Intro Advanced Math, test 1 answers.

1. (a) Write down the **converse** of the statement $p \implies q$.

   $q \implies p$

   (b) Write down the **contrapositive** of the statement $p \implies q$.

   $\neg q \implies \neg p$

   (c) Let x and y be real numbers: Consider the statement

   $$(\forall_{\epsilon>0} \quad |x-y| \leq \epsilon) \implies x = y$$

   Write down the **contrapositive** of this statement and **simplify**.

   $$x \neq y \quad \implies \quad \neg\,(\forall_{\epsilon>0} \quad |x-y| \leq \epsilon)$$

   Simplifying gives:

   $$x \neq y \quad \implies \quad \exists_{\epsilon>0}\, |x-y| > \epsilon$$

   [Giving just this last statement is sufficient to get full credit.]

   (d) Can you prove the statement from part (c)?

   [WP#1 tells us to write the next line:]
   Assume $x \neq y$. To prove: $\exists_{\epsilon>0} \quad |x-y| > \epsilon$.
   [WP#6 tells us to write: Take $\epsilon := \ldots$ so all we have to do is to figure out how to fill in the dots:]
   Proof: Take $\epsilon := |x-y|/2$ [then $\epsilon > 0$ because $x \neq y$, and $|x-y| > \epsilon$]

2. Let A, B, C be sets.

   (a) Write the defining formula for the **difference of two sets**:

   $$x \in A \setminus B \iff \ldots$$

   $x \in A \wedge x \notin B$

   (b) Write the defining formula for a **power set**:

   $$A \in P(C) \iff \ldots$$

   $A \subseteq C$ [Also correct $A \subset C$ because that means the same thing]

   (c) Use your answer on part (a),(b) to prove

   $$A \in P(C) \quad \implies \quad (A \setminus B) \in P(C).$$

   Let $A \in P(C)$. To prove: $A \setminus B \in P(C)$.
   $A \in P(C)$ means $A \subseteq C$.
   $A \setminus B \subseteq A \subseteq C$ so $A \setminus B \subseteq C$ which means $A \setminus B \in P(C)$.

3. Suppose that A, B, I are sets, and C_i is a set for every $i \in I$. Suppose that $C_i \subseteq B$ for every $i \in I$. Show that

$$A \setminus B \subseteq \bigcap_{i \in I} A \setminus C_i$$

To prove: $x \in A \setminus B \implies x \in \bigcap_{i \in I} A \setminus C_i$.
Direct proof: Assume $x \in A \setminus B$ in other words $x \in A$ and $x \notin B$.
To prove: $x \in \bigcap_{i \in I} A \setminus C_i$ in other words to prove: $x \in A \setminus C_i$ for all $i \in I$.

Proof: Since $C_i \subseteq B$ and $x \notin B$ it follows that $x \notin C_i$. Combine this with $x \in A$ gives $x \in A \setminus C_i$.

[Check this proof by checking that we didn't use to-prove statements and only used given/assumed statements like $C_i \subseteq B$ and $x \in A$ and $x \notin B$].

4. Let $x > -1$. Use induction to prove

$$(1 + x)^n \geq 1 + nx$$

for every positive integer n.

For $n = 1$ both sides are $1 + x$ so then the statement is true.
Now assume $(1 + x)^n \geq 1 + nx$ (the induction hyphothesis).
To prove: $(1 + x)^{n+1} \geq 1 + (n + 1)x$.

$(1 + x)^{n+1} = (1 + x)(1 + x)^n$. The assumptions tell us that $1 + x$ is not negative and $(1 + x)^n \geq 1 + nx$. From that we get $(1 + x)(1 + x)^n \geq (1 + x)(1 + nx) = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x$.

5. Let $S = \{3k+1 \ : \ k \in \mathbb{Z}\}$, this is the set of integers that have remainder 1 after dividing by 3. If $n \in S$ and $m \in S$ then show that $nm \in S$.

$n, m \in S$ so we can write $n = 3k + 1$ and $m = 3l + 1$ for some $k, l \in \mathbb{Z}$. Then $nm = 9kl + 3k + 3l + 1 = 3(3kl + k + l) + 1 = 3(\text{an integer}) + 1 \in S$.

Remarks on quantifiers:

Quantifiers are the symbols $\forall$ and $\exists$, as well as words/phrases that have the same meaning as $\forall$ or $\exists$

- $\forall$. Phrases with the same meaning: "every", "all", "any".
  Note: the phrase "Let $x \in A$" allows x to be *any* element of A, so this is how one typically starts a proof of a $\forall_{x \in A} \ldots$ statement. Likewise, the phrase "Let x be a real number" means that x could be *any* element of $\mathbb{R}$.

- $\exists$. Phrases with the same meaning: 'there is", "there exists", "for some", "for at least one" .

Turn in Exercise: Is this statement true or false? $\forall_{x \in \emptyset} \ (x < 1 \land x > 2)$.

Before you decide this, first compute the negation of this statement, then decide which one is true and which one is false.

Common errors:

- **Misinterpreting exists versus for-all.**

  Suppose someone tells you: if n is even then n^2 is even.
  What did they mean by that? Did it mean:

  1. There is **at least one** even number whose square is even. Or:
  2. The square of **every** even number is even.

  The statement: "n even $\Longrightarrow n^2$ even" is generally interpreted to mean that the square of **every** (i.e. $\forall$) even number is even rather than just saying that the square of **some** (**at least one**) ($\exists$) even number is even. In general, symbol(s) that do not appear in quantifiers ($\forall$, $\exists$, or phrases with the same meaning) are interpreted as $\forall$.

- **Common error: Proving a for-all statement with an example.**
  Example. Prove: for every integer there is a larger integer.
  In other words: $\forall_{x\in\mathbb{Z}}\exists_{y\in\mathbb{Z}}\ y > x$.

  Bad proof: Take $x = 5$ and $y = 6$. Then $y > x$.

  This "proof" gave an example for both x and y, so it folllowed WP#6 for both x and y. This means that it only proved $\exists_{x\in\mathbb{Z}}\exists_{y\in\mathbb{Z}}\ y > x$.

  To prove $\forall_{x\in\mathbb{Z}}\exists_{y\in\mathbb{Z}}\ y > x$ follow WP#5 for $\forall$ and then WP#6 for $\exists$.

  Valid proof:
  Let $x \in \mathbb{Z}$. [This means x is allowed to be *any* integer, see WP#5].
  Then take $y := x + 1$. Then $y > x$. [See WP#6].

- Exercise 1(d) starts with "Let x and y be real numbers". Note that it didn't say *which* real numbers. This means that x and y could be **any** real number (so the quantifier for x and y is $\forall$). This means you are not supposed to prove this:

  $$\exists_{x,y\in\mathbb{R}} \quad (x \neq y \quad \Longrightarrow \quad \exists_{\epsilon>0}\ |x-y| > \epsilon) \tag{1}$$

  but rather, you are supposed to prove this:

  $$\forall_{x,y\in\mathbb{R}} \quad (x \neq y \quad \Longrightarrow \quad \exists_{\epsilon>0}\ |x-y| > \epsilon) \tag{2}$$

  The WP handout says choosing an example for x, y is OK for proving (**??**) but not OK for proving (**??**). To prove (**??**) you must allow x, y to be *any* real numbers (so do not take an example for x, y). You do have to give an example for ϵ because ϵ sits under an $\exists$ quantifier. Since we do not know what the numbers x, y are, this ϵ will have to depend on x, y. But you may assume almost nothing about x, y other than the premise $x \neq y$.

- **Use induction for integers but not for real numbers.** In question 4, use induction w.r.t. n but not w.r.t. x because that would only prove the statement for $x = 0, 1, 2, \ldots$ but not for $x = 0.347$ or $x = 5.11$ etc.

- **Common error: Not indicating which statements are assumed, derived, or TP, and then mixing them up**.
  Example: Say x is some number, and we are given some statement $S(x)$ about x, and now you're asked to prove that $x = 4$.

  Bad (but remarkably common!) "Proof":
  L1: $x = 4$ [Failed to indicate that this is a T.P. statement]
  L2: Add 1 to both sides.
  L3: $x + 1 = 5$
  L4: Multiply by 2
  L5: $2x + 2 = 10$
  L6: subtract 2 [At this point the author forgot that these lines were]
  L7: $2x = 8$ [derived from the T.P. statement so the author does]
  L8: divide by 2 [not see that none of the lines have been proved]
  L9: $x = 4$. Done!

  This violated Rule #1 in the OP handout (assuming the T.P. statement). There are also situations where it is less obvious that Rule #1 is violated. How to prevent that from happening? Read the first paragraph of the OP handout. Organization is key. The first problem with the above "proof" is that the first line "$x = 4$" should have been "T.P. $x = 4$". That way it is clear that we are not allowed to use that line. We are allowed to rewrite it to an equivalent line (like $x + 1 = 5$ in line L3) but in that case we should indicate that that line L3 is also a T.P. statement. When these indications are missing, eventually a T.P. statement is confused for a "derived" statement causing the author to believe that line L9 is proven (after all, L9 does follow from L7, which in turn follows from L5, which follows from L3, and the author decides to not read back further ...).

  So please, check your proofs and make sure that it is always clear which line is "T.P." or "assumed" or "follows from the previous statement" (use words like hence, thus,... see OP#5) or follows from a combination of previous statements (in that case, indicate which previous statements, e.g. using labels).

  Notice by the way that what was "derived" on line L9 was already on line L1. So in the above "proof" you may delete lines L2 – L9 without changing its (in)validity.

- The previous error is also common in induction proofs. Say you want to give a direct proof for $S(k) \Longrightarrow S(k+1)$. Make sure that when you spell out $S(k)$, you write "Assume" in front of it, and when you spell out $S(k+1)$, you write T.P. in front of it. That helps to prevent many common errors. For derived statements use OP#5, actually, read all of handout OP.

- Students often continue after proving the T.P. statement. When you proved the T.P. statement, you're done! (for clarity, it helps to add something like "this proved $S(k+1)$" or "this proved (1)" etc.).