Injective and surjective, two more exercises, Answers

Turn in exercises: Let $f : S \to T$, $g : T \to U$, and $h = g \circ f$. Prove:

1. (h onto $\wedge$ g injective) $\Longrightarrow$ f onto.

   Proof:
   Assume h onto: (a): $\forall_{u \in U} \exists_{s \in S}\ h(s) = u$
   and g injective: (b): $\forall_{t_1, t_2 \in T}\ g(t_1) = g(t_2) \implies t_1 = t_2$.
   To prove: f onto: (c): $\forall_{t \in T}\ \exists_{s \in S}\ f(s) = t$.

   Let $t \in T$. To prove: (d): $\exists_{s \in S}\ f(s) = t$.

   Proof for (d):
   Apply (a) to $u = g(t)$ and we get $h(s) = g(t)$ for some $s \in S$.
   But $h(s) = g(f(s))$, so $g(f(s)) = g(t)$. Then $f(s) = t$ since g is injective[1].

   [[1] We may directly conclude $\ldots = \ldots$ from $g(\ldots) = g(\ldots)$ and g injective. It is not necessary to spell out that we used (b) for $t_1 := f(s)$ and $t_2 := t$.]

2. (h injective $\wedge$ f onto) $\Longrightarrow$ g injective.

   Proof: Assume h injective so (a): $h(s_1) = h(s_2) \Longrightarrow s_1 = s_2$,
   and f is onto, so (b): $\forall_{t \in T}\ \exists_{s \in S}\ f(s) = t$.
   To prove: g injective, i.e. (c): $g(t_1) = g(t_2) \Longrightarrow t_1 = t_2$.
   Assume (d): $g(t_1) = g(t_2)$. To prove (e): $t_1 = t_2$.

   Proof for (e):
   By (b): $\exists_{s_1 \in S}\ f(s_1) = t_1$ and $\exists_{s_2 \in S}\ f(s_2) = t_2$.
   Then $h(s_1) = g(f(s_1) = g(t_1) =$ [use (d)] $= g(t_2) = g(f(s_2)) = h(s_2)$.
   So $h(s_1) = h(s_2)$. Then $s_1 = s_2$ by (a). But then $t_1 = f(s_1) = f(s_2) = t_2$.

Do not accept/reject any proof (including the ones above) simply because it looks plausible/implausible. Instead, check if it follow the rules. Check that the only things that were assumed were things that the rules allow us to assume. Check that we wrote the right T.P. statement and check that we proved it. No theorem in math should be accepted without such verification. Conversely, once we checked a proof, we must accept its conclusion regardless of whether it sounds plausible or not.

For example, Cantor wrote that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q}$ have the same cardinality, and yet, there exist different infinite cardinalities! For that to be accepted in math, it does not matter if it is plausible or not. What matters is: can you make this precise (write precise definitions!) and give a valid proof (one that follows the rules!).

Practice these and previous exercises to make sure you can do them. Remember that minor changes to a text can drastically change its meaning ("dog bites man" vs "man bites dog"). So make sure your work does not have errors that appear to be minor, but really aren't.

Proofs for a few HW questions:

- Ex 26, p. 103. T.P.: $\{\{a\},\{a,b\}\} = \{\{a'\},\{a',b'\}\} \iff (a = a' \ \wedge\ b = b')$

  The direction $\Longleftarrow$ is trivial. Remains to prove $\Longrightarrow$.
  Let $S = \{\{a\},\{a,b\}\} = \{\{a'\},\{a',b'\}\}$. To prove $a = a'$ and $b = b'$.

  Pitfalls: Many homework answers implicitly assumed that S has two distinct elements: $\{a\}$ and $\{a,b\}$. But these need not be distinct; there are two cases. Most of your answers only covered case (2).

  Proof: Case (1): assume $a = b$. Then S has only one element. That one element equals $\{a\}$ but it also equals $\{a',b'\}$. So $\{a\} = \{a',b'\}$ and thus $a = a'$ and $a = b'$ so $b = b'$.
  Case (2): assume $a \neq b$. Then S has two elements, one of which has 1 element and one of which has 2 elements. Comparing the element of S with 1 element we get $\{a\} = \{a'\}$ so $a = a'$. Then take the element of S with two elements and remove a from it, and we get $\{b\} = \{b'\}$ and hence $b = b'$.

- Let $f : S \to T$. Suppose that g_1 is an inverse of f, and g_2 is also an inverse of f. Show that $g_1 = g_2$.

  [Any correct answer **must use the definition** of inverse. Read Definition 4.4.6 on page 85. If your answer did not contain at least a part of that definition, then it can not be a valid answer.]

  The first part of definition 4.4.6 says $(f \circ g_1)(t) = t$ for all $t \in T$.
  In other words $f(g_1(t)) = t$ for all $t \in T$.
  Likewise $f(g_2(t)) = t$ for all $t \in T$.

  So $f(g_1(t)) = f(g_2(t))$ for all $t \in T$. But f is injective and so $g_1(t) = g_2(t)$ for all $t \in T$. Then g_1 and g_2 are the same function.

- Before you write any proof:

  Make sure you have all definitions in front of you!

  Do not start working on any proof until you collected all relevant definitions. Finding a proof often means finding some connection between assumed/given/TP statements and definitions. Those statements need to be **visible in front of you so that your eyes can spot a connection**.

  A drawing or an intuitive description often helps to understand the material, and I may put some on the blackboard. Occasionally, they may help to find a proof. However, put drawings and intuitive descriptions only on scratch paper and **not on the work you turn in**. Work you turn in for tests/quizzes/HW should **only use formal and precise statements**.