Answers Test 2, Intro Advanced Math, Oct 18 2019.

1. For each, simplify the cardinality to one of: $0, 1, 2, \ldots, \aleph_0, c, 2^c, 2^{2^c}, \ldots$
   No explanation is needed for (a)–(e). Do explain your answers for (f),(g).

   (a) $\{3, 3, 3\}$. Cardinality is 1.

   (b) $\mathbb{R} \setminus \mathbb{Q}$. Cardinality is c. [Any other answer would violate item 22]

   (c) $\mathbb{N} \times \mathbb{Q}$. Cardinality is $\aleph_0 \cdot \aleph_0 = \aleph_0$.

   (d) $P(\mathbb{Q})$. Cardinality is $2^{\aleph_0} = c$. [See items 23 and 24].

   (e) $\mathbb{R}^{\mathbb{R}}$. Cardinality is $c^c = (2^{\aleph_0})^c = 2^{\aleph_0 c} = 2^c$.

   (f) $\mathbb{R}^{\mathbb{N}}$. Cardinality is $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0} = c$.

   (g) Does there exist a bijection between $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}$?
   Yes because question (f) showed that both have the same cardinality.

2. Give the definitions:

   (a) A relation R on a set S is an *equivalence relation* when:

   i. [Reflexive] $\forall_{x \in S} \;\; xRx$
   ii. [Symmetric] $\forall_{x,y \in S} \;\; xRy \Longrightarrow yRx$
   iii. [Transitive] $\forall_{x,y,z \in S} \;\; xRy \wedge yRz \Longrightarrow xRz$

   [Here it is OK if you omitted the quantifier $\forall$.
   It is also OK to write $(x, y) \in R$ instead of xRy]

   (b) A function $f : A \to B$ is *onto* (*surjective*) when:

   $\forall_{b \in B} \exists_{a \in A} \; f(a) = b$.

   [Here it is not OK if you omitted quantifiers, but it is always OK to write text like "for all" and "exists" instead of symbols like $\forall$ and $\exists$. It is also OK to use other letters than a, b.]

3. Suppose that $f : P(A) \to B$ is injective.
   Prove that there is no injective function $g : B \to A$.

   $\text{card}(A) < \text{card}(P(A)) \leq \text{card}(B)$ (1).

   [The $<$ is item 7, while the $\leq$ is because f is injective]

   Then an injective function $g : B \to A$ can not exist because that would imply $\text{card}(B) \leq \text{card}(A)$ contradicting (1).

4. Suppose that A_q is a set for every $q \in \mathbb{Q}$.
   Suppose that for every $r \in \mathbb{R}$ there is some $q \in \mathbb{Q}$ for which $r \in A_q$.
   Must there be some $q \in \mathbb{Q}$ for which A_q is uncountable?
   Why or why not?

   Yes, because if every A_q were countable, then $S := \bigcup_{q \in \mathbb{Q}} A_q$ would be a countable union of countable sets. Then S would be countable [item 19], contradicting the fact that every $r \in \mathbb{R}$ is in S.

5. Let C be a set, let A be a subset of C, and let $B = C \setminus A$. Suppose there is an injective function from B to A but not from C to A. Prove that C must be a finite set.

   There is an injective function from B to A, so (a): $\text{card}(B) \leq \text{card}(A)$. Suppose C is infinite. To Prove: a contradiction.

   $C = A \bigcup B$ and $A \bigcap B = \emptyset$ so [see item 21:]

   $$\begin{aligned}\text{card}(C) &= \text{card}(A) + \text{card}(B) \quad \text{[we can use item 22 because } C \text{ is infinite:]}\\ &= \max(\text{card}(A), \text{card}(B))\\ &= \text{card}(A) \quad \text{by (a).}\end{aligned}$$

   But $\text{card}(C) = \text{card}(A)$ contradicts the given statement that there is no injective function from C to A.

Items needed from the "List of facts on cardinal numbers" included in the test:

2. $\text{card}(A) \leq \text{card}(B)$ means $\exists f : A \to B$ with f one-to-one [Needed to know this in Ex 3 and Ex 5]
7. $\text{card}(A) < \text{card}(P(A))$ [**Must use this in Ex 3**]
19. If you have countably many sets, and if each of these sets is countable, then their union is also countable [**Must use this** or item 26 **in Ex 4**].
20. $\mathbb{R}$ is uncountable [Need to know this in Ex 4]
21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$ [**Must use this in Ex 5**].
22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$ [**Must use this in Ex 5**].
23. $\text{card}(P(A)) = 2^{\text{card}(A)}$ [Was used in Ex 1]
24. $c = \text{card}(\mathbb{R}) = 2^{\aleph_0}$ [Was used in Ex 1]
25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$ [Was used in Ex 1]