List of definitions and facts.

1. We say that p and x are **ϵ-close** when $|p - x| < \epsilon$. In other words, when the distance between p and x is less than ϵ
2. The set of all points that are ϵ-close to x is $(x - \epsilon, x + \epsilon)$.
3. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** when $\forall_{x \in \mathcal{O}} \exists_{\epsilon > 0} \, (x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
   In other words, for any $x \in \mathcal{O}$ there is some $\epsilon > 0$ such that all points ϵ-close to x are again in $\mathcal{O}$.
4. $\mathbb{R}$ and $\emptyset$ are open (check this!). We proved in class that (a, b) is open.
5. **Any** union of open sets is always open (even infinite unions!).
6. The intersection of **finitely many** open sets is again open.
7. Let $a_1, a_2, \ldots$ be a sequence. A **tail** is a subsequence of the form $a_{K+1}, a_{K+2}, \ldots$
   So a tail is: all terms beyond some cutoff point K.
8. $a_1, a_2, \ldots$ **converges** to α when $\forall_{\epsilon > 0} \, \exists_K \, \forall_{j > K} \, |a_i - \alpha| < \epsilon$. In other words, or every $\epsilon > 0$ the sequence has a tail contained in $(\alpha - \epsilon, \alpha + \epsilon)$.
   In this case we call α the **limit** of the sequence $a_1, a_2, \ldots$.
9. α is called a **limit point** of V when (i) there is a sequence in $V - \{\alpha\}$ that converges to α. This is equivalent to (ii) $\forall_{\epsilon > 0} \, (\alpha - \epsilon, \alpha + \epsilon) \bigcap (V - \{\alpha\}) \neq \emptyset$.
10. A set $V \subseteq \mathbb{R}$ is closed when
    (a) The complement of V is open.
    (b) If a sequence $a_1, a_2, \ldots$ in V converges to α then $\alpha \in V$.
    (c) V contains all of its limit points.
    (d) If $(\alpha - \epsilon, \alpha + \epsilon) \bigcap V$ is not empty for every $\epsilon > 0$ then $\alpha \in V$.
11. Notation: $\overline{S}$ is called the **closure** of the set S
    (a) $\overline{S}$ is the union of S and all of its limit points.
    (b) $\overline{S}$ is the smallest closed set that contains S.
    (c) $\overline{S}$ is the intersection of all closed sets that contain S.
    (d) $x \in \overline{S} \iff \forall_{\epsilon > 0}$ there is a point in S that is ϵ-close to x.
    (e) $x \in \overline{S} \iff \forall_{\epsilon > 0} \, (x - \epsilon, x + \epsilon)$ intersects S.
    (f) $x \in \overline{S} \iff \exists$ a sequence $a_1, a_2, \ldots \in S$ that converges to x.
12. α is a limit point of S if α is in the closure of $S - \{\alpha\}$.
13. The union of *finitely many* closed sets is again closed.
14. The intersection of closed sets (even infinitely many closes sets) is closed.
15. If $S \subseteq \mathbb{R}$ has finitely many elements then S is closed.
16. An **interior point** of S is a point s for which $\exists_{\epsilon > 0} \, (s - \epsilon, s + \epsilon) \subseteq S$.
    Denote $\text{Int}(S)$ as the set of interior points of S.
    (a) S is open $\iff$ every element of S is an interior point of S.
    In other words S is open $\iff S = \text{Int}(S)$.
    (b) $\text{Int}(S)$ is open.

(c) $\text{Int}(S)$ is the union of all open subsets of S.

(d) If S is the complement of U then $\overline{S}$ is the complement of $\text{Int}(U)$.

17. A point $s \in \mathbb{R}$ is a boundary point of S if

$$\forall_{\epsilon>0} \left((s-\epsilon, s+\epsilon) \bigcap S \neq \emptyset \text{ and } (s-\epsilon, s+\epsilon) \bigcap {}^c S \neq \emptyset \right)$$

The boundary of S is the set of all boundary points of S. It also equals the intersection of $\overline{S}$ and $\overline{{}^c S}$.

18. (Ex 13 in the book). The set of limit points of any set S is closed.

19. A subset $S \subseteq \mathbb{R}$ is called *dense* if (definition is also in Ex 12 in the book) for every $x \in \mathbb{R}$ and every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains an element of S.

    (a) S dense $\Longleftrightarrow \overline{S} = \mathbb{R}$

    (b) S dense $\Longleftrightarrow$ For every $\alpha \in \mathbb{R}$ there exists a sequence $a_1, a_2, \ldots \in S$ that converges to α.

    (c) S dense $\Longleftrightarrow$ Every non-empty open set intersects S (meaning, every non-empty open set has at least one element of S).

    (d) S dense $\Longleftrightarrow$ Every non-empty open set contains infinitely many elements of S.

    (e) $\mathbb{Q}$ is dense

    (f) If A is countable then $\mathbb{R} - A$ is dense.

20. S is a discrete set if $\forall_{s \in S} \exists_{\epsilon>0}\ (s-\epsilon, s+\epsilon) \bigcap S = \{s\}$.

21. Every finite set is discrete.

22. Every discrete set is countable.

23. NEW EXERCISES. Prove these:

    (a) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

    (b) If $C = A \bigcup B$ then $\overline{C} = \overline{A} \bigcup \overline{B}$.

    (c) If A is an open and B is closed set then $A - B$ is open.

    (d) If S is a discrete set, and if $s \in S$, and if $a_1, a_2, a_3, \ldots$ is a sequence in S and if that sequence converges to s, then a tail of that sequence must be $s, s, s, \ldots$

    (e) If $a_1, a_2, a_3, \ldots$ is a sequence that converges to s, and if s is an interior point of a set S, then show that some tail of the sequence lies in S.

    (f) Now suppose that p is **not** an interior point of S. Then show that there exists a sequence $a_1, a_2, a_3, \ldots$ inside the complement of S that converges to p.

    (g) If S is a discrete set and x is any number then there is a sequence $a_1, a_2, a_3, \ldots$ in the complement of S that converges to x.

    (h) If S is a discrete set then its interior $\text{Int}(S)$ is empty.

I'll add more questions/answers on Saturday.