A method for the Integration of Solutions of Ore Equations

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Abstract

We introduce the notion of the adjoint Ore ring and give a definition of adjoint polynomial, operator and equation. We apply this for integrating solutions of Ore equations.

1 Introduction

The study of Ore rings [10, 4] is attractive because, first of all, it allows statement concerning operators of various kinds to be treated simultaneously. In addition, the use of this theory lets one create general purpose algorithms and corresponding programs adjustable to a specific form of operators and equations.

The goal of this paper is integration (in the difference case this is: summation) of solutions of Ore equations. For this purpose we first define an adjoint for an Ore ring, similar to the well known adjoint for differential operators, and also similar to ideas in [7].

For solving linear differential equations one often applies "reduction of order" in case one of the solutions was found. Reduction of order leads to the problem of integrating solutions of a differential equation. In this paper we give a simple and easy to implement method for this problem. Given an operator L, our algorithm computes an operator $\tilde{L}$ of minimal order such that all solutions of L are derivatives of solutions of $\tilde{L}$. In the case that the order of $\tilde{L}$ equals the order of L, this effectively removes, at low computational cost, one integration symbol from the symbolic solutions of the original differential equation. The use of Ore rings makes our algorithm more general, so that it can be applied for the case of difference and q-difference equations as well.

2 Integrating factors and adjoints

Let k be a field and let K be a ring that contains k. We consider Ore rings $k[\nabla]$ and $K[\nabla]$ for two different types of ∇:

- Case 1: ∇ is a derivation on K, i.e. $\nabla(ab) = b\nabla(a) + a\nabla(b)$ for all $a, b \in K$, and it is also a derivation on k (so $\nabla(a) \in k$ for $a \in k$).
- Case 2: $\nabla = \sigma - 1$ where σ is an automorphism of k and of K.

In both cases we will assume that the set of constants Const $= \{a \in K | \nabla(a) = 0\}$ is a subfield of k. Let $k[\nabla]$ be the ring of all operators $\sum_{i=0}^{n} a_i \nabla^i$. Similarly define $K[\nabla]$. An element L of $K[\nabla]$ or $k[\nabla]$ can be viewed as a Const-linear map from K to K, $L(y) = \sum_{i=0}^{n} a_i \nabla^i(y) \in K$. We will assume that $\sum a_i \nabla^i \in K[\nabla]$ acts as the zero map on K if and only if all a_i are zero. A common situation for such Ore rings is that one is given an equation $L(y) = 0$ for some $L \in k[\nabla]$ and one is interested in finding solutions y of this equation in some (field or ring) extension K of k. By "rational solutions" of L we mean solutions $y \in k$.

Now $k[\nabla]$ and $K[\nabla]$ are rings. The multiplication in these ring corresponds to composition of operators. Using the relation

- Case 1: $\nabla \circ a = a\nabla + \nabla(a)$
- Case 2: $\nabla \circ a = \sigma(a)\nabla + \nabla(a)$

any product of elements in $K[\nabla]$ can be written in a standard form $\sum_{i=0}^{n} a_i \nabla^i$. We define ∇^* as follows:

- Case 1: $\nabla^* = -\nabla$
- Case 2: $\nabla^* = \sigma^{-1} - 1$

and we can define the *adjoint ring* of $k[\nabla]$ as $k[\nabla^*]$. Note that in case 1 we have $k[\nabla] = k[\nabla^*]$ but in case 2 $k[\nabla]$ need not be equal to its adjoint ring. Now we can define the adjoint map

$$\text{ad}: \ k[\nabla] \to k[\nabla^*]$$

for an operator $L = \sum_i a_i \nabla^i$ as follows:

$$\text{ad } L = \sum_i (\nabla^*)^i \circ a_i.$$

This can be rewritten to the standard form ad $L = \sum_i b_i (\nabla^*)^i$ for some $b_i \in k$. For brevity we will often write L^* instead of ad L.

Now one can verify that the adjoint is a Const-linear bijective map and that

$$(L \circ M)^* = M^* \circ L^*$$

for all $L, M \in k[\nabla]$.

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Proposition 1

$$\nabla(f) = 0 \iff f \in \text{Const} \iff \nabla^*(f) = 0.$$

Additionally for any $L \in K[\nabla]$

$$L(1) = 0 \iff \exists_M\ L = M \circ \nabla$$

and

$$L^*(1) = 0 \iff \exists_M\ L = \nabla \circ M.$$

Proof: The first statement follows from the definitions. Write $L = \sum_i a_i \nabla^i$ for some $a_i \in K$. Now $L = M \circ \nabla$ for some M if and only if $a_0 = 0$. Now the second statement followings because $a_0 = L(1)$. For the third statement, write $L^* \in K[\nabla^*]$ (note that in general L^* need not be an element of $K[\nabla]$) as $L^* = \sum_i a_i (\nabla^*)^i$ for some $a_i \in K$. Now $L = \sum_i \nabla^i \circ a_i$ and $L = \nabla \circ M$ for some M iff $a_0 = L^*(1) = 0$. □

An element $l \in K$ is an *integrating factor* for $L \in K[\nabla]$ if $lL = \nabla \circ M$, where $M \in K[\nabla]$.

The following Proposition shows that in the general case the adjoint operator has an important feature which is well known in the differential case.

Proposition 2 *$l \in K$ is an integrating factor for L iff $L^*(l) = 0$.*

Proof: By Proposition 1 we have $lL = \nabla \circ M$ for some M iff $(lL)^*(1) = 0$. Since $l \in K$ we have $l^*(1) = l(1) = l$ and so $(lL)^*(1) = L^*(l)$. □

3 Accurate integration

An element $g \in K$ is a *primitive* of $f \in K$ if $\nabla(g) = f$. Consider the following problem:

Let $f \in K$ and the minimal annihilating operator $L \in k[\nabla]$ for f be given. So $n = \operatorname{ord} L$ is minimal with the property that $L \in k[\nabla]$ and $L(f) = 0$. Decide whether there exists a primitive g of f such that the minimal annihilating operator $\widetilde{L}$ for g has order n. If so, then construct all such g together with their minimal annihilating operators.

We show that this problem (the problem of the *accurate integration*) can be solved with the help of finding integrating factors.

Let g be any primitive of f and $\widetilde{L} \in k[\nabla]$ be the minimal annihilating operator for g. Now $L \circ \nabla(g) = L(f) = 0$ hence by the minimality of $\widetilde{L}$ (and by the fact that $k[\nabla]$ is a Euclidean ring, c.f. [10]) it follows that $\widetilde{L}$ is a right-hand factor of $L \circ \nabla$. Hence

$$\operatorname{ord} \widetilde{L} \le \operatorname{ord} L \circ \nabla = n{+}1 \text{ and if } \operatorname{ord} \widetilde{L} = n{+}1 \text{ then } \widetilde{L} = L \circ \nabla.$$

Consider the least common left multiple ($LCLM$) of $\widetilde{L}$ and ∇ presented in the form

$$LCLM(\widetilde{L}, \nabla) = L_1 \circ \nabla, \tag{1}$$

$L_1 \in k[\nabla], \operatorname{ord} L_1 \le \operatorname{ord} \widetilde{L}$. We have $\widetilde{L}(g) = 0$, so $L_1 \circ \nabla(g) = 0$, hence $L_1(f) = 0$ and so $\operatorname{ord} L_1 \ge n$ by the minimality of L. So $\operatorname{ord} LCLM(\widetilde{L}, \nabla) \ge n + 1$ and hence

$$\operatorname{ord} \widetilde{L} \ge n \quad \text{and} \quad \text{if} \quad \operatorname{ord} \widetilde{L} = n \quad \text{then} \quad GCRD(\widetilde{L}, \nabla) = 1 \tag{2}$$

where $GCRD$ stands for greatest common right divisor.

Thus there are two possibilities for $\operatorname{ord} \widetilde{L}$: n and $n + 1$. The questions are: when is $\operatorname{ord} \widetilde{L} = n$ and what is $\widetilde{L}$ in this case.

If $\operatorname{ord} \widetilde{L} = n$ then from equation (2) it follows that

$$r \circ \nabla + \widetilde{l} \circ \widetilde{L} = 1 \tag{3}$$

for some $\widetilde{l}, r \in k[\nabla]$ with $\operatorname{ord} r < \operatorname{ord} \widetilde{L} = n$ and $\operatorname{ord} \widetilde{l} < \operatorname{ord} \nabla = 1$. Applying equation (3) on g results in

$$r(f) = g.$$

Applying ∇ on this equation yields $\nabla \circ r(f) = f$ so $(1 - \nabla \circ r)(f) = 0$. By the minimality of L it follows that $1 - \nabla \circ r = l \circ L$ for some operator l hence

$$\nabla \circ r + l \circ L = 1. \tag{4}$$

Conversely, if equations (3),(4), $\operatorname{ord} r < n$ and $\operatorname{ord} \widetilde{l} < 1$ hold then one can easily verify that $\operatorname{ord} \widetilde{L} = n$, that $\widetilde{L}$ is the minimal annihilating operator for $r(f)$ and that $\nabla(r(f)) = f$. Hence equations (3),(4) with the conditions on $\operatorname{ord} r$ and $\operatorname{ord} \widetilde{l}$ are equivalent with the problem of accurate integration.

The inequality $\operatorname{ord} \widetilde{l} < 1$ implies $\operatorname{ord} l = \operatorname{ord} \widetilde{l} = 0$, i.e. $l, \widetilde{l} \in k$. Both sides of (4) are operators and if we take the adjoints we get

$$r^* \circ \nabla^* + L^* \circ l^* = 1. \tag{5}$$

Applying the left- and the right-hand side of (5) to 1 we obtain

$$L^*(l) = 1. \tag{6}$$

For each solution $l \in k$ of (6) we have $(1 - lL)^*(1) = 1 - L^*(l) = 0$ and so by Proposition 1 it follows that equation (4) allows a unique solution r. The minimal annihilating operator of g is defined up to a left-hand factor in k. Therefore we can take $\widetilde{l} = 1$ and

$$\widetilde{L} = 1 - r \circ \nabla. \tag{7}$$

This operator annihilates one-unique primitive $r(f)$ of f. If operators r_0 and r_1 correspond to different solutions l_0 and l_1 of (6) then the primitives $r_0(f)$ and $r_1(f)$ of f are also different (otherwise the operator $r_0 - r_1$ of order $< n$ would annihilate f). Since primitives are determined up to constants it follows that $(r_0 - r_1)(f)$ must be a constant.

$\widetilde{L}$ maps the primitive $r(f)$ of f to 0. Furthermore it maps any constant to itself. Hence it maps any primitive of f to a constant.

The preceding can be formulated as the following

Proposition 3 *Let $L \in k[\nabla]$ be the minimal annihilating operator for $f \in K$ and $L^*(l) = 1, l \in k$. Then the equality $\nabla \circ r + l \circ L = 1$ uniquely determines r. In turns r lets find the operator $\widetilde{L} \in k[\nabla]$ (up to a factor in k) annihilating the primitive*

$$g = r(f) \tag{8}$$

of f. If formula (7) is used to construct $\widetilde{L}$ then $\widetilde{L}(g_1) \in$ Const for any primitive g_1 of f. □

If (6) has no solution in k then no primitive of f has a minimal annihilating operator over k of order n. If (6) has a unique solution in k then a primitive and its minimal annihilating operator can be defined uniquely by (8),(7).

Proposition 4 *Let $\mathcal{M}$ be the set of all solutions of (6) in k. Then $\mathcal{M}$ either is empty, or $\mathcal{M}$ has only one element, or $\mathcal{M}$ has the form*

$$l_0 + Ch, \tag{9}$$

where $l_0, h \in k, h \neq 0, C$ runs through Const. *In the last case any primitive of f has a minimal annihilating operator of order n.*

Proof: Suppose there exists a solution l_0 of (6). Then the solution space of (6) is of the form $l_0 + V$ where V is the solution space of $L^*(l) = 0$. The map $l \mapsto r(f)$ (r depends on l by (4)) is an injective (here we use that L is minimal) linear map from $l_0 + V$ to the set of primitives of f. Since the set of primitives is an affine space of dimension 1, V must have dimension ≤ 1. □

Note that if the solution space of $L^*(l) = 0$ has dimension > 1 then the map $l \mapsto r(f)$ can not be injective because the image of this map has dimension ≤ 1. The fact that the map is not injective means that there exists an r, $\operatorname{ord} r < \operatorname{ord} L$, with $r(f) = 0$ which contradicts our assumption that L is minimal.

Let now $\mathcal{M}$ have the form (9). Denote by l_C the solution $l_0 + Ch$ of (6) and by r_C and $\widetilde{L}_C$ the operators which are found starting with l_C. Since h is an integrating factor for L we have $hL = \nabla \circ M, \operatorname{ord} M = n - 1$. Now from (4) and (7) we obtain

$$r_C = r_0 - CM \tag{10}$$

$$\widetilde{L}_C = \widetilde{L}_0 + CM \circ \nabla \tag{11}$$

where r_0 and $\widetilde{L}_0$ correspond to the solution l_0 of (6). The operator $\widetilde{L}_C$ is the minimal annihilating operator for the primitive

$$g_C = r_C(f) \tag{12}$$

of f.

Let g be a primitive of f and $C \in$ Const. Then $\widetilde{L}_C(g) = \widetilde{L}_0(g) + CM(\nabla(g)) = \widetilde{L}_0(g) + CM(f)$ and $M(\nabla(g)) \in$ Const because $\widetilde{L}_C(g), \widetilde{L}_0(g) \in$ Const. Additionally $M(f) \neq 0$ because $\operatorname{ord} M < \operatorname{ord} L$. Taking

$$C = -\frac{\widetilde{L}_0(g)}{M(f)} \tag{13}$$

we obtain the value of C such that $g = r_C(f)$.

The price which we pay for solving the problem of the accurate integration is finding solutions in k of the equation $L^*(y) = 1$. If k is a rational function field, then the last problem can be solved effectively in all the cases mentioned in the examples below (c.f. [1, 2, 3]).

Examples Let $k = \mathbf{C}(x), \nabla = D = \frac{d}{dx}$. Applying the described approach to $f = \ln x$,

$$L = xD^2 + D \tag{14}$$

gives $L^* = xD^2 + D$, and the general rational solution of the equation $L^*(y) = 1$ is $l_C = x + C$. Therefore $l_0 = x$, $h = 1$. So any primitive of $\ln x$ is annihilated by a second order operator. We obtain

$$\widetilde{L}_C = (x^2 + Cx)D^2 - xD + 1, \; r_C = (-x^2 - Cx)D + x.$$

The algorithm proposed above lets in some cases integrate special functions. The minimal annihilating operator for Bessel function J_1 is $x^2D^2 + xD + (x^2 - 1)$. Applying the algorithm we get $L^* = x^2D^2 + 3xD + x^2$, and $L^*(y) = 1$ has one-unique rational solution $\frac{1}{x^2}$. We obtain

$$\widetilde{L} = D^2 + \frac{1}{x}D + 1, \; r = -D - \frac{1}{x}.$$

Thus we get a primitive of J_1 in the form

$$r(J_1) = (-D - \frac{1}{x})(J_1),$$

with the minimal annihilating operator $D^2 + \frac{1}{x}D + 1$. Other primitives can be annihilated by the operator $L \circ D$ of order 3. This result is in agreement with the Bessel function theory.

Let $k = \mathbf{Q}(n)$, $\nabla = E - 1$ where $E(n) = n + 1$. Let $u_0, u_1, \ldots$ be Fibonacci numbers. Apply the described approach to u_n^2, $L = E^3 - 2E^2 - 2E + 1$. We obtain $L^* = E^{-3} - 2E^{-2} - 2E^{-1} + 1$, and the equation $L^*(y) = 1$ has the unique rational solution $-\frac{1}{2}$. It shows that one-unique primitive of u_n^2 can be annihilated by an operator the order 3: $\widetilde{L} = -\frac{1}{2}L$, while $r = \frac{1}{2}(E^2 - E - 3)$. This primitive is

$$\frac{1}{2}(u_{n+2}^2 - u_{n+1}^2 - 3u_n^2)$$

and in particular

$$\sum_{i=0}^{n} u_i^2 = \frac{1}{2}(u_{n+3}^2 - u_{n+2}^2 - 3u_{n+1}^2).$$

□

The algorithm described in this Section, is a generalization of well known Gosper's algorithm ([8]) which, given a first order (i.e. hyper-geometric) sequence over $\mathbf{Q}(n)$, lets recognize whether there exists another sequence of such a kind that is a primitive for the given sequence. The algorithm, proposed above, solves the analogous problem for a wide class of equations of any order n. Using (8) we can express the mentioned primitive explicitly.

Remark that from the results in this paper it follows that the following 3 problems are equivalent.

- Find the solutions $l \in k$ of $L^*(l) = 1$.
- Let $f \in K$. Let the minimal annihilating operator $L \in k[\nabla]$ for f be given. Decide whether there exists $r \in k[\nabla]$ such that $r(f)$ is a primitive of f. If so, then construct such r.
- The problem of accurate integration.
- Computing solutions (r, l) of equation (4). Note that according to section 3.1 in [9] this is equivalent to computing a complement of Const in the solution space of $L \circ \nabla$.

Similar problems, but in a more general situation, are studied in [5, 6]. Our approach is less general but it has the advantage of simplicity, it only uses an adjoint and rational solutions, which are quite efficient.

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