Answers and Questions

1. If S is a finite set then S is closed (note: this one is not new).

   The goal here is to prove item 15 in "List of definitions and facts". First, suppose that S has only one element: $S = \{s\}$. The definition of closed, item 10(b), says that S is closed when every convergent sequence in S converges to an element of S. But there is only one sequence in $\{s\}$ namely $s, s, s, \ldots$ and that clearly converges to an element of S.
   Thus: sets with a single element are closed. Then using item 13 we get that every set with finitely many elements is closed.

2. If S is a finite subset of $\mathbb{R}$ then show that S is discrete.

   The definition of discrete is in Ex 9 in the book: If $S \subseteq \mathbb{R}$ then
   S is *discrete* if $\forall_{s \in S} \exists_{\epsilon > 0}\, (s - \epsilon, s + \epsilon) \bigcap S = \{s\}$.

   Assume S is a finite set. To prove: S is discrete.
   To prove: $\forall_{s \in S} \exists_{\epsilon > 0}\, (s - \epsilon, s + \epsilon) \bigcap S = \{s\}$.
   [Read items 5 and 6 in the Writing Proofs guidelines, this tells you that the proof should look like this:]
   Let $s \in S$. Take $\epsilon := \ldots$.

   [Once we figure out what to write on the dots, the proof will be done. Our s is an element of a finite set. So if we take the distances between s and the elements of $S - \{s\}$ then we get finitely many non-zero numbers. Then we can take the minimum.]

   Let $s \in S$. If s is the only element of S then $\epsilon :=$ any positive number.
   Otherwise $\epsilon :=$ the smallest $|s - s'|$ for all $s' \in S - \{s\}$.

   (Why does this proof fail for infinite sets? Well, because then you have infinitely many positive numbers $|s - s'|$ and it could happen that none of them is the smallest one.) (If you have infinitely many numbers, then it can happen that whichever one you pick, there might be another one that is smaller.)

3. If S is not closed, then show that its closure is not a discrete set.

   You need to read the handout, otherwise it would be difficult to come up with:

   If S is not closed, then $S \neq \overline{S}$ because $\overline{S}$ is always closed.
   But $\overline{S}$ is the union of S and all of its limit points, so if $\overline{S} \neq S$ then there exists $s \in \overline{S} - S$ and that s is a limit point of S.
   Then for any $\epsilon > 0$ we get $(s - \epsilon, s + \epsilon) \bigcap (S - \{s\}) \neq \emptyset$
   so then $(s - \epsilon, s + \epsilon) \bigcap (\overline{S} - \{s\}) \neq \emptyset$
   so then $(s - \epsilon, s + \epsilon) \bigcap \overline{S} \neq \{s\}$ (for any $\epsilon > 0$)
   so then $\overline{S}$ is not discrete.

4. A subset $S \subseteq \mathbb{R}$ is called *dense* if (definition is also in Ex 12 in the book) for every $x \in \mathbb{R}$ and every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ contains an element of S.

   (a) Show that S dense $\iff \overline{S} = \mathbb{R}$

      S dense [look at definition:]
      $\iff \forall_{x \in \mathbb{R}} \forall_{\epsilon > 0}$ $(x - \epsilon, x + \epsilon)$ intersects S [Now look at 11(e):]
      $\iff \forall_{x \in \mathbb{R}}\ x \in \overline{S}$
      $\iff \mathbb{R} = \overline{S}$

(b) Show that S dense $\iff$ For every $\alpha \in \mathbb{R}$ there exists a sequence $a_1, a_2, \ldots \in S$ that converges to α.

Item 11(f) says that the phrase "there exists a sequence $a_1, a_2, \ldots \in S$ that converges to α" is equivalent to saying $\alpha \in \overline{S}$. Then our To Prove statement becomes: T.P. S dense $\iff$ For every $\alpha \in \mathbb{R}$ $\alpha \in \overline{S}$.
But that is the same as part (a) of this question.

(c) Show that S dense $\iff$ Every non-empty open set intersects S.

Comment: The following mean the same:
(1) A intersects B.
(2) A contains at least one element of B.
(3) $A \bigcap B \neq \emptyset$.
(4) $A \not\subseteq {}^cB$ (the ^{c} means complement)

"$\Longrightarrow$" If S is dense then every set of the form $(x-\epsilon, x+\epsilon)$ intersects S. But every non-empty open set contains a set of that form, and thus intersects S.
"$\Longleftarrow$" If every non-empty open set intersects S, then that includes all sets of the form $(x-\epsilon, x+\epsilon)$. But then S is dense.

(d) Show that S dense $\iff$ Every non-empty open set contains infinitely many elements of S.

"$\Longleftarrow$" follows from the previous item.
"$\Longrightarrow$" Every non-empty open set S contains some interval (a, b). Inside that interval are two open intervals (a, c) and (c, b) where $c = (a+b)/2$, each of which contain at least one point from S by the previous item. That means that (a, b) contains at least 2 points from S. But the same argument works for any open interval, including (a, c) and (c, b), so those must also contain at least 2 points each from S, but then (a, b) contains at least 4 points from S. Continuing, we see that (a, b) contains at least 2, 4, 8, 16, $\ldots$ points from S.

(e) Show that $\mathbb{Q}$ is dense.

We showed in class (more details below) that for every real number $x \in \mathbb{R}$ there is a sequence $a_1, a_2, \ldots \in \mathbb{Q}$ that converges to x. But then item 11(f) says that x is then in $\overline{\mathbb{Q}}$. So every real number x is in $\overline{\mathbb{Q}}$ so $\mathbb{R} = \overline{\mathbb{Q}}$.

Now why is there, for any real number x, a rational sequence that converges to it? The easiest way to understand it is by an example. Take any real number, for instance $x = 3.141592\ldots$ Then the sequence of rationals is: 3, 31/10, 314/100, 3141/1000, $\ldots$.

Its clear that the same would have worked for any other real number too. But in some sense this argument is backwards; the real numbers were constructed precisely so that they would be limits of rational sequences.

(f) Show that if A is countable then $\mathbb{R} - A$ is dense.

Take any open interval $I = (x-\epsilon, x+\epsilon)$. We have seen previously that I is uncountable. Therefore, I can not be a subset of A when A is countable. Therefore, I intersects $\mathbb{R} - A$ and so we see from the definition of dense that $\mathbb{R} - A$ is dense.

In case you forgot, why is any interval of the form (a, b) with $a < b$ uncountable? Well, you can find a bijection $(a, b) \to (0, 1)$ by sending x to $(x-a)/(b-a)$. Then you find a bijection from $(0, 1)$ to $(-\pi/2, \pi/2)$ by sending x to $\pi \cdot (x - 1/2)$. Then you find a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$, namely the arctan function. So all of those intervals have bijections between them, so they all have the same cardinality. Its best to simply rememeber that any open interval is uncountable.

Now lets look at the questions at the end of the file "Material for test 3".

(a) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

Suppose $A \subseteq B$. To prove $\overline{A} \subseteq \overline{B}$. You **MUST KNOW** that to prove a $\subseteq$ statement you start by writing this:

Assume $x \in \overline{A}$. To prove $x \in \overline{B}$.

Proof: If $x \in \overline{A}$ then that means there is a sequence in A converging to x (see item 11(f)). But since $A \subseteq B$, that same sequence is also in B. But then $x \in \overline{B}$ by once again using 11(f).

(b) If $C = A \bigcup B$ then $\overline{C} = \overline{A} \bigcup \overline{B}$.

$A \subseteq C$ so $\overline{A} \subseteq \overline{C}$ by the previous item. Likewise $\overline{B} \subseteq \overline{C}$. Combining this we get $\overline{A} \bigcup \overline{B} \subseteq \overline{C}$. Remains to prove the opposite inclusion.
So assume $x \in \overline{C}$. Then there is a sequence $a_1, a_2, \ldots$ in $C = A \bigcup B$ converging to x. That sequence has infinitely many terms in $A \bigcup B$, so at least one of A (case 1) or B (case 2) must have infinitely many terms of our sequence. That means that at least one of A (case 1) or B (case 2) contains a subsequence of $a_1, a_2, \ldots$ that still has infinitely many terms. That subsequence converges to the same limit x. In case 1 we have $x \in \overline{A}$ and in case 2 we have $x \in \overline{B}$, either way we have $x \in \overline{A} \bigcup \overline{B}$.

Another proof based on 11(b)(c): Any closed set that contains $C = A \bigcup B$ must also contain A, and any closed set that contains A must also contain $\overline{A}$ since $\overline{A}$ is the intersection of all closed sets that contain A. So any closed set that contains C must contain $\overline{A}$, and for the same reason $\overline{B}$, so any closed set that contains C must contain $\overline{A} \bigcup \overline{B}$. That means $\overline{A} \bigcup \overline{B}$ is the smallest closed set that contains C, and so it equals $\overline{C}$.

(c) If A is an open and B is closed set then $A - B$ is open.

When B is closed, its complement ${}^cB$ is open. Now $A - B = A \bigcap {}^cB$ is an intersection of two open sets, which is open (item 6).

(d) If S is a discrete set, and if $s \in S$, and if $a_1, a_2, a_3, \ldots$ is a sequence in S and if that sequence converges to s, then a tail of that sequence must be $s, s, s, \ldots$

If $s \in S$ and S is discrete then there is $\epsilon > 0$ such that $(s - \epsilon, s + \epsilon) \bigcap S = \{s\}$. If $a_1, a_2, \ldots$ converges to s then a tail of this sequence is in $(s - \epsilon, s + \epsilon)$ (read item 8). But our sequence was also in S so that tail is in $(s - \epsilon, s + \epsilon) \bigcap S = \{s\}$. So that tail is $s, s, s, \ldots$.

(e) If $a_1, a_2, a_3, \ldots$ is a sequence that converges to s, and if s is an interior point of a set S, then show that some tail of the sequence lies in S.

Hint: in items 3 and 8, replace the x and α there by s, then combine them to find a proof.

(f) Now suppose that x is **not** an interior point of S. Then show that there exists a sequence $a_1, a_2, a_3, \ldots$ inside the complement of S that converges to x.

Hints: you have to negate the definition $\exists_{\epsilon>0}\ldots$ found in item 16. When you do that, you get something of the form $\ldots \not\subseteq S$. But that is equivalent to ... intersects ${}^cS$. But that is the right-hand side in 11(e) with S replaced by ${}^cS$. So you conclude the left-hand side of 11(e) with S replaced by ${}^cS$ Which part of item 11 can you use now?

Note: I'm sure that there are other proofs too for many of these exercises.

(g) If S is a discrete set and x is any number then there is a sequence $a_1, a_2, a_3, \ldots$ in the complement of S that converges to x.

Hint: how about using item 22 in the file "Material for test 3"? Now find another item that allows you to make use of this hint.

(h) If S is a discrete set then its interior $\mathrm{Int}(S)$ is empty.

A non-empty open set is uncountable, while a discrete set S is countable, so if $\mathrm{Int}(S)$ is not empty, then it can't be a subset of S giving a contradiction. But there are also other proofs directly from the definitions. For instance,
Let $x \in \mathrm{Int}(S)$. That's an open set so for some $\epsilon > 0$ we have $(x-\epsilon, x+\epsilon) \subseteq \mathrm{Int}(S) \subseteq S$. Now find several ways to prove that $(x-\epsilon, x+\epsilon)$ can not be a subset of a discrete set!

(i) Write down the definition of an open set: Memorize item 3!
Then prove, using only the definition and the rules of logic, that the empty set is open.

We have to prove: $x \in \emptyset \Longrightarrow \exists_{\epsilon>0}\ (x-\epsilon, x+\epsilon) \subseteq \emptyset$.

Recall that "False$\Longrightarrow$Anything" is always True.
And that "$x \in \emptyset$" is always False.

(j) Show that the complement of a discrete set is dense.

Items 22 and 19(f).

(k) If S is a closed set, then show that every boundary point of S is already in S.

Item 17 says that boundary points are in $\overline{S} \bigcap \overline{{}^cS}$ which is a subset of $\overline{S}$, which is S because S is closed.

(l) If S is an open set, then show that none of its boundary points are in S.

Item 17 says that boundary points are in $\overline{S} \bigcap \overline{{}^cS}$ which is a subset of $\overline{{}^cS}$, which is ${}^cS$ because ${}^cS$ is closed since we assumed S to be open. But saying that the boundary points are in ${}^cS$ is the same as saying that they are not in S.