Name: SSN:

1	2	3	4	5	6	7	8	9	10

11	12	13	14	15

Do not write here!

MAC 2312–8
Final Exam
April 19, 2002

The use of calculators is not allowed!

There are fifteen problems, and each problem is worth 4 points. Everything you write will be taken at face value, so when you give your answers, always motivate them and write full work. (This will help your grade in case of wrong or incomplete answers.) **No credit will be given for an answer, even if correct, if you give no justification for it.** Write your answers down in a clean, neat form: no scratchwork, please. (Do your scratchwork on separate sheets of paper.)

Please, fill in your name and SSN when required. If you want your grades posted, or otherwise handled in any "public" way (such as used for graphs, etc.) please choose an alphanumeric 6 characters code that will be used to identify you anonymously (according to FSU guidelines): make it yourself by picking a string of 6 characters (i.e. letters and/or digits) and write it below. Do *not* use parts of your SSN! Your grades will not be posted if you do not choose the code.

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1. Find

$$\int \sin^3(x)\,dx$$

(HINT: consider writing $\sin^3 x = \sin x \sin^2 x$ and then use the identity $\cos^2 x + \sin^2 x = 1$.)

SOLUTION: We have:

$$\begin{aligned}\int \sin^3(x)\,dx &= \int \sin(x)(1-\cos^2(x))\,dx\\ &= \int \sin x\,dx - \int \sin x \cos^2 x\,dx\\ &= -\cos x + \int u^2\,du\,.\end{aligned}$$

where for the last one we used the substitution $u = \cos x$, so that $du = -\sin x\,dx$. So we have

$$\int u^2\,du = \frac{1}{3}u^3 + C$$

and substituting back the original expression for u we obtain

$$\int \sin^3(x)\,dx = -\cos x + \frac{1}{3}\cos^3 x + C\,.$$

2. Find

$$\int \frac{1}{(y-1)(y-2)}\,dy\,.$$

SOLUTION: By partial fractions:

$$\frac{A}{y-1} + \frac{B}{y-2} = \frac{1}{(y-1)(y-2)}$$

if

$$A(y-2) + B(y-1) = (A+B)y - 2A - B = 1\,,$$

which gives $A = -1$ and $B = 1$. Thus

$$\frac{1}{(y-1)(y-2)} = \frac{1}{y-2} - \frac{1}{y-1}\,,$$

and the integral will be:

$$\begin{aligned}\int \frac{1}{(y-1)(y-2)}\,dy &= \int \frac{dy}{y-2} - \int \frac{dy}{y-1}\\ &= \ln|y-2| - \ln|y-1| + C = \ln\left|\frac{y-2}{y-1}\right| + C\,.\end{aligned}$$

3. TRUE OR FALSE: If the function f is increasing over the interval $[a, b]$, then

$$\text{LEFT}(n) \geq \int_a^b f(x)\,dx \geq \text{RIGHT}(n)\,.$$

Explain why or why not.

4. Determine if the following integral is convergent, and if yes, find its value. If not, explain why not.

$$\int_e^\infty \frac{1}{x\ln x}\,dx\,.$$

SOLUTION: First, by definition

$$\int_e^\infty \frac{1}{x\ln x}\,dx = \lim_{a\to\infty}\int_e^a \frac{1}{x\ln x}\,dx\,,$$

where at least $a > e$. By substitution with $y = \ln x$, we have

$$\int_e^a \frac{1}{x\ln x}\,dx = \int_1^{\ln a}\frac{dy}{y} = \ln|y|\Big|_1^{\ln a} = \ln|\ln a|\ .$$

Now, $\ln a$ grows without bounds as $a \to \infty$, so

$$\lim_{a\to\infty}\ln|\ln a|$$

does not exist, so the improper integral is divergent.

5. Suppose the function $f(x)$ is approximated near $x = 0$ by the fifth degree polynomial

$$P_5(x) = x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5\,.$$

What are the values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, $f^{(4)}(0)$, and $f^{(5)}(0)$?

SOLUTION: The coefficients are in general $C_n = f^{(n)}(0)/n!$. So we use

$$f^{(n)}(0) = n!\,C_n$$

and find

$$f(0) = 0\,,\ f'(0) = 1\,,\ f''(0) = 2!(1/4) = 1/2\,,\ f'''(0) = 3!(1/9) = 2/3$$
$$f^{(4)}(0) = 4!(1/16) = 3/2\,,\ f^{(5)}(0) = 5!(1/25) = 24/5\,.$$

6. Determine the radius of convergence of the series

$$1 + x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \cdots + \frac{1}{n^2}x^n + \cdots$$

SOLUTION: We have $C_n = 1/n^2$, so

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1\,.$$

7. Find the Taylor series of the function $f(y) = 1/(1+y)$. Also, knowing that the radius of convergence of this series is 1, find the radius of convergence of the Taylor series of $g(x) = 1/(1+2x^2)$. (HINT: Use substitution.)

SOLUTION: From the geometric series

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

we obtain, with $z = -y$:

$$\frac{1}{1+y} = 1 - y + y^2 - y^3 + \cdots + (-y)^n + \cdots$$

We now use the substitution $y = 2x^2$ and find

$$\frac{1}{1+2x^2} = 1 - 2x^2 + 4x^4 - 8x^6 + \cdots + (-2)^n x^{2n} + \cdots$$

Now we must have $\left|2x^2\right| = |y| < 1$ which implies $|x| < 1/\sqrt{2}$, so the radius of convergence is $1/\sqrt{2}$.

8. Find the Taylor series of $f(x) = e^{2x}$.

Solution: The Taylor series for e^y is

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!},$$

so with the substitution $y = 2x$ we get

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

9. Determine if the following statement could be true or not: f is a periodic *odd* function over the interval $[-\pi, \pi]$ and its average (i.e. the a_0 coefficient of the Fourier series) is different from zero.

10. Viruses are infecting a certain computer network. The amount $C(t)$ of viruses as a function of time t (measured in seconds) is increasing at a rate exactly equal to the amount present. Determine the number of seconds T after which the amount doubles. (Hint: assume C_0 is the initial amount present at $t = 0$.)

Solution: The differential equation must be

$$\frac{dC}{dt} = C,$$

with solution $P(t) = P_0 e^t$. After T seconds we have $P(T) = P_0 e^T = 2P_0$, so $e^T = 2$, that is the doubling time is $T = \ln 2$ seconds.

11. Find the solution of

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

Solution: By separation of variables we have

$$\int \frac{dy}{y^2} = -\frac{1}{y} = \int dt = t + C$$

where C is an arbitrary constant. Solving for y gives

$$y(t) = -\frac{1}{t+C},$$

and $y(0) = -1/C = 1$, so $C = -1$ and the particular solution is

$$y(t) = -\frac{1}{t-1} = \frac{1}{1-t}.$$

12. Determine the slope field, the equilibrium solutions and their stability for the differential equation

$$\frac{dy}{dt} = y(2-y)\,.$$

Without solving the differential equation, determine the points where the slope attains its maximum.

SOLUTION: The equilibrium solutions are those corresponding to the values of y such that $y(2-y) = 0$, that is $y = 0$ and $y = 2$.

Since $y(2-y)$ is positive for $0 < y < 2$ and negative for either $y < 0$ or $y > 2$, we get that $y = 0$ is unstable and $y = 2$ is stable.

The maximum of $f(y) = y(2-y)$ occurs at the value of y such that $f'(y) = 2 - 2y = 0$, that is, $y = 1$. This is the maximum *slope.*

13. Verify that $y(t) = te^{-2t}$ is a solution of

$$y'' + 4y' + 4y = 0\,.$$

Is there any other solution?

SOLUTION: The verification is done by plugging the expression for $y(t)$ in the differential equation and verifying that the result is identically zero.

The other solution is obtained by setting $y(t) = e^{-2t}$.

14. Solve

$$\frac{d^2 s}{dt^2} = -s\,; \quad s(0) = 0\,,\ s'(0) = 2\,.$$

SOLUTION: The general solution is a combination

$$s(t) = C_1 \cos t + C_2 \sin t\,.$$

Then $s(0) = C_1$, so $C_1 = 0$. Computing the first derivative:

$$s'(t) = C_2 \cos t\,,$$

and $s'(0) = C_2 = 2$, so the particular solution is

$$s(t) = 2 \sin t\,.$$

15. Explain what is the characteristic equation for the differential equation

$$\frac{d^2 y}{dt^2} + b\frac{dy}{dt} + cy = 0\,.$$

(b and c are constants.)