More sample questions, ANSWERS

1. Simplify the following cardinal numbers to the point where they look like $0, 1, 2, \ldots$, or $\aleph_0$, or c, or 2^c, or $2^{(2^c)}$, etc.

   $\text{card}(\{3,3,3\}) = 1$
   $\text{card}(P(\emptyset)) = 2^0 = 1$
   $\text{card}(\mathbb{Q}) = \aleph_0$
   $\text{card}(\mathbb{R} \times \mathbb{N}) = c \cdot \aleph_0 = c$
   $\aleph_0^{\aleph_0} = c$ (explanation: $2^{\aleph}_0 \leq \aleph_0^{\aleph_0} \leq c^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$)
   $\text{card}(P(\mathbb{R})) = 2^c$
   $c^{\aleph_0} = c$ (explanation: $2^{\aleph}_0 \leq c^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$)

   Does there exist an injective function from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$?
   Yes, $\text{card}(\mathbb{R}^{\mathbb{N}}) = c^{\aleph_0} = c$ (see above) so both have cardinality c so there is even a bijection.

2. Give the definitions:

   (a) A function $f : A \to B$ is **injective** (same as "one-to-one") when:

   $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$ (for all $a_1, a_2 \in A$)

   (b) Sets A and B have the **same cardinality** when:

   When there exists a bijection from A to B.

3. If $f : A \to B$ and $g : B \to C$ are both **injective** (exercise 2a) then prove that the composition $g \circ f : A \to C$ is also **injective**.

   Let $a_1, a_2 \in A$ and assume $g(f(a_1)) = g(f(a_2))$. To prove: $a_1 = a_2$.
   Given: g is injective so $g(b_1) = g(b_2) \Longrightarrow b_1 = b_2$ for any $b_1, b_2 \in B$.
   Applying that to $b_1 = f(a_1),\ b_2 = f(a_2)$ gives $f(a_1) = f(a_2)$.
   Then the statement in exercise 2a gives $a_1 = a_2$.

4. Using **only** the definition from exercise 2b, show that $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ have the **same cardinality**.

   To prove: there exists a bijection from $\mathbb{N}$ to $\mathbb{N}^*$.
   Proof: take the function $f(n) := n + 1$.

   [Shouldn't we also prove that this is a bijection? In principle yes, but if you skip that on the test then you will still get full credit. The most important part about "show that there exists a bijection" is to *actually write down what that bijection is!* Students will get full credit if and *only if* they write that bijection. To emphasize this, in my answer sheet I only write what f is and skip the proof that f is a bijection even though that is in principle part of a complete proof.]

5. Let $A \subseteq B \subseteq C$ and suppose $\exists$ injective function from C to A. Then prove that there is a surjective function from B to C.

   $\text{card}(A) \leq \text{card}(B) \leq \text{card}(C) \leq \text{card}(A)$ (the first two $\leq$ are by item 13 and the last one is by item 2). In particular $\text{card}(B) \leq \text{card}(C)$ and $\text{card}(C) \leq \text{card}(B)$ but then $\text{card}(B) = \text{card}(C)$ by item 14. Then there exists a bijection from B to C (item 1). Any bijection is surjective [as well as injective.]

6. Let A, B be sets and let $C = A \times B$. Show that **at least one** of these must be true:
   (1) C is a **finite** set.
   or (2) There exists a **bijection** from A to C.
   or (3) There exists a **bijection** from B to C.

   If not (1), then C is infinite and $\text{card}(C) = \text{card}(A) \cdot \text{card}(B) = \max(\text{card}(A), \text{card}(B))$ by item 22. Then $\text{card}(C)$ equals $\text{card}(A)$ or $\text{card}(B)$, hence (2) or (3).