Final, Intro Advanced Math, Dec 10 2019. NAME:

1. Give the definition of:
   (for (a)–(d) give the definition, not a statement about cardinalities)

   (a) A function $f : A \to B$ is called *injective* when:
   (b) A function $f : A \to B$ is called *surjective* when:
   (c) The *power set* $P(A)$ of a set A is:
   (d) The product $A \times B$ of two sets is the set of all:
   (e) The *contrapositive* of a statement $p \implies q$ is:

   Answer:

   (a) $f(a_1) = f(a_2) \implies a_1 = a_2 \quad$ (for all $a_1, a_2 \in A$)
   (b) $\forall_{b \in B}\, \exists_{a \in A}\; f(a) = b$
   (c) the set of all subsets of A.
   (d) the set of all pairs (a, b) for all $a \in A$ and all $b \in B$.
   (e) $\neg q \implies \neg p$

2. Let $f : A \to B$ and consider the following statement:

$$S : \quad \exists_{b \in B}\, \forall_{a \in A}\; f(a) \neq b$$

   Compute $\neg S$ (the negation of S). What does $\neg S$ say about f?

   Answer: $\forall_{b \in B}\, \exists_{a \in A}\; f(a) = b$. This says that f is onto (i.e. surjective).

3. Let $x \in \mathbb{R}$. Write down the *contrapositive* of the following statement:

$$S : \quad (\forall_{\epsilon > 0}\; |x| < \epsilon) \implies x = 0.$$

   Is S true? (Prove or disprove).

   $x \neq 0 \implies \exists_{\epsilon > 0}\; |x| \geq \epsilon.$
   Proof: Assume $x \neq 0$. To prove: $\exists_{\epsilon > 0}\; |x| \geq \epsilon$. Proof: Take $\epsilon = |x|$.

4. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

   For the last two, justify your answer by showing your steps.

   (a) $\mathbb{Q} - \mathbb{Z}$: $\aleph_0$ ($\mathbb{Q} - \mathbb{Z}$ is an infinite subset of a countably infinite set $\mathbb{Q}$)
   (b) $P(\mathbb{N})$: $2^{\aleph_0} = c$
   (c) $\{2, 2\}$: 1
   (d) $\mathbb{R}^{\mathbb{N}}$: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c$
   (e) $\mathbb{R}^{\mathbb{R}}$: $c^c = (2^{\aleph_0})^c = 2^{\aleph_0 c} = 2^c$

5. For each of the following subsets of $\mathbb{R}$, mention if it is open, closed, both, or neither. For each set A that is not closed, write down its closure $\overline{A}$:

   $\emptyset$: both
   $[0, \infty)$: closed
   $\mathbb{R} - \{0\}$: open, closure $= \mathbb{R}$
   $(0,1) \bigcap \mathbb{Q}$: neither. Closure $= [0,1]$
   $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{1/n \mid n \in \mathbb{N}^*\}$: neither. Closure $= \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\} \bigcup \{0\}$.

6. Suppose that A, B are infinite sets and that $f : P(A) \to A \times B$ is injective. Must there exist an injective function from $P(A)$ to B? (Prove or disprove).

   Yes. $\text{card}(P(A)) \leq \text{card}(A \times B) = \text{card}(A)\text{card}(B) = \max(\text{card}(A), \text{card}(B))$ using items 2, 21, 22. If this max is $\text{card}(A)$ then $\text{card}(P(A)) \leq \text{card}(A)$ contradicting item 7. So this max is $\text{card}(B)$ and so $\text{card}(P(A)) \leq \text{card}(B)$.

7. Let $\text{Int}(A)$ denote the set of interior points of A.

   (a) If $A \subseteq B$ then prove $\text{Int}(A) \subseteq \text{Int}(B)$.

   Let $x \in \text{Int}(A)$. Then $(x - \epsilon, x + \epsilon) \subseteq A$ for some $\epsilon > 0$ by item 16. Then $(x - \epsilon, x + \epsilon) \subseteq B$ because $A \subseteq B$. Then $x \in \text{Int}(B)$.

   (b) If $\text{Int}(A) \subseteq B$ then prove that $\text{Int}(A) \subseteq \text{Int}(B)$.

   Assume $\text{Int}(A) \subseteq B$. To prove: $\text{Int}(A) \subseteq \text{Int}(B)$.
   One proof is to use part (a) with A replaced by $\text{Int}(A)$ and to note that $\text{Int}(\text{Int}(A)) = \text{Int}(A)$. Another proof: Item 16(c) says that $\text{Int}(B)$ is the union of *all open subsets* of B. One of those is $\text{Int}(A) \subseteq B$, see item 16(b).

8. (a) Suppose that (1) for every a in A and every $\epsilon > 0$ there exists b in B with $|a - b| < \epsilon$. Then show that (2): $A \subseteq \overline{B}$.

   Let $a \in A$. To prove: $a \in \overline{B}$. By item 11(d) that means showing $\forall_{\epsilon>0} \exists_{b \in B}$ with b ϵ-close to a. But that is precisely what (1) says.

   (b) Suppose for every $a \in A$ there is a sequence in B that converges to a. Show that $\overline{A} \subseteq \overline{B}$. (hint: first show $A \subseteq \overline{B}$).

   To prove the hint, let $a \in A$, to prove $a \in \overline{B}$. We are given that there is a sequence in B that converges to a, but then $a \in \overline{B}$ by item 11(f). Hence $A \subseteq \overline{B}$. Item 11(c) says that $\overline{A}$ is the intersection of all closed sets that contain A, but we saw that one of those is $\overline{B}$, so $\overline{A} \subseteq \overline{B}$.

   (c) Suppose $0 \notin \overline{A}$. Show that there exists $\epsilon > 0$ with $(-\epsilon, \epsilon) \bigcap A = \emptyset$.

   One proof is to compute the negation of item 11(e). Another proof: Let U be the complement of $\overline{A}$. Then U is open (see item 10(a)) and $0 \in U$ so by item 3 there exists $\epsilon > 0$ with $(0 - \epsilon, 0 + \epsilon) \subseteq U$.
   Then $(-\epsilon, \epsilon) \bigcap U^c = \emptyset$. Now note that $A \subseteq \overline{A} = U^c$.