Final, Intro Advanced Math, Dec 10 2019. NAME:

1. Give the definition of:
   (for (a)–(d) give the definition, not a statement about cardinalities)
   (a) A function $f : A \to B$ is called *injective* when:
   (b) A function $f : A \to B$ is called *surjective* when:
   (c) The *power set* $P(A)$ of a set A is:
   (d) The product $A \times B$ of two sets is the set of all:
   (e) The *contrapositive* of a statement $p \implies q$ is:

   Answer:

2. Let $f : A \to B$ and consider the following statement:

$$S : \quad \exists_{b \in B} \forall_{a \in A} \ f(a) \neq b$$

   Compute $\neg S$ (the negation of S). What does $\neg S$ say about f?

3. Let $x \in \mathbb{R}$. Write down the *contrapositive* of the following statement:

$$S : \quad (\forall_{\epsilon > 0} \ |x| < \epsilon) \implies x = 0.$$

   Is S true? (Prove or disprove).

4. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

   For the last two, justify your answer by showing your steps.

   (a) $\mathbb{Q} - \mathbb{Z}$
   (b) $P(\mathbb{N})$
   (c) $\{2, 2\}$
   (d) $\mathbb{R}^{\mathbb{N}}$
   (e) $\mathbb{R}^{\mathbb{R}}$

5. For each of the following subsets of $\mathbb{R}$, mention if it is open, closed, both, or neither. For each set A that is not closed, write down its closure $\overline{A}$:

   $\emptyset$

   $[0, \infty)$

   $\mathbb{R} - \{0\}$

   $(0, 1) \bigcap \mathbb{Q}$

   $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = \{1/n \mid n \in \mathbb{N}^*\}$

6. Suppose that A, B are infinite sets and that $f : P(A) \to A \times B$ is injective. Must there exist an injective function from $P(A)$ to B? (Prove or disprove).

7. Let $\mathrm{Int}(A)$ denote the set of interior points of A.

   (a) If $A \subseteq B$ then prove $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

   (b) If $\mathrm{Int}(A) \subseteq B$ then prove that $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

8. Note: If the test turns out to be too long for many students then 8(b),(c) will become bonus questions (in that case do Ex 1–7 and 8(a) first).

   (a) Suppose that for every a in A and every $\epsilon > 0$ there exists b in B with $|a - b| < \epsilon$. Then show that $A \subseteq \overline{B}$.

   (b) Suppose for every $a \in A$ there is a sequence in B that converges to a. Show that $\overline{A} \subseteq \overline{B}$. (hint: first show $A \subseteq \overline{B}$).

   (c) Suppose $0 \notin \overline{A}$. Show that there exists $\epsilon > 0$ with $(-\epsilon, \epsilon) \bigcap A = \emptyset$.

List of facts on cardinal numbers, shortened version.

1. $\text{card}(A) = \text{card}(B)$ means $\exists f : A \to B$ with f bijection.

2. $\text{card}(A) \leq \text{card}(B)$ means $\exists f : A \to B$ with f one-to-one.

3. $\aleph_0$ is short notation for $\text{card}(\mathbb{N}^*)$.

4. c is short notation for $\text{card}(\mathbb{R})$.

5. The set A is *countably infinite* when: $\text{card}(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.

6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.

7. $\text{card}(A) < \text{card}(P(A))$.

8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $\text{card}(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!

9. If $f : A \to B$ is onto then $\text{card}(B) \leq \text{card}(A)$.

10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.

11. A is countable when $\text{card}(A) \leq \aleph_0$.

12. A subset of a countable set is again countable.

13. If $A \subseteq B$ then $\text{card}(A) \leq \text{card}(B)$.

14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
    The proof is not easy! (Schroeder-Bernstein theorem on p 88–89).

15. The ordering $\leq$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \leq e$ or $d \geq e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.

16. Set A is uncountable when $\text{card}(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $\text{card}(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*))$.

21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$.
d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $\text{card}(P(A)) = \text{card}(\{0,1\}^A) = \text{card}(\{0,1\})^{\text{card}(A)} = 2^{\text{card}(A)}$.

24. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*)) = \text{card}(\{0,1\}^{\mathbb{N}^*}) = 2^{\text{card}(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $\text{card}(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by max(d,e), see item 22.

27. So far we have encountered these increasing cardinals:

$$0,\ 1,\ 2,\ 3,\ \ldots\ \aleph_0,\ \ c = 2^{\aleph_0},\ \ 2^c,\ \ 2^{2^c},\ \ldots$$

    and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

    From the axioms of set theory (= the only statements mathematicians accept without a proof) it is impossible to prove or disprove this.

List of definitions and facts. Writing clear proofs means you should cite each item number(letter) that you use, precisely at the step where you use it.

1. We say that p and x are **ϵ-close** when $|p - x| < \epsilon$.
2. The set of all points that are ϵ-close to x is $(x - \epsilon, x + \epsilon)$.
3. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** when $\forall_{x \in \mathcal{O}} \exists_{\epsilon > 0} \, (x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
4. $\mathbb{R}$ and $\emptyset$ and any interval of the form (a, b) is open.
5. **Any** union of open sets is always open (even infinite unions!).
6. The intersection of **finitely many** open sets is again open.
7. Let $a_1, a_2, \ldots$ be a sequence. A **tail** is a subsequence of the form $a_{K+1}, a_{K+2}, \ldots$ So a tail is: all terms beyond some cutoff point K.
8. $a_1, a_2, \ldots$ **converges** to α when $\forall_{\epsilon>0} \, \exists_K \, \forall_{j>K} \, |a_i - \alpha| < \epsilon$. In other words, or every $\epsilon > 0$ the sequence has a tail contained in $(\alpha - \epsilon, \alpha + \epsilon)$.
   In this case we call α the **limit** of the sequence $a_1, a_2, \ldots$.
9. α is called a **limit point** of V when (i) there is a sequence in $V - \{\alpha\}$ that converges to α. This is equivalent to (ii) $\forall_{\epsilon>0} \, (\alpha - \epsilon, \alpha + \epsilon) \bigcap (V - \{\alpha\}) \neq \emptyset$.
10. A set $V \subseteq \mathbb{R}$ is **closed** when
    (a) The complement of V is open.
    (b) If a sequence $a_1, a_2, \ldots$ in V converges to α then $\alpha \in V$.
    (c) V contains all of its limit points.
    (d) If $(\alpha - \epsilon, \alpha + \epsilon) \bigcap V$ is not empty for every $\epsilon > 0$ then $\alpha \in V$.
11. Notation: $\overline{S}$ is called the **closure** of the set S
    (a) $\overline{S}$ is the union of S and all of its limit points.
    (b) $\overline{S}$ is the smallest closed set that contains S.
    (c) $\overline{S}$ is the intersection of all closed sets that contain S.
    (d) $x \in \overline{S} \iff \forall_{\epsilon>0}$ there is a point in S that is ϵ-close to x.
    (e) $x \in \overline{S} \iff \forall_{\epsilon>0} \, (x - \epsilon, x + \epsilon)$ intersects S.
    (f) $x \in \overline{S} \iff \exists$ a sequence $a_1, a_2, \ldots \in S$ that converges to x.
12. α is a limit point of S if α is in the closure of $S - \{\alpha\}$.
13. The union of **finitely many** closed sets is again closed.
14. **Any** intersection of closed sets (even infinitely many) is closed.
15. If $S \subseteq \mathbb{R}$ has finitely many elements then S is closed.

16. An **interior point** of S is a point s for which $\exists_{\epsilon>0}\,(s-\epsilon, s+\epsilon) \subseteq S$. Denote $\mathrm{Int}(S)$ as the set of interior points of S.

    (a) S is open $\iff$ every element of S is an interior point of S.

    (b) $\mathrm{Int}(S)$ is open.

    (c) $\mathrm{Int}(S)$ is the union of all open subsets of S.

    (d) If S is the complement of U then $\overline{S}$ is the complement of $\mathrm{Int}(U)$.

17. A point $s \in \mathbb{R}$ is a **boundary point** of S if

$$\forall_{\epsilon>0} \left((s-\epsilon, s+\epsilon) \bigcap S \neq \emptyset \ \text{ and } \ (s-\epsilon, s+\epsilon) \bigcap {}^cS \neq \emptyset \right)$$

    The boundary of S is the set of all boundary points of S. It also equals the intersection of $\overline{S}$ and $\overline{{}^cS}$.

18. (Ex 13 in the book). The set of limit points of any set S is closed.

19. A subset $S \subseteq \mathbb{R}$ is called **dense** if (definition is also in Ex 12 in the book) for every $x \in \mathbb{R}$ and every $\epsilon > 0$ the interval $(x-\epsilon, x+\epsilon)$ contains an element of S.

    (a) S dense $\iff \overline{S} = \mathbb{R}$

    (b) S dense $\iff$ For every $\alpha \in \mathbb{R}$ there exists a sequence $a_1, a_2, \ldots \in S$ that converges to α.

    (c) S dense $\iff$ Every non-empty open set has at least one element of S.

    (d) S dense $\iff$ Every non-empty open set contains infinitely many elements of S.

    (e) $\mathbb{Q}$ is dense

    (f) If A is countable then $\mathbb{R} - A$ is dense.

20. S is a **discrete** set if $\forall_{s\in S}\exists_{\epsilon>0}\ (s-\epsilon, s+\epsilon) \bigcap S = \{s\}$.

21. Every finite set is discrete.

22. Every discrete set is countable.

Writing Proofs.

1. **Direct proof for $p \Longrightarrow q$.**
   Assume: p. To prove: q.

2. **Proving $p \Longrightarrow q$ by contrapositive**.
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving S by contradiction**.
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving $p \Longrightarrow q$ by contradiction**.
   Assume: p and $\neg q$. To prove: a contradiction.

5. **Direct proof for a $\forall_{x \in A} P(x)$ statement**.
   To ensure you prove $P(x)$ for *all* (rather than for *some*) x in A, do this:
   Start your proof with: Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for $\exists_{x \in A} P(x)$ statement**.
   Take $x :=$ [write down an expression that is in A, and satisfies $P(x)$].

7. **Proving $\forall_{x \in A} P(x)$ by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving $\exists_{x \in A} P(x)$ by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving S by cases**.
   Suppose for example a statement p can help to prove S. Write two proofs:
   Case 1: Assume p. To prove: S.
   Case 2: Assume $\neg p$. To prove S.

10. **Proving $p \wedge q$**
    Write two separate proofs: To prove: p. To prove: q.

11. **Proving $p \Longleftrightarrow q$**
    Write two proofs. To prove: $p \Longrightarrow q$ To prove: $q \Longrightarrow p$.

12. **Proving $p \vee q$**
    Method (1): Assume $\neg p$. To prove: q.
    Method (2): Assume $\neg q$. To prove: p.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using $p \vee q$ to prove another statement r**.
    Write two proofs:
    Assume p. To prove r.
    Assume q. To prove r.

14. **How to use a for-all statement $\forall_{x \in A} P(x)$**.
    You need to produce an element of A, then use P for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* x. All you know is $x \in A$ and $P(x)$.