Injective and surjective with answers to exercises

Let S, T be sets, let $f : S \to T$ be a function.

f is injective means:

$$\forall_{s_1,s_2\in S} \quad f(s_1) = f(s_2) \quad \Longrightarrow \quad s_1 = s_2 \tag{1}$$

Note: the book uses s, s' instead of s_1, s_2 but that means the same thing (here s' is just another symbol, it does not mean the derivative of f). Also, the book on page 81 still uses the notation $(s,t) \in f$ to highlight the fact that functions can be defined in terms of sets, but once we know this, we can replace $(s,t) \in f$ by the more common notation $f(s) = t$.

On page 82 the book uses the common notation $f(s) = t$. Read the definition of surjective (a.k.a. onto) on page 82. Replacing the phrases "for each" and "there is an" by $\forall$ and $\exists$, the definition of f is surjective is this:

$$\forall_{t\in T} \; \exists_{s\in S} \quad f(s) = t \tag{2}$$

Next, we take sets S, T, U and some functions $f : S \to T$ and $g : T \to U$. Now let $h = g \circ f$ be the composition, so

$$h(s) = g(f(s)) \tag{3}$$

Make sure to interpret these expressions carefully. When $f : S \to T$ then any time you see something like $f(\text{something})$ then that something must be an element of S because otherwise $f(\text{something})$ is an error. If $f : S \to T$ then it means that the input of f should be an element of S, and the output must be an element of T. Likewise, if $g : T \to U$ and if x is any expression, if you see $g(x)$ then x must be an element of T (if it isn't, then there is an error in the notation) and likewise $g(x)$ must be in U.

So in the formula (3) you see element of S (namely s), an element of T (namely $f(s)$) and an element of U (namely $g(f(s))$).

Turn in exercises: Let $f : S \to T$, $g : T \to U$, and $h = g \circ f$. Prove:

1. h onto $\Longrightarrow$ g onto.

   Proof: Assume h onto, so (a): $\forall_{u\in U}\exists_{s\in S}\; h(s) = u$.
   To prove: g onto, i.e. (b): $\forall_{u\in U}\exists_{t\in T}\; g(t) = u$. [WP#5 tells us to do this:]
   Let $u \in U$. T.P. (c): $\exists_{t\in T}\; g(t) = u$. [WP#6 tells us to write: take $t := \ldots$]

   Proof: From (a) we see that there is an $s \in S$ for which $h(s) = u$. But $h(s) = g(f(s))$. So we can prove (c) by taking $t := f(s)$.

2. h injective $\Longrightarrow$ f injective

   Proof: Assume h injective so (a): $h(s_1) = h(s_2) \Longrightarrow s_1 = s_2$.
   To prove: f injective, i.e. (b): $f(s_1) = f(s_2) \Longrightarrow s_1 = s_2$.
   Assume (c): $f(s_1) = f(s_2)$. To prove (d): $s_1 = s_2$.

   Applying g to both sides of (c) gives $g(f(s_1)) = g(f(s_2))$ which is the same as $h(s_1) = h(s_2)$. Then we can apply (a) to conclude $s_1 = s_2$.