Two HW questions that were turned in on Dec 2

1. Suppose S is discrete and closed. Suppose sequence $a_1, a_2, a_3, \ldots \in S$ converges. Show that a tail of the sequence is constant.

   Proof: Say that the sequence converges to s. Then $s \in S$ because S is closed, see 10(b). Since S is discrete, item 20 tells us that $(s-\epsilon, s+\epsilon) \bigcap S = \{s\}$ for some $\epsilon > 0$. A tail of our sequence is in $(s-\epsilon, s+\epsilon)$, see item 8. Since our sequence is in S, that tail will be in $(s-\epsilon, s+\epsilon) \bigcap S = \{s\}$. So that tail will be $s, s, s, \ldots$.

2. Let U be a subset of $\mathbb{R}$ and suppose that: (a) every sequence that converges to a point in U has a tail in U.
   Then show that (b): U must be open.

   Proof #1: Assume (a). Let S be the complement of U. (Why am I taking a complement? Because in the list of facts, convergent sequences tend to appear in the context of closed sets rather than open sets.) We will prove that $S = \overline{S}$. To do this, pick $x \in \overline{S}$. Then 11(f) says that there is a sequence in S that converges to x. But S is the complement of U so no tail of that sequence is in U. Then the contrapositive of assumption (a) tells us that that sequence does not converge to a point in U. But it converges to x. So $x \notin U$, hence $x \in S$.
   This proved $\overline{S} \subseteq S$, hence $\overline{S} = S$, hence S is closed, hence U is open.

   Proof #2: Lets do a proof by contrapositive, so we suppose that U is not open, and we have to prove that there is a sequence that converges to a point in U, a sequence that does not have a tail in U.

   Negating item 3 we get that there is $x \in U$ such that $\forall_{\epsilon>0}\ (x-\epsilon, x+\epsilon) \not\subseteq U$.

   Proof #2a:
   (Note: $A \not\subseteq B$ is the same as saying that there exists an element $a \in A \setminus B$.)
   Now for each n, take $\epsilon = 1/n$ and take a point $a_n \in (x - 1/n, x + 1/n) \setminus U$. That sequence $a_1, a_2, \ldots$ converges to x but has no tail in U because $a_n \notin U$ for each n.

   Proof #2b: $(x-\epsilon, x+\epsilon) \not\subseteq U$ is the same as saying that $(x-\epsilon, x+\epsilon)$ intersects ${}^{c}U$, which is equivalent (see 11(e)) to saying that x is in the closure of ${}^{c}U$, which is equivalent (see 11(f)) to saying that there is a sequence in ${}^{c}U$ that converges to x. No tail (or element) of that sequence will be in U.