ in terms of ${}_2F_1(a, b; c \mid f)$, with $f \in C(x)$ of degree 3.

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Output: A non zero solution $y = e^{\int r}(r_0 S(f) + r_1 S(f)')$, if it exists, such that $L_{inp}(y) = 0$, where $S(f) = {}_2F_1(a, b; c \mid f)$, $f, r, r_0, r_1 \in C(x)$ and f has degree 3.

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That produces a set of lists $[f, (e_0, e_1, e_\infty)]$.

- **STEP 3 :** $H^{a,b}_{c,x} = x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab$.
From each element in *'Candidates'* above (a) compute a, b, c (b) substitute the values of a, b, c in $H^{a,b}_{c,x}$ and (c) apply the change of variable $x \mapsto f$ on $H^{a,b}_{c,x}$. That produces a list of operators $H^{a,b}_{c,f}$.

Main Algorithm

- **STEP 4 :** Compute the transformations $\xrightarrow{r_0, r_1}_G \xrightarrow{r}_E$ between each operator $H_{c,f}^{a,b}$ in **STEP 3** and L_{inp}. We get a map of the form:
  $G = e^{\int r}(r_0 + r_1 \partial), \quad$ where $r, r_0, r_1 \in C(x)$ and $\partial = \frac{d}{dx}$.

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  STEP 6: Repeat the procedure to get a list of solutions of L_{inp}. Choose the best solution (with shortest length) from the list.

THANK YOU!