Intro Advanced Math, test 1. **Your name:**

1. (a) Write down the **converse** of the statement $p \implies q$.
   (b) Write down the **contrapositive** of the statement $p \implies q$.
   (c) Let x and y be real numbers: Consider the statement

$$(\forall_{\epsilon>0} \ \ |x-y| \leq \epsilon) \implies x = y$$

   Write down the **contrapositive** of this statement and **simplify** your answer so that you have no negation symbol in front of a quantifier (quantifiers are the symbols $\forall$ and $\exists$).
   (d) Can you prove the statement from part (c)?

2. Let A, B, C be sets.

   (a) Write the defining formula for the **difference of two sets**:

$$x \in A \setminus B \iff \dots$$

   (b) Write the defining formula for a **power set**:

$$A \in P(C) \iff \dots$$

   (c) Use your answer on part (a),(b) to prove

$$A \in P(C) \implies (A \setminus B) \in P(C).$$

3. Suppose that A, B, I are sets, and C_i is a set for every $i \in I$. Suppose that $C_i \subseteq B$ for every $i \in I$. Show that

$$A \setminus B \subseteq \bigcap_{i \in I} A \setminus C_i$$

4. Let $x > -1$. Use induction to prove

$$(1+x)^n \geq 1 + nx$$

   for every positive integer n.

5. Let $S = \{3k+1 \ : \ k \in \mathbb{Z}\}$, this is the set of integers that have remainder 1 after dividing by 3. If $n \in S$ and $m \in S$ then show that $nm \in S$.

   (Hint 1: induction is not a good idea here. Hint 2: do not use the same symbol for different numbers, instead use something like k, l or k_1, k_2).

If you run out of time, then mark either Exercise 4 or Exercise 5 as **take-home** and turn in that one on Monday.

Defining formulas

(only some of these are useful for the test, and some have been deleted)

$$A = B \quad \text{means} \quad x \in A \Longleftrightarrow x \in B. \tag{1}$$

$$A \subseteq B \quad \text{means} \quad x \in A \Longrightarrow x \in B. \tag{2}$$

$$A = B \quad \text{is equivalent to} \quad A \subseteq B \ \wedge \ B \subseteq A \tag{3}$$

$$A = \emptyset \quad \text{means} \quad \forall_x \ x \notin A. \tag{4}$$

$$x \in A \bigcap B \quad \text{means} \quad (x \in A \ \wedge \ x \in B) \tag{5}$$

$$x \in A \bigcup B \quad \text{means} \quad (x \in A \ \vee \ x \in B) \tag{6}$$

$$x \in \bigcap_{i \in I} A_i \quad \text{means} \quad \forall_{i \in I} \ x \in A_i \tag{7}$$

$$x \in \bigcup_{i \in I} A_i \quad \text{means} \quad \exists_{i \in I} \ x \in A_i \tag{8}$$

Writing Proofs.

1. **Direct proof for $p \Longrightarrow q$.**
   Assume: p. To prove: q.

2. **Proving $p \Longrightarrow q$ by contrapositive.**
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving S by contradiction.**
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving $p \Longrightarrow q$ by contradiction.**
   Assume: p and $\neg q$. To prove: a contradiction.

5. **Direct proof for a $\forall_{x \in A} P(x)$ statement.**

   To ensure you prove $P(x)$ for *all* (rather than for *some*) x in A, do this:

   Start your proof with: Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for $\exists_{x \in A} P(x)$ statement.**
   Take $x :=$ [write down an expression that is in A, and satisfies $P(x)$].

7. **Proving $\forall_{x \in A} P(x)$ by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving $\exists_{x \in A} P(x)$ by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving S by cases.**
   Suppose for example a statement p can help to prove S. Write two proofs:
   Case 1: Assume p. To prove: S.
   Case 2: Assume $\neg p$. To prove S.

10. **Proving $p \wedge q$**
    Write two separate proofs: To prove: p. To prove: q.

11. **Proving $p \Longleftrightarrow q$**
    Write two proofs. To prove: $p \Longrightarrow q$ To prove: $q \Longrightarrow p$.

12. **Proving $p \vee q$**
    Method (1): Assume $\neg p$. To prove: q.
    Method (2): Assume $\neg q$. To prove: p.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using $p \vee q$ to prove another statement r.**
    Write two proofs:
    Assume p. To prove r.
    Assume q. To prove r.

14. **How to use a for-all statement $\forall_{x \in A} P(x)$.**
    You need to produce an element of A, then use P for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* x. All you know is $x \in A$ and $P(x)$.