Intro Advanced Math, test 1 answers.

1. (a) Write down the converse of the statement $p \implies q$.

$q \implies p$

(b) Write down the contrapositive of the statement $p \implies q$.

$\neg q \implies \neg p$

(c) Let $x$ and $y$ be real numbers: Consider the statement

$\left( \forall \epsilon > 0 \right) |x - y| \leq \epsilon \implies x = y$

Write down the contrapositive of this statement and simplify.

$x \neq y \implies \neg \left( \forall \epsilon > 0 \right) |x - y| \leq \epsilon$

Simplifying gives:

$x \neq y \implies \exists \epsilon > 0 \left| x - y \right| > \epsilon$

[Giving just this last statement is sufficient to get full credit.]

(d) Can you prove the statement from part (c)?

[WP#1 tells us to write the next line:]
Assume $x \neq y$. To prove: $\exists \epsilon > 0 \left| x - y \right| > \epsilon$.

[WP#6 tells us to write: Take $\epsilon := \ldots$ so all we have to do is to figure out how to fill in the dots:]

Proof: Take $\epsilon := |x-y|/2$ [then $\epsilon > 0$ because $x \neq y$, and $|x-y| > \epsilon$]

2. Let $A, B, C$ be sets.

(a) Write the defining formula for the difference of two sets:

$x \in A \setminus B \iff \ldots$

$x \in A \land x \notin B$

(b) Write the defining formula for a power set:

$A \in P(C) \iff \ldots$

$A \subseteq C$ [Also correct $A \subset C$ because that means the same thing]

(c) Use your answer on part (a), (b) to prove

$A \in P(C) \implies (A \setminus B) \in P(C)$.

Let $A \in P(C)$. To prove: $A \setminus B \in P(C)$.

$A \in P(C)$ means $A \subseteq C$.

$A \setminus B \subseteq A \subseteq C$ so $A \setminus B \in C$ which means $A \setminus B \in P(C)$.

3. Suppose that $A, B, I$ are sets, and $C_i$ is a set for every $i \in I$. Suppose that $C_i \subseteq B$ for every $i \in I$. Show that

$A \setminus B \subseteq \bigcap_{i \in I} A \setminus C_i$
To prove: $x \in A \setminus B \implies x \in \bigcap_{i \in I} A \setminus C_i$.

Direct proof: Assume $x \in A \setminus B$ in other words $x \in A$ and $x \notin B$.

To prove: $x \in \bigcap_{i \in I} A \setminus C_i$ in other words to prove: $x \in A \setminus C_i$ for all $i \in I$.

Proof: Since $C_i \subseteq B$ and $x \notin B$ it follows that $x \notin C_i$. Combine this with $x \in A$ gives $x \in A \setminus C_i$.

[Check this proof by checking that we didn’t use to-prove statements and only used given/assumed statements like $C_i \subseteq B$ and $x \in A$ and $x \notin B$].

4. Let $x > -1$. Use induction to prove

$$(1 + x)^n \geq 1 + nx$$

for every positive integer $n$.

5. Let $S = \{3k + 1 : k \in \mathbb{Z}\}$, this is the set of integers that have remainder 1 after dividing by 3. If $n \in S$ and $m \in S$ then show that $nm \in S$.

(Hint 1: induction is not a good idea here. Hint 2: do not use the same symbol for different numbers, instead use something like $k, l$ or $k_1, k_2$).

If you run out of time, then mark either Exercise 4 or Exercise 5 as take-home and turn in that one on Monday.
Defining formulas
(only some of these are useful for the test, and some have been deleted)

\[ A = B \quad \text{means} \quad x \in A \iff x \in B. \] (1)

\[ A \subseteq B \quad \text{means} \quad x \in A \Rightarrow x \in B. \] (2)

\[ A = B \quad \text{is equivalent to} \quad A \subseteq B \land B \subseteq A \] (3)

\[ A = \emptyset \quad \text{means} \quad \forall x \ x \notin A. \] (4)

\[ x \in A \bigcap B \quad \text{means} \quad (x \in A \land x \in B) \] (5)

\[ x \in A \bigcup B \quad \text{means} \quad (x \in A \lor x \in B) \] (6)

\[ x \in \bigcap_{i \in I} A_i \quad \text{means} \quad \forall_{i \in I} x \in A_i \] (7)

\[ x \in \bigcup_{i \in I} A_i \quad \text{means} \quad \exists_{i \in I} x \in A_i \] (8)
Writing Proofs.

1. Direct proof for \( p \implies q \).
   Assume: \( p \). To prove: \( q \).

2. Proving \( p \implies q \) by contrapositive.
   Assume: \( \neg q \). To prove: \( \neg p \).

3. Proving \( S \) by contradiction.
   Assume: \( \neg S \). To prove: a contradiction.

4. Proving \( p \implies q \) by contradiction.
   Assume: \( p \) and \( \neg q \). To prove: a contradiction.

5. Direct proof for a \( \forall x \in A \ P(x) \) statement.
   To ensure you prove \( P(x) \) for all (rather than for some) \( x \) in \( A \), do this:
   Start your proof with: Let \( x \in A \). To prove: \( P(x) \).

6. Direct proof for a \( \exists x \in A \ P(x) \) statement.
   Take \( x := \) [write down an expression that is in \( A \), and satisfies \( P(x) \)].

7. Proving \( \forall x \in A \ P(x) \) by contradiction.
   Assume: \( x \in A \) and \( \neg P(x) \). To prove: a contradiction.

8. Proving \( \exists x \in A \ P(x) \) by contradiction.
   Assume: \( \neg P(x) \) for every \( x \in A \). To prove: a contradiction.

9. Proving \( S \) by cases.
   Suppose for example a statement \( p \) can help to prove \( S \). Write two proofs:
   Case 1: Assume \( p \). To prove: \( S \).
   Case 2: Assume \( \neg p \). To prove: \( S \).

10. Proving \( p \land q \)
    Write two separate proofs: To prove: \( p \). To prove: \( q \).

11. Proving \( p \iff q \)
    Write two proofs. To prove: \( p \implies q \) To prove: \( q \implies p \).

12. Proving \( p \lor q \)
    Method (1): Assume \( \neg p \). To prove: \( q \).
    Method (2): Assume \( \neg q \). To prove: \( p \).
    Method (3): Assume \( \neg p \) and \( \neg q \). To prove: a contradiction.

13. Using \( p \lor q \) to prove another statement \( r \).
    Write two proofs:
    Assume \( p \). To prove \( r \).
    Assume \( q \). To prove \( r \).

14. How to use a for-all statement \( \forall x \in A \ P(x) \).
    You need to produce an element of \( A \), then use \( P \) for that element.

15. If you want to use an exists statement like \( \exists x \in A \ P(x) \) to prove another statement, then you may not choose \( x \). All you know is \( x \in A \) and \( P(x) \).