Injective and surjective, two more exercises

Consider the following two sentences:

(A) Man bites dog.
(B) Dog bites man.

These sentences have the **same words** but **different meanings**.

For the "Write the definition" questions in quiz 3, many of you had answers with the right symbols or words, but in the wrong order. That changes the meaning just like (A) is not the same as (B).

To write proofs you must know **precise definitions**. If you write intuitive descriptions instead of exact definitions, then it's almost impossible to write correct proofs.

Let S, T be sets, let $f : S \to T$ be a function.

f is injective (a.k.a. one-to-one) means:

$$\forall_{s_1, s_2 \in S} \;\; f(s_1) = f(s_2) \quad \Longrightarrow \quad s_1 = s_2 \tag{1}$$

f is surjective (a.k.a. onto) means:

$$\forall_{t \in T} \; \exists_{s \in S} \;\; f(s) = t \tag{2}$$

(Don't write $\forall_{s \in S} \; \exists_{t \in T} \;\; f(s) = t$ because that's true for every function so it doesn't say anything about f).

Composition: If $h = g \circ f$ then $h(s) = g(f(s))$.

Turn in exercises: Let $f : S \to T$, $g : T \to U$, and $h = g \circ f$. Prove:

1. (h onto $\wedge$ g injective) $\Longrightarrow$ f onto
2. (h injective $\wedge$ f onto) $\Longrightarrow$ g injective

Hints on the next page.

Partial proof for 1: Assume h onto, so (a): $\forall_{u \in U} \exists_{s \in S}\ h(s) = u$
and g injective, so for (b): $\forall_{t_1, t_2 \in T}\ g(t_1) = g(t_2) \implies t_1 = t_2$.
To prove: f onto, i.e. (c): $\forall_{t \in T} \exists_{s \in S}\ f(s) = t$.

Hints to finish the proof:
WP#5 says that to prove (c) we need to start like this:
Let $t \in T$. T.P. (d): $\exists_{s \in S}\ f(s) = t$.

There are now two tasks. First, we have to use (a),(b) and f, g, h, t to come up with some element of S. Then, we have to use (a),(b) to show that f of that is t. For the first task, notice that for every $u \in U$, statement (a) gives us an element of S. The only way that is useful (read WP#14) is if we have an element of U. But what do we have at this stage in the proof? Well, we have an element $t \in T$ and we have functions $f : S \to T$ and $g : T \to U$ and $h : S \to U$. Do you see a way to combine those expressions to produce an element of U? After you answer that question, you can then use (a) to get some $s \in S$. Then it remains to show that f of that is t, and you will need to use (b) to prove that.

Partial proof for 2: Assume h injective so (a): $h(s_1) = h(s_2) \Longrightarrow s_1 = s_2$,
and f is onto, so (b): $\forall_{t \in T}\ \exists_{s \in S}\ \ f(s) = t$.
To prove: g injective, i.e. (c): $g(t_1) = g(t_2) \Longrightarrow t_1 = t_2$.
Assume (d): $g(t_1) = g(t_2)$. To prove (e): $t_1 = t_2$.

We will have to use both (a) and (b). WP#14 tells us that we can use (a) once we have encountered elements of S. We have not encountered elements of S yet. So we will have to use (b) first. Notice that (b) gives us an element of S for every element of T and so far, we have encountered two elements of T in statement (d).