
1. For each, simplify the cardinality to one of: 0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots
   No explanation is needed for (a)–(e). Do explain your answers for (f),(g).

   (a) \{3,3,3\}. Cardinality is 1.
   (b) \mathbb{R} \setminus \mathbb{Q}. Cardinality is \mathcal{C}. [Any other answer would violate item 22]
   (c) \mathbb{N} \times \mathbb{Q}. Cardinality is \aleph_0 \cdot \aleph_0 = \aleph_0.
   (d) \mathcal{P}(\mathbb{Q}). Cardinality is 2^{\aleph_0} = \mathcal{C}. [See items 23 and 24].
   (e) \mathbb{R}^\mathbb{R}. Cardinality is c^c = (2^{\aleph_0})^c = 2^{\aleph_0} = 2^c.
   (f) \mathbb{R}^\mathbb{N}. Cardinality is c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathcal{C}.
   (g) Does there exist a bijection between \mathbb{R}^\mathbb{N} and \mathbb{R}?
      Yes because question (f) showed that both have the same cardinality.

2. Give the definitions:

   (a) A relation \( R \) on a set \( S \) is an equivalence relation when:
      i. [Reflexive] \( \forall x \in S \ xRx \)
      ii. [Symmetric] \( \forall x,y \in S \ xRy \implies yRx \)
      iii. [Transitive] \( \forall x,y,z \in S \ xRy \land yRz \implies xRz \)
         [Here it is OK if you omitted the quantifier \( \forall \).
         It is also OK to write \( (x,y) \in R \) instead of \( xRy \)]
   (b) A function \( f : A \rightarrow B \) is onto (surjective) when:
      \( \forall b \in B \exists a \in A \ f(a) = b. \)
      [Here it is not OK if you omitted quantifiers, but it is always OK to write text like “for all” and “exists” instead of symbols like \( \forall \) and \( \exists \).
      It is also OK to use other letters than \( a, b \).]

3. Suppose that \( f : \mathcal{P}(A) \rightarrow B \) is injective.
   Prove that there is no injective function \( g : B \rightarrow A \).
   \( \text{card}(A) < \text{card}(\mathcal{P}(A)) \leq \text{card}(B) \) (1).
   [The \( < \) is item 7, while the \( \leq \) is because \( f \) is injective]
   Then an injective function \( g : B \rightarrow A \) can not exist because that would imply \( \text{card}(B) \leq \text{card}(A) \) contradicting (1).

4. Suppose that \( A_q \) is a set for every \( q \in \mathbb{Q} \).
   Suppose that for every \( r \in \mathbb{R} \) there is some \( q \in \mathbb{Q} \) for which \( r \in A_q \).
   Must there be some \( q \in \mathbb{Q} \) for which \( A_q \) is uncountable?
   Why or why not?
   Yes, because if every \( A_q \) were countable, then \( S := \bigcup_{q \in \mathbb{Q}} A_q \) would be a countable union of countable sets. Then \( S \) would be countable [item 19],
   contradicting the fact that every \( r \in \mathbb{R} \) is in \( S \).
5. Let $C$ be a set, let $A$ be a subset of $C$, and let $B = C \setminus A$. Suppose there is an injective function from $B$ to $A$ but not from $C$ to $A$. Prove that $C$ must be a finite set.

There is an injective function from $B$ to $A$, so (a): $\text{card}(B) \leq \text{card}(A)$. Suppose $C$ is infinite. To Prove: a contradiction.

$C = A \cup B$ and $A \cap B = \emptyset$ so [see item 21:]

$\text{card}(C) = \text{card}(A) + \text{card}(B)$ [we can use item 22 because $C$ is infinite:]

$= \max(\text{card}(A), \text{card}(B))$

$= \text{card}(A)$ by (a).

But $\text{card}(C) = \text{card}(A)$ contradicts the given statement that there is no injective function from $C$ to $A$.

Items needed from the “List of facts on cardinal numbers” included in the test:

2. $\text{card}(A) \leq \text{card}(B)$ means $\exists f : A \to B$ with $f$ one-to-one [Needed to know this in Ex 3 and Ex 5]

7. $\text{card}(A) < \text{card}(P(A))$ [Must use this in Ex 3]

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable [Must use this or item 26 in Ex 4].

20. $\mathbb{R}$ is uncountable [Need to know this in Ex 4]

21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \cup E$ if we assume that $D \cap E = \emptyset$ [Must use this in Ex 5].

22. If $d, e$ are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$ [Must use this in Ex 5].

23. $\text{card}(P(A)) = 2^{\text{card}(A)}$ [Was used in Ex 1]

24. $c = \text{card}(\mathbb{R}) = 2^{\aleph_0}$ [Was used in Ex 1]

25. $(d_1d_2)^e = d_1^ed_2^e$, $d^{e_1 + e_2} = d^{e_1}d^{e_2}$, $(d^e)^f = d^{ef}$ [Was used in Ex 1]