# Project: Numerically solve a differential equation that describes the motion of the shocks of a car.

Calc I, MAC2311-05, Fall 2000

November 20, 2000

Here is a description of the project we talked about on Friday. We consider one of the shocks of a car, which carries a certain weight $M$. The behavior of this shock is characterized by two positive constants, the damping constant $C_D$ and the spring constant $C_S$. Manufacturers can produce shocks with any value for $C_D$ and $C_S$ that you want, but of course some values are better for car safety or comfort than other values. For example, if you drive over a bump, and if the damping constant $C_D$ is too small, then the car will keep on going up and down and up and down for a while because this up and down motion is not damped enough, which is obviously not good for traffic safety. If shocks are worn out then they may display this behavior. You can sometimes observe this on roads in the US.

On the other hand, if $C_D$ is much too large, then the motion of the shocks is slowed down too much, and it will take longer for the car to go back to its balance. So the value of $C_D$ should not be too high either.

A low value for the spring constant $C_S$ means a more comfortable ride. A higher value for $C_S$ is less comfortable, but it makes the shocks react faster and harder. A high value for $C_S$ causes the wheels to go back quicker to the road when you hit some bumps, so that the road-tire contact is restored quicker after the bump. More road-tire contact leads to better grip. So: low $C_S$ leads to more comfort (smooth ride), high $C_S$ leads to more grip (you may conclude that a race car will have a very high value for $C_S$ and will not be comfortable).

Let $y(t)$ be the height at time $t$. We normalize such that height 0 is the height in which the shock is in balance, so the force of the spring and the force of gravity are exactly in balance at height $y(t) = 0$. Then we have seen on Friday that: $y''(t) = -\frac{C_D}{M}y'(t) - \frac{C_S}{M}y(t)$. For simplicity we will take $M = 1$ so we get the following *differential equation:*

$$y''(t) = -C_D\ y'(t) - C_S\ y(t).$$

We will also take the following *initial data*

$$y(0) = 0, y'(0) = 1.$$

Note that: $y(t)$ is the height, $v(t) = y'(t)$ is the velocity (positive means going up, negative means going down), and $a(t) = y''(t)$ is the acceleration. So our initial data says:
At time $t = 0$ the height is $y(0) = 0$, and the velocity is $y'(0) = v(0) = 1$. So we are starting with height in the balance, but with an upward motion. You can imagine this as follows: suppose the car was standing on the parking lot (so the height is in balance, the initial height is zero) and you give it a kick from below so that it is moving upwards, starting from height 0 with initial velocity 1. Then it will then go up and down a few times, but the damping would slow down this motion so the car is soon standing almost still again (unless $C_D$ is much too small, because then it may take a while before it is standing almost still).

I will now take $C_D = 1$ and $C_S = 2$ and take time steps $\Delta t = 0.1$ and compute values for $y(t), y'(t), y''(t)$ for a few time steps. We will start with $t = 0$, then $t = \Delta t$, then $t = 2\Delta t$, then $t = 3\Delta t$ etc.

Suppose we want to compute from $t = 0$ until $t = 4$. Then obviously, if we make $\Delta t$ smaller, we will need more time steps. With $\Delta t = 0.1$ we need $0, 0.1, 0.2, ..., 4.0$ so thats 40 steps (41 steps if you also count $t = 0$). If we took $\Delta t = 0.2$ instead of 0.1 we would need only 20 steps to get from 0 to 4. So: smaller values for $\Delta t$ means: more work to get from $t = 0$ to $t = 4$, but also: more accuracy. If you make $\Delta t$ twice as small, you'll have twice as much work, but the error in the computation will be 4 times smaller. So smaller $\Delta t$ means: more work for you but also more accuracy. I should point out that it is possible to compute $y(t)$ exactly, but this will be too difficult for us because it is well outside of the scope of Calc I. Therefore, we will only be able to do an approximate calculation. Just like sometimes when you have to compute $\int_a^b f(x)\mathrm{d}x$ you can calculate exact answers, when you can find $F(x)$ for which $F'(x) = f(x)$. But sometimes we can not find such $F(x)$, then all we can do is compute an approximate answer with left and right sums.

When you do the computation, you will need $y''(t)$ to compute the next value for $y'(t)$, and you need $y'(t)$ to compute the next value for $y(t)$, and then the differential equation will tell you what $y''(t)$ is, which you can use for the next iteration. This will hopefully help you to better understand how $y''(t)$ and $y'(t)$ are related, and how $y'(t)$ and $y(t)$ are related, which is the primary goal of this project. The other goal is to understand that physical laws are often written as a differential equation, which means: an equation that involves a function $y(t)$ and also derivative(s) of $y(t)$. The physics that describes the motion of the shocks of a car is given by the differential equation above is just one of the many examples. The vast majority of the laws of physics are differential equations. Equations that describe electro-magnetism (Maxwell's laws) or mechanics, or waves of any kind (light, sound, water waves). Without understanding derivatives, it is impossible to understand most of these physical laws.

Let start computing. I will take $C_D = 1$ and $C_S = 2$, so the differential equation is: $y''(t) = -y'(t) - 2y(t)$, and the initial data is: $y(0) = 0, y'(0) = 1$. Lets fill in the initial data and we get the following table:

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | ? |
| 0.1 | ? | ? | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Now for every number $t$, as soon as we know: $y(t)$ and $y'(t)$, we can calculate: $y''(t)$ from the differential equation: $y'' = -y' - 2y''$ So we get:

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1 - 2 \cdot 0 = -1$ |
| 0.1 | ? | ? | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Now $v(t) = y'(t)$ is the velocity. And $v'(t) = y''(t) = a(t)$ is the derivative of the velocity, it is the acceleration. And:

$$f(t_0 + \Delta t) \approx f(t_0) + f'(t_0) \cdot \Delta t. \tag{1}$$

This equation is true for any function $f(t)$, and the smaller the value of $\Delta t$, the more accurate this approximation is. That is why, when we use smaller values for $\Delta t$, the result of our computation will be more accurate. We will use this equation for $f(t) = v(t) = y'(t)$ so we get:

$$y'(t_0 + \Delta t) \approx y'(t_0) + y''(t_0) \cdot \Delta t. \tag{2}$$

If we take $t_0 = 0$ then (note that we fixed $\Delta t$ to be 0.1) we get: $y'(0+0.1) \approx y'(0)+0.1 \cdot y''(0) = 1+0.1 \cdot (-1) = 0.90$.

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1$ |
| 0.1 | ? | $1 + (-1) \cdot \Delta t = 0.90$ | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Now plug in $y(t)$ for $f(t)$ in equation 1 and you get:

$$y(t_0 + \Delta t) \approx y(t_0) + y'(t_0) \cdot \Delta t. \tag{3}$$

Take $t_0 = 0$ so $y(0 + 0.1) \approx y(0) + 0.1 \cdot y'(0) = 0 + 0.1 \cdot 1 = 0.10$. So:

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1$ |
| 0.1 | $0 + 1 \cdot \Delta t = 0.10$ | 0.90 | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Now apply the differential equation: $y'' = -y' - 2y$, so we get: $-0.90 - 2 \cdot 0.10 = -1.10$.

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1$ |
| 0.1 | 0.10 | 0.90 | $-1.10$ |
| 0.2 | ? | ? | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Keep on going this way: $t_0 = 0.1$, use the formula 2 and we get $y'(0.1 + \Delta t) \approx y'(0.1) + y''(0.1) \cdot \Delta t = 0.90 + (-1.10) \cdot 0.1 = 0.79$. And use formula 3 to find: $y(0.1 + \Delta t) \approx y(0.1) + y'(0.1) \cdot \Delta t = 0.10 + 0.90 \cdot 0.1 = 0.19$. Then we have:

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1$ |
| 0.1 | 0.10 | 0.90 | $-1.10$ |
| 0.2 | 0.19 | 0.79 | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Then compute $y''(0.2)$ by the differential equation: $y''(0.2) = -y'(0.2) - 2y(0.2) = -0.79 - 2 \cdot 0.19 = -1.17$. Then: $y'(0.2 + \Delta t) \approx y'(0.2) + y''(0.2) \cdot \Delta t = 0.79 + 0.1 \cdot (-1.17) = 0.673$. Then $y(0.2 + \Delta t) \approx y(0.2) + y'(0.2) \cdot \Delta t = 0.19 + 0.1 \cdot 0.79 = 0.269$. And we have:

| $t$ | $y$ | $y'$ | $y'' = -y' - 2y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1$ |
| 0.1 | 0.10 | 0.90 | $-1.10$ |
| 0.2 | 0.19 | 0.79 | $-1.17$ |
| 0.3 | 0.269 | 0.673 | ? |
| 0.4 | ? | ? | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 4.0 | ? | ? | ? |

Now complete the above table. Note: you do not need to use more than 3 digits after the dot. Using more digits will not help much for the accuracy, because to get a better accuracy one should also make $\Delta t$ smaller. Then, make a plot of $y(t)$ for $t \in [0, 4]$. Make estimates for:

a) The value of $t$ for which $y(t)$ is maximal, and also that value of $y(t)$.
b) The value of $t$ for which $y(t)$ is minimal, and also that value of $y(t)$.
c) The value of $t$ where $y(t)$ switches from positive to negative.

Note: In class I mentioned that when you compute $y(t_0 + \Delta t)$ you can take: $y(t_0) + y'(t_0) \cdot \Delta t$ like we did above (I will now refer to this as *method one*), and if you do so, then in fact you are using the following estimate for the velocity $v(t) = y'(t)$ during the time $[t_0, t_0 + \Delta t]$, namely you estimate $v(t) = y'(t)$ to be approximately $v(t_0) = y'(t_0)$ during the time interval from $t_0$ to $t_0 + \Delta t$. If $\Delta t$ is small, then this is not a bad estimate because the true value of $v(t)$ will not change much in this time interval when $\Delta t$ is small. But, it is possible to make a more accurate estimate: Instead of estimating $v(t) = y'(t)$ during $[t_0, t_0 + \Delta t]$ by the already calculated number $y'(t_0)$, we can also estimate it by taking the average of the numbers $y'(t_0)$ and $y'(t_0 + \Delta t)$. I will refer to this as *method two*. We can do this because both numbers have already been calculated. In other words: when you compute $y(0.1)$ in method one, you take $y(0) + 0.1 \cdot y'(0)$ but in method two you take $y(0) + 0.1 \cdot S$ where $S$ is the average of $y'(0)$ and $y'(0.1)$. And if you compute $y(0.2)$, you take $y(0.1) + 0.1 \cdot y'(0.1)$ in method one, but you take $y(0.1) + 0.1 \cdot S$ in method two where $S$ is the average of the (already calculated at that point) numbers $y'(0.1)$ and $y'(0.2)$.

You should work in groups of 1, 2 or 3 people. If your group has 1 or 2 people, then you only need to do method one; the table I have given above for $t = 0, \ldots, 0.3$ you have to complete it up to $t = 0, \ldots, 4.0$. If your group has 3 people, then compute the table for method one as well as for method two, where the numbers will be slightly different. How big (in %) is the difference for questions a,b,c in the two methods?

The *joint work* in this project is the following:
(*) Compute the case $\Delta t = 0.1$, $C_D = 1$, $C_S = 2$ with method one for $t \in [0.0, 4.0]$. This is already given above for $t \in [0.0, 0.3]$. Then answer questions a,b,c.
(*) If your group consists of more than 2 people, then do $\Delta t = 0.1$, $C_D = 1$, $C_S = 2$ also with method two.

There is also an individual part, as follows: Take the number of the first letter of your first name (A is 1, B=2,..., Z=26), divide it by 10, and add 1. That's your value for $C_D$. So if your first letter of your first name is E then you have $C_D = 1 + 5/10 = 1.5$. Take the first letter of your last name, divide it by 10, and add 2, and that's your value for $C_S$. So if your last name starts with Z, then you have $C_S = 2 + 26/10 = 4.6$. Now take again $\Delta t = 0.1$, use method one as illustrated with all these tables above, and compute $y(t)$ for $t$ from 0 to 3. So if your first name starts with E and last name with Z, then you have:

| $t$ | $y$ | $y'$ | $y'' = -1.5y' - 4.6y$ |
|---|---|---|---|
| 0 | 0 | 1 | $-1.5 \cdot 1 - 4.6 \cdot 0$ |
| 0.1 | ? | ? | ? |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| 3.0 | ? | ? | ? |

Does $y(t)$ switch from positive down to negative in the time interval $[0, 3]$? If so, when? If not: then what is the highest value of $y(t)$ for $t$ between 0 and 3? Note: the answer depends on $C_S$ and $C_D$. If you have large $C_S$ and small $C_D$, you will see that $y(t)$ goes back through 0 at some point (so $y(t)$ goes from positive to negative at some point). But for small $C_S$ and large $C_D$, you will see that $y(t)$ will never go back through 0, so with those settings the shocks behave quite different.

In summary, there should be: one individual table for each person in the group, and furthermore one joint graph plus one joint table (or two joint tables, if your group has more than 2 people). The joint table(s) is slighly longer because it goes from 0 to 4, whereas the individual table only goes from 0 to 3 with the same step size $\Delta t = 0.1$. You have two weeks for this project. Hand in your work on or before Tuesday December 5. That way I will have time to grade it and hand it back to you in the last week of classes.