THE LANDAU-MIGNOTTE ESTIMATE

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We start with a definition. For all $f = \sum_{i=0}^{m} f_i x^i \in \mathbb{C}[x]$ we define the *norm* of f by

$$\| f \| = \sqrt{\sum_{i=0}^{m} |f_i|^2}.$$

One easily checks that the norm has the following properties:

1. $\| f \| = 0$ if and only if $f = 0$.
2. $\| af \| = |a| \, \| f \|$ for all $c \in \mathbb{C}$ and all f.
3. $\| f + g \| \leq \| f \| + \| g \|$ for all $f, g \in \mathbb{C}[x]$.
4. $\| x^r f \| = \| f \|$ for all $r \in \mathbb{N}$ and all $f \in \mathbb{C}[x]$.

$\| fg \| \leq \| f \| \, \| g \|$ does not hold in general. Counter-example: $f = g = x + 1$. However, the following lemma is valid:

Lemma 1. *For all $f \in \mathbb{C}[x]$ and all $c \in \mathbb{C}, c \neq 0$ one has*

$$\| (x + c) f \| = |c| \; \| (x + \bar{c}^{-1}) f \| \, .$$

Proof. Let $f = \sum_{i=0}^{m} f_i x^i$. Define $f_{-1} = f_{m+1} = 0$. Then

$$\begin{aligned} (x+c)f &= \sum_{i=0}^{m} (x+c) f_i x^i = \sum_{i=0}^{m} f_i x^{i+1} + \sum_{i=0}^{m} c f_i x^i \\ &= \sum_{i=0}^{m+1} f_{i-1} x^i + \sum_{i=0}^{m+1} c f_i x^i = \sum_{i=0}^{m+1} (f_{i-1} + c f_i) x^i. \end{aligned}$$

Hence

$$\begin{aligned} \| (x+c) f \|^2 &= \sum_{i=0}^{m+1} |f_{i-1} + c f_i|^2 \\ &= \sum_{i=0}^{m+1} (|f_{i-1}|^2 + |c|^2 |f_i|^2 + c \bar{f}_{i-1} f_i + \bar{c} f_{i-1} \bar{f}_i) \\ &= (1 + |c|^2) \, \| f \|^2 + \sum_{i=0}^{m+1} (c \bar{f}_{i-1} f_i + \bar{c} f_{i-1} \bar{f}_i). \end{aligned}$$

In a similar way:

$$\| (x + \bar{c}^{-1}) f \|^2 = (1 + |c|^{-2}) \, \| f \|^2 + \sum_{i=0}^{m+1} (\bar{c}^{-1} \bar{f}_{i-1} f_i + c^{-1} f_{i-1} \bar{f}_i).$$

The last two relations imply the statement of the lemma.

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Lemma 2. *For all* $x_1, \dots, x_m \in \mathbb{C}$ *and* $t \in \{1, \dots, m\}$ *such that* $x_1 \neq 0, \dots, x_t \neq 0$ *one has*

$$\| (x - x_1) \cdots (x - x_t)(x - x_{t+1}) \cdots (x - x_m) \| = \\ |x_1 \cdots x_t| \, \| (x - \bar{x}_1^{-1}) \cdots (x - \bar{x}_t^{-1})(x - x_{t+1}) \cdots (x - x_m)) \| .$$

Proof. Repeated application of Lemma 1 yields

$$\| (x - x_1) \cdots (x - x_t)(x - x_{t+1}) \cdots (x - x_m) \| = \\ |x_1| \, \| (x - \bar{x}_1^{-1})(x - x_2) \cdots (x - x_m) \| = \\ |x_1||x_2| \, \| (x - \bar{x}_1^{-1})(x - \bar{x}_2^{-1})(x - x_3) \cdots (x - x_m) \| = \\ \cdots \\ |x_1 \cdots x_t| \, \| (x - \bar{x}_1^{-1}) \cdots (x - \bar{x}_t^{-1})(x - x_{t+1}) \cdots (x - x_m) \| .$$

Lemma 3. *For all* $x_1, \dots, x_m \in \mathbb{C}$ *and* $t \in \{1 \dots, m\}$ *one has*

$$\| \prod_{i=1}^{m} (x - x_i) \|^2 \geq |x_1 \cdots x_t|^2 + |x_{t+1} \cdots x_m|^2. \tag{1}$$

Proof. First assume that $x_1 \neq 0, \dots, x_t \neq 0$. Then Lemma 2 implies

$$\| \prod_{i=1}^{m} (x - x_i) \|^2 = |x_1 \cdots x_t|^2 \, \| h \|^2,$$

where

$$h = (x - \bar{x}_1^{-1}) \cdots (x - \bar{x}_t^{-1})(x - x_{t+1}) \cdots (x - x_m).$$

If h_i is the coefficient of x^i in h, then

$$h_m = 1 \text{ and } h_0 = (-1)^m \bar{x}_1^{-1} \cdots \bar{x}_t^{-1} x_{t+1} \cdots x_m.$$

Furthermore,

$$\| h \|^2 \geq |h_m|^2 + |h_0|^2 \geq 1 + |x_1 \cdots x_t|^{-2} |x_{t+1} \cdots x_m|^2.$$

(1) is an immediate consequence.

Now assume that $x_1 = x_2 = \cdots = x_s = 0$ and $x_{s+1} \neq 0, \dots, x_t \neq 0$. In this case we must show

$$\| \prod_{i=1}^{m} (x - x_i) \|^2 \geq |x_{t+1} \cdots x_m|^2.$$

One has $\| f x^s \| = \| f \|$ (cf. property 4.). Hence

$$\| \prod_{i=1}^{m} (x - x_i) \|^2 = \| \prod_{i=s+1}^{m} (x - x_i) \| \geq |x_{s+1} \cdots x_t|^2 + |x_{t+1} \cdots x_m|^2$$

by the first part of the proof.

Corollary 1. $\| f \| \geq |f_m||x_{t+1} \cdots x_m|$.

Lemma 4. *Let* $x_{t+1}, \dots, x_m$ *be the zeroes of* f *having absolute value* ≥ 1. *Then for all* $i \in \{0, \dots, m\}$

$$|f_i| \leq \binom{m}{i} |f_m||x_{t+1} \cdots x_m|.$$

Proof. Left to the reader.

Theorem 1. *If $f = \sum_{i=0}^{m} f_i x^i, g = \sum_{i=0}^{n} g_i x^i \in \mathbb{C}[x]$ and g divides f, then the following inequality holds for all $i \in \{0, \ldots, n\}$*

$$|g_i| \leq \frac{|g_n|}{|f_m|} \binom{n}{i} \parallel f \parallel .$$

Proof. Left to the reader.

Corollary 2. *Let $f, g \in \mathbb{Z}[x]$ be polynomials of degrees $m, n, resp.$. Let $d = \sum_{i=0}^{r} d_i x^i \in \mathbb{Z}[x]$ be the greatest common divisor of f and g. Then the following inequality is valid for all $i \in \{0, \ldots, r\}$*

$$|d_i| \leq 2^{\min(m,n)} \gcd(lc(f), lc(g)) \min\left(\frac{||f||}{|lc(f)|}, \frac{||g||}{|lc(g)|}\right) \tag{2}$$

(here $lc(f)$ denotes the heighest coefficient of f).

Proof. Left to the reader.