The point of this handout is to show that, if one accepts the statements in the "list of facts on cardinal numbers" handout, then one can compute the cardinality of almost any set. In turn, that will provide the motivation to prove some of the items in that handout (proving all of them will take too much time).

1. Exercise 20 on page 102.

   (a) $\mathrm{card}(\mathbb{N} \times \mathbb{Q}) =$ [see item 21] $= \mathrm{card}(\mathbb{N}) \cdot \mathrm{card}(\mathbb{Q}) =$ [see item 22] $= \max(\mathrm{card}(\mathbb{N}), \mathrm{card}(\mathbb{Q})) =$ [items 3 and 18] $= \max(\aleph_0, \aleph_0) = \aleph_0$

   (b) similar to (a)

   (c) $\mathrm{card}(\mathbb{R} \times \mathbb{Q}) =$ [see item 21] $= \mathrm{card}(\mathbb{R}) \cdot \mathrm{card}(\mathbb{Q}) =$ [see item 22] $= \max(\mathrm{card}(\mathbb{R}), \mathrm{card}(\mathbb{Q})) =$ [items 18 and 20] $= \max(c, \aleph_0) = c$

   (d) $\mathrm{card}(P(\mathbb{Q})) =$ [see item 23] $= 2^{\mathrm{card}(\mathbb{Q})} =$ [see item 18] $= 2^{\aleph_0} =$ [see item 24] $= c$

   (e) $\mathbb{C} = \{a + bi \,|\, a, b \in \mathbb{R}\}$ has a bijection to $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ which has cardinality $c \cdot c$ [see item 21] which equals c [item 22].

   (f) Let $S = \mathbb{R} \setminus \mathbb{N}$ and $s = \mathrm{card}(S)$. Now S and $\mathbb{N}$ are disjoint and $S \bigcup \mathbb{N} = \mathbb{R}$ and so by item 21 we have $s + \mathrm{card}(\mathbb{N}) = \mathrm{card}(\mathbb{R})$. So $s + \aleph_0 = c$. Then $\max(s, \aleph_0) = c$ [item 22]. But $\aleph_0 < c$ so $s = c$.

   (g) Any infinite set S has cardinality $\geq \aleph_0$ [this follows from items 17 and 13 but can also be shown more directly]. Let $S = \mathbb{Q} \setminus \mathbb{N}$ and $s = \mathrm{card}(S)$. The set S is infinite, so $s \geq \aleph_0$. But $S \subseteq \mathbb{Q}$ and $\mathrm{card}(\mathbb{Q}) = \aleph_0$ and so $s \leq \aleph_0$ by item 13. Then $s = \aleph_0$ by item 14.

   (h) The set S in this question is a subset of $\mathbb{R}$ so $\mathrm{card}(S) \leq c$ by item 13. We will now give an injective map from $P(\mathbb{N}^*)$ to S, proving that $\mathrm{card}(P(\mathbb{N}^*)) \leq \mathrm{card}(S)$, i.e. $c \leq \mathrm{card}(S)$ [see item 24] and then $c = \mathrm{card}(S)$ by item 14.
   Define $f : P(\mathbb{N}^*) \to S$ as follows. If $A \in P(\mathbb{N}^*)$ then $A \subseteq \mathbb{N}^*$. Now let $f(A) = 0.\,d_1\,d_2\,d_3\ldots$ where $d_i = 3$ if $i \in A$ and $d_i = 7$ if $i \notin A$. For example, if $A = \{1, 3, 4\}$ then $f(A) = 0.3733777777777\ldots$
   This function f is injective, so $\mathrm{card}(P(\mathbb{N}^*)) \leq \mathrm{card}(S)$.

   (i) Terminating decimal expansions only produce rational numbers. So this set S is a subset of $\mathbb{Q}$ and hence its cardinality is $\leq \aleph_0$. But S is also infinite so its cardinality is $\geq \aleph_0$. So the cardinality is $\aleph_0$.

   (j) $\aleph_0$

   (k) c

   (l) $\aleph_0$

   (m) c

Next page: How to recognize a countable set?

Quick way to recognize countable sets

It may be surprising that there is such a wide variety of countably-infinite sets (see answers to Ex 20 the previous page), and yet, somehow there are also uncountable sets. Is there a quick way to tell if a set is countable?

Rule: Suppose that S is a set. Suppose that any element $s \in S$ can be represented with a finite amount of text. Then S is countable!

Examples:

- The set in Ex 20(a) is $S := \mathbb{N} \times \mathbb{Q}$. Any element $s \in S$ is of the form: $s = (a, b)$ where $a \in \mathbb{N}$ and $b \in \mathbb{Q}$. No matter how you choose $s \in S$ (for example: example $s = (160058, -180223/5001)$) you can always write any $s \in S$ with a finite amount of text.

- Some elements of $\mathbb{R}$ can be written with a finite amount of text, but most elements can not. If you take a real number like $s = 0.3587659765\ldots$ then you always print only finitely many digits and use $\ldots$ to indicate that there are more digits; you simply can not write down all digits with a finite amount of text.

  The number $\pi = 3.141592\ldots$ has been computed to more than 30 trillion decimals. However, 30 trillion is next to nothing compared to the infinitely many digits that have not been computed.

- Now notice that in Ex 20, the sets that were countable were precisely the ones where the rule applied.

- But how do we prove the rule? Make a sequence $a_1, a_2, a_3, a_4, \ldots$ where each a_i is a text, in the following way: First write down all texts with one character $a_1 =$"a", $a_2 =$"b", etc. $a_{26} =$"z", $a_{27} =$"(", $a_{28} =$"," $a_{29} =$"/" etc, then all texts with 2 characters, then all texts with 3 characters, etc. (Note: For any length $l \in \mathbb{N}^*$, there are only finitely many texts of length l). Then $A := \{a_1, a_2, a_3, \ldots\}$ is the set of all finite non-empty texts, and the function $\mathbb{N}^* \to A$ that sends n to a_n is a bijection. So A is countable. Then any subset of A is also countable.

- What about $S := P(\mathbb{N}^*)$? Well, lets take an element $s \in S$. Then $s \subseteq \mathbb{N}^*$. How would I write down s? Well, I have to indicate if $1 \in s$ or not (that takes 1 bit of data). Same if 2 is in s or not (also 1 bit of data). So representing s takes 1 bit of data for every $n \in \mathbb{N}^*$. In other words: an infinite amount of data is needed to write down s. No wonder $P(\mathbb{N}^*)$ has a larger cardinality than $\mathbb{N}^*$.

- The rule only works in one direction! If you are unable to write some $s \in S$ with a finite text, it does not prove that S is not countable. For example, take $S = \{3.141592\ldots\}$. This set has only 1 element so it is a countable set. But that one element requires an infinite amount of text.