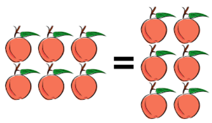

Demonstrates $3 \cdot 2 = 2 \cdot 3$ by sending apple (x, y) to (y, x). The same idea is used in Exercise 6.

Intro Advanced Math. Answers to questions 1–8

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$

   For (a)–(h) you do not need to show your work, but for (i),(j) you need to justify your answer by showing all steps.

   (a) $\mathbb{N}$: $\aleph_0$

   (b) $\emptyset \times \mathbb{R}$: $0 \cdot c = 0$

   (c) $\mathbb{Q}$: $\aleph_0$

   (d) $\mathbb{R} \times P(\mathbb{Q})$: $c \cdot 2^{\aleph_0} = c \cdot c = c$

   (e) $\mathbb{Q} - \mathbb{Z}$: $\aleph_0$

   (f) $P(\mathbb{N})$: $2^{\aleph_0} = c$

   (g) $P(\mathbb{R})$: 2^c

   (h) $\{2, 2\}$: 1

   (i) $\mathbb{R}^{\mathbb{N}}$: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c$

   (j) $\mathbb{R}^{\mathbb{R}}$: $c^c = (2^{\aleph_0})^c = 2^{\aleph_0 c} = 2^c$

2. Based on the answer in your previous question, does there exist: (it suffices to write yes/no):

   (a) an injective function from $\mathbb{R}^{\mathbb{R}}$ to $P(\mathbb{R})$? Yes

   (b) an injective function from $\mathbb{Q}$ to $\mathbb{N}$? Yes

   (c) an injective function from $P(\mathbb{N})$ to $\mathbb{N}$? No

3. Prove, using only the definition, that the intervals $(0, 1)$ and $(0, 2)$ have the same cardinality.

   The function $f(x) = 2x$ is a bijection from $(0, 1)$ to $(0, 2)$.

4. Let A, B be sets and let $C = A \bigcup B$. Suppose that $A \bigcap B = \emptyset$ and:

   - (1) there is **no** bijection from A to C
   - (2) there is **no** bijection from B to C

   Prove that A and B are finite sets.

   Let a, b, c_0 be the cardinalities of A, B, C. Then $c_0 = a + b$.
   Suppose a or b is infinite. Then $c_0 = \max(a, b)$ so $c_0 = a$ or $c_0 = b$.
   But $c_0 \neq a$ by (1) and $c_0 \neq b$ by (2), contradiction.

5. Let A be any set. Prove that there is no bijection from $\mathbb{N}$ to $P(A)$.

   Suppose there is a bijection. Then $\text{card}(P(A)) = \aleph_0$. But $\text{card}(A) < \text{card}(P(A))$. So $\text{card}(A) < \aleph_0$. Then A is finite. Then $P(A)$ is finite, contradiction.

6. TURN IN: We know that if d, e are natural numbers then $d \cdot e = e \cdot d$. But do you remember how to prove that? Lets prove this not only for natural numbers, but for all cardinal numbers! I will type the first line in the proof, and you finish it:

   Proof: Let D, E be sets for which $d = \text{card}(D)$ and $e = \text{card}(E)$.

   (a) Give the definition of $D \times E$.

   This is the set of all pairs (x, y) with $x \in D$ and $y \in E$.
   The shortest way to denote that is: $\{(x, y) : x \in D, y \in E\}$.

   (b) Give a bijection from $D \times E$ to $E \times D$.

   Let $f : D \times E \to E \times D$ be the function that sends (x, y) to (y, x).

   [Explanation: Any element of $D \times E$ can be written as (x, y) with $x \in D$ and $y \in E$. But then the pair (y, x) will be an element of $E \times D$. Clearly this is a bijection.
   Notation: If we write $f : (x, y) \mapsto (y, x)$ then this indicates that (x, y) is an element of the domain of f and that (y, x) is an element of the codomain (range) of f. In other words, this notation means that f sends this element (x, y) in $D \times E$ to the element (y, x) in $E \times D$.]

   (c) Why does this bijection prove $d \cdot e = e \cdot d$?

   Because $d \cdot e$ is by definition $\text{card}(D \times E)$. But giving a bijection from $D \times E$ to $E \times D$ proves that $\text{card}(D \times E) = \text{card}(E \times D)$ but the latter is by definition equal to $e \cdot d$.

7. TURN IN: Find all sets A for which the following is true: Every element of A is equal to 1.

   Answer: $A = \emptyset$ and $A = \{1\}$.

8. TURN IN:
   Item 22 says that if d, e are cardinals, and if at least one of them is infinite, then $d + e = \max(d, e)$. It is quite hard to prove this in general. Lets prove it in a special case, when $d = e = \aleph_0$, as follows:
   Let $\mathbb{N}^* = \{1, 2, 3, 4, \ldots\}$, $E = \{2, 4, 6, 8, \ldots\}$, $D = \{1, 3, 5, 7, \ldots\}$.
   So $E = \{\text{all even positive integers}\}$, and $D = \{\text{all odd positive integers}\}$.

   (a) Give a bijection $f : \mathbb{N}^* \to E$ (write down: $f(n) = \ldots$)

   This function: $f(n) = 2n$ is a bijection (there are other correct answers, but this one is the most obvious one).

(b) Give a bijection $g : \mathbb{N}^* \to D$.

This function: $g(n) = 2n - 1$ is a bijection.

(c) Explain why parts (a),(b) prove that $\aleph_0 + \aleph_0 = \aleph_0$.

If D, E are disjoint sets, each with cardinality $\aleph_0$, then $\aleph_0 + \aleph_0$ is by definition the cardinality of the union $D \bigcup E$. So $\aleph_0 + \aleph_0 = \text{card}(D \bigcup E) = \text{card}(\mathbb{N}^*) = \aleph_0$.

[Note: this is exactly like the second part of Hotel Infinity.]

List of facts on cardinal numbers, shortened version.

Note: During the actual test, basic definitions that everyone must know (such as items 1–7) may be deleted!

1. $\text{card}(A) = \text{card}(B)$ means $\exists f : A \to B$ with f bijection.
2. $\text{card}(A) \leq \text{card}(B)$ means $\exists f : A \to B$ with f one-to-one.
3. $\aleph_0$ is short notation for $\text{card}(\mathbb{N}^*)$.
4. c is short notation for $\text{card}(\mathbb{R})$.
5. The set A is *countably infinite* when: $\text{card}(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.
6. Notation: $x < y$ is short for: $x \leq y \wedge x \neq y$.
7. $\text{card}(A) < \text{card}(P(A))$.
8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $\text{card}(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!
9. If $f : A \to B$ is onto then $\text{card}(B) \leq \text{card}(A)$.
10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.
11. A is countable when $\text{card}(A) \leq \aleph_0$.
12. A subset of a countable set is again countable.
13. If $A \subseteq B$ then $\text{card}(A) \leq \text{card}(B)$.
14. The ordering $\leq$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \leq e$ and $e \leq d$ we may conclude $d = e$.
    The proof is not easy! (Schroeder-Bernstein theorem on p 88–89).

15. The ordering $\leq$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \leq e$ or $d \geq e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.

16. Set A is uncountable when $\text{card}(A) \not\leq \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $\text{card}(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*))$.

21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$. d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0,1\}^A$, and hence $\text{card}(P(A)) = \text{card}(\{0,1\}^A) = \text{card}(\{0,1\})^{\text{card}(A)} = 2^{\text{card}(A)}$.

24. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*)) = \text{card}(\{0,1\}^{\mathbb{N}^*}) = 2^{\text{card}(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $\text{card}(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by $\max(d,e)$, see item 22.

27. So far we have encountered these increasing cardinals:

    $$0,\ 1,\ 2,\ 3,\ \ldots\ \ \aleph_0,\ \ c = 2^{\aleph_0},\ \ 2^c,\ \ 2^{2^c},\ \ldots$$

    and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

    From the axioms of set theory (= the only statements mathematicians accept without a proof) it is impossible to prove or disprove this.