Injective and surjective

Let S, T be sets, let $f : S \to T$ be a function.

f is injective means:

$$\forall_{s_1, s_2 \in S} \;\; f(s_1) = f(s_2) \quad \Longrightarrow \quad s_1 = s_2 \tag{1}$$

Note: the book uses s, s' instead of s_1, s_2 but that means the same thing (here s' is just another symbol, it does not mean the derivative of f). Also, the book on page 81 still uses the notation $(s, t) \in f$ to highlight the fact that functions can be defined in terms of sets, but once we know this, we can replace $(s, t) \in f$ by the more common notation $f(s) = t$.

On page 82 the book uses the common notation $f(s) = t$. Read the definition of surjective (a.k.a. onto) on page 82. Replacing the phrases "for each" and "there is an" by $\forall$ and $\exists$, the definition of f is surjective is this:

$$\forall_{t \in T} \; \exists_{s \in S} \;\; f(s) = t \tag{2}$$

Next, we take sets S, T, U and some functions $f : S \to T$ and $g : T \to U$. Now let $h = g \circ f$ be the composition, so

$$h(s) = g(f(s)) \tag{3}$$

Make sure to interpret these expressions carefully. When $f : S \to T$ then any time you see something like $f(\text{something})$ then that something must be an element of S because otherwise $f(\text{something})$ is an error. If $f : S \to T$ then it means that the input of f should be an element of S, and the output must be an element of T. Likewise, if $g : T \to U$ and if x is any expression, if you see $g(x)$ then x must be an element of T (if it isn't, then there is an error in the notation) and likewise $g(x)$ must be in U.

So in the formula (3) you see element of S (namely s), an element of T (namely $f(s)$) and an element of U (namely $g(f(s))$).

Turn in exercises: Let $f : S \to T$, $g : T \to U$, and $h = g \circ f$. Prove:

1. h onto $\Longrightarrow$ g onto
2. h injective $\Longrightarrow$ f injective

Hints on the next page.

Partial proof for 1: Assume h onto, so (a): $\forall_{u\in U}\exists_{s\in S}\ h(s) = u$.
To prove: g onto, i.e. (b): $\forall_{u\in U}\exists_{t\in T}\ g(t) = u$.

Hints to finish the proof:
[WP#5 says that to prove (b) we need to start like this:
Let $u \in U$. T.P. (c): $\exists_{t\in T}\ g(t) = u$.
WP#6 says that to prove (c) we have to write: Take $t := \ldots$ But what do we put on those dots? When stuck, look at given/assumed statements that have not yet been used. So have to use (a). It tells us that there exists some s with some property. But that s is an element of S whereas the $t := \ldots$ that we need to spell out should be an element of T. So the thing you need to figure out is: how do you get an element of T when the only expressions we encountered are: $s \in S$, $u \in U$, $f : S \to T$, $g : T \to U$, and $h : S \to U$. Once you figure that out how to get some element of T from that, you have something you can write in the line: Take $t := \ldots$ Check if that leads to a proof]

Partial proof for 2: Assume h injective so (a): $h(s_1) = h(s_2) \Longrightarrow s_1 = s_2$.
To prove: f injective, i.e. (b): $f(s_1) = f(s_2) \Longrightarrow s_1 = s_2$.
Assume (c): $f(s_1) = f(s_2)$. To prove (d): $s_1 = s_2$.

When stuck: we have to use assumption (a) in some way. We have encountered expressions $f, g, h, s_1, s_2, f(s_1), f(s_2)$ and (c) and $h = g \circ f$. Find some way to use that to produce something where (a) becomes usable.