Finding ${}_2F_1$ Type Solutions of Differential Equations with 5 Singularities

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Introduction

Differential equations with ${}_2F_1$ type solutions are very common in Mathematics and they occur quite frequently in Combinatorics and Physics. We are interested in solving differential equations with $n = 5$ *non removable regular* singularities. ($n = 3$ is easy, and $n = 4$ is done by *M. van Hoeij* and *R. Vidunas* (paper in progress)).

An Example

Consider the following differential equation:

$$-\frac{x\,(x-1)\,y}{(x+1)^2\,(3\,x-1)^2\,(3\,x+1)} + \frac{\left(27\,x^3+19\,x^2+x+1\right)}{2\,x\,(x+1)\,(3\,x-1)\,(3\,x+1)}\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0. \qquad (1)$$

This equation has 5 *regular* singularities $\{-1, -\frac{1}{3}, 0, \frac{1}{3}, \infty\}$; among which the singularities $\{-1, \frac{1}{3}\}$ are *logarithmic*. Current Computer Algebra systems do not solve it. $y = {}_2F_1(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}|\, f)$, with $f = \frac{(x-1)^3(3\,x+1)}{(3\,x-1)(x+1)^3}$ is a solution of (1).
Our goal is to build a complete table of all rational functions f that can occur in this context, and then to develop a differential solver from it.

Gauss Hypergeometric Equation

Gauss hypergeometric differential equation (GHE) has the form:

$$x(1-x)\frac{d^2y}{dx^2} + (c-(a+b+1)x)\frac{dy}{dx} - aby = 0. \qquad (2)$$

It has regular singularities at $0, 1$ and ∞ with local exponents $\{0, 1-c\}$ at $x = 0$, $\{0, c-a-b\}$ at $x = 1$ and $\{a, b\}$ at $x = \infty$. $y = {}_2F_1(a, b; c|x)$ is one of its two independent solutions at $x = 0$. Computing a ${}_2F_1$ type solution of (1) is the same as computing transformations from (2) to (1).

Problem Statement

If a second order differential equation L_{inp} has:

(i) 5 *non removable regular* singularities.

(ii) At least one of the singularities is *logarithmic*.

Then we want to find its solution if it can be expressed in terms of ${}_2F_1$ Hypergeometric function. More precisely, we want to find a solution of L_{inp} in the form:

$$y = e^{\int r}(r_0 y_1 + r_1 y_1') \qquad (3)$$

where $y_1 = {}_2F_1(a, b; c|f)$ and f,r,r_0,$r_1 \in \mathbb{C}(x)$.

Why *logarithmic* singularities ?

In the above example, the degree of f was 4. For arbitrary a, b and c (without restriction (ii) above), the degree bound for f is 60 when $n = 4$, and 96 when $n = 5$. For $n = 4$, there are 926 *Belyi* maps (up to Möbius equivalence) and a small number of *near Belyi* maps that can occur as f. For $n = 5$, we decided to restrict to differential equations L_{inp} that have at least one *logarithmic* singularity, for two reasons:

1. That lowers the degree bound for f from 96 to a more manageable 18.
2. *Logarithmic* singularities are very common in practice.

Among the differential equations with 5 *non removable regular* singularities, most of those which are ${}_2F_1$ solvable, arise from (2) with exponent differences $(1-c, c-a-b, b-a) = (1/k, 1/2, 0)$ where $k \in \{3, 4, 6\}$. We want to treat $(1/3, 1/2, 0)$ first, as that covers the majority of such cases. Denote the GHE with exponent differences $(1/3, 1/2, 0)$ at $(0, 1, \infty)$ as L_{320}.

Idea

We define the following transformations [1] on any second order differential equation:

1. $y(x) \to y(f)$, $\quad f \in \mathbb{C}(x) \setminus \mathbb{C}$ $\quad$ (Change of variable)
2. $y \to r_0 y + r_1 y'$, $\quad r_0, r_1 \in \mathbb{C}(x)$ $\quad$ (Gauge transformation)
3. $y \to e^{\int r} y$, $\quad r \in \mathbb{C}(x)$ $\quad$ (Exponential product)

These transformations preserve the order of differential equations, and are denoted as: $\longrightarrow_C$, $\longrightarrow_G$ and $\longrightarrow_E$ respectively. To solve L_{inp} in terms of ${}_2F_1$ Hypergeometric function is equivalent to find if there exists any sequence of above transformations that transforms L_{320} to L_{inp}. More precisely, this problem reduces (see [5]) to the following:

$$L_{320} \longrightarrow_C L_f \longrightarrow_{EG} L_{inp}.$$

If such transformations exist, then we get a solution of L_{inp} in the same fashion as:

$$y_{320} = {}_2F_1(\tfrac{1}{12}, \tfrac{1}{12}; \tfrac{2}{3}|x) \longrightarrow_C y_f = {}_2F_1(\tfrac{1}{12}, \tfrac{1}{12}; \tfrac{2}{3}|f) \longrightarrow_{EG} y_{inp} = e^{\int r}(r_0 y_f + r_1 y_f').$$

Once we find such f, then [3] takes care of the second part. Hence the crucial part is to compute f. We computed a table of all such f's.

The Correspondence

Given a *Belyi* map f, the corresponding *dessin* is the graph of $f^{-1}([0, 1])$. There is a correspondence [4] between *dessins* with $n/2$ edges (or n half-edges) and *Belyi* maps of degree n (up to Möbius equivalence).
A *dessin* can be represented by an ordered triple (g_0, g_1, g_∞) of permutations in S_n such that:

$a)$ the group generated by g_0 and g_1 acts transitively on $\{1, 2, .., n\}$.

$b)$ $g_0 g_1 g_\infty = 1$.

Any two conjugated triples represent the same *dessin*. Here is an example of a *dessin* from (1/3,1/2,0):

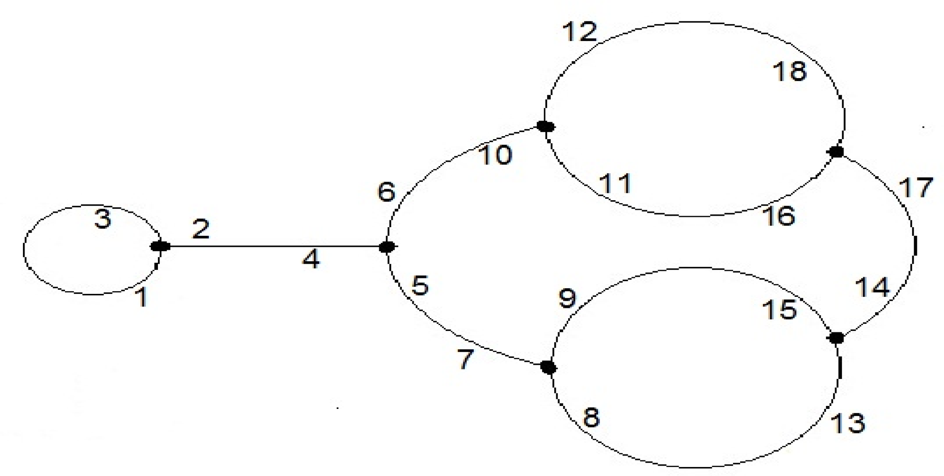

Figure 1: A clean planar dessin

This *dessin* has 6 **vertices** (points above 0), 9 **edges** (correspond to the points above 1) and 5 **faces** (correspond to the points above ∞). This is a *clean* (each point above 1 has ramification order 2) and *planar* (genus 0) *dessin*. In terms of permutations $g_0, g_1, g_\infty \in S_{18}$ (up to conjugation), it can be expressed as:

$g_0 = (1\ 2\ 3)\ (4\ 5\ 6)\ (7\ 8\ 9)\ (10\ 11\ 12)\ (13\ 14\ 15)\ (16\ 17\ 18).$

$g_1 = (1\ 3)\ (2\ 4)\ (5\ 7)\ (6\ 10)\ (8\ 13)\ (9\ 15)\ (11\ 16)\ (12\ 18)\ (14\ 17).$

Recall: $g_\infty = (g_0 g_1)^{-1}$.

Each *planar dessin* determines a *Belyi* map $f : \mathbb{P}^1 \to \mathbb{P}^1$ up to Möbius equivalence. The above *dessin* corresponds to the following degree 18 *Belyi* map (up to Möbius equivalence) from our table:

$$f := \frac{4}{27} \cdot \frac{\left(x^6 - 4\,x^5 + 5\,x^2 + 4\,x + 4\right)^3}{(x-4)\left(5\,x^2 + 4\,x + 4\right)^2 x^5}. \qquad (4)$$

${}_2F_1(1/12, 1/12; 2/3|f)$ satisfies a differential equation with 5 *regular* singularities. Our goal is to tabulate all such f's.

Our Results

For a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n, total amount of ramification is given by:

$$\sum_{p \in \mathbb{P}^1} (e_p - 1) = 2n - 2 \quad \text{(Riemann-Hurwitz)} \qquad (5)$$

where e_p is the ramification order of f at p. Let the amount of ramification of f be $R_{01\infty}$ (above $\{0, 1, \infty\}$) and R_{out} (above $\mathbb{P}^1 \setminus \{0, 1, \infty\}$). As in [2], using (5), we can find the bound on the degree of f and R_{out}. For $(1/3, 1/2, 0)$, we find:

$\deg(f) \leq 18$ and $R_{out} \leq 2$.

We computed all rational functions (up to Möbius equivalence) that can occur as f in the solution (3) of L_{inp}. For $(1/3, 1/2, 0)$, we computed a table with the following numbers of entries:

R_{out}	Name	Degrees	Number of f's (up to Möbius equivalence)	Remarks
0	*Belyi*	$3-16, 18$	260	0 dimensional families
1	*Belyi*$_{-1}$	$2-10, 12$	68	1 dimensional families
2	*Belyi*$_{-2}$	$4, 6$	2	2 dimensional families

Our solver for L_{inp} will be complete if our table is complete. To prove the completeness, we do a combinatorial search to find all *dessins* and *near dessins* that are compatible with conditions (i) and (ii). If every *dessin* and *near dessin* in this search corresponds to a member of our table of *Belyi* maps and *near Belyi* maps, then the table is complete.

Main Algorithm

Step 1: Compute the singularity structure and a 5 point invariant (a function for sets of 5 points that is invariant under Möbius transformation) of L_{inp}.

Step 2.a: Compare the 5 point invariant of L_{inp} with the ones in our table of *Belyi* maps. If they match, then compute Möbius transformation from singularities of the *Belyi* map to the singularities of L_{inp}. The *Belyi* map composed with the Möbius transformation gives *Candidate(s)* for f.

Step 2.b: For $Belyi_{-1}$ maps $f(x, t)$, compare 5 point invariants between singularity structures with matching exponent differences. That gives the value of t and thus, gives *Candidate(s)* for f.

Step 2.c: For $Belyi_{-2}$ maps, we have programs that compute *Candidate(s)* for f from the singularity structure of L_{inp}.

Step 3: For each *Candidate* f, we compute L_f (apply $x \mapsto f$ on L_{320}) and finally, use [3] to compute r, r_0, r_1 in (3) if they exist.

What Comes Next ?

Our next task is to build similar tables for the remaining *logarithmic* cases (those tables have fewer entries with lower degree bounds for f) and to implement the above algorithm.

References

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