List of facts on cardinal numbers, shortened version.

Note: During the actual test, basic definitions that everyone must know (such as items 1–7) may be deleted!

1. $\text{card}(A) = \text{card}(B)$ means $\exists f : A \to B$ with f bijection.

2. $\text{card}(A) \le \text{card}(B)$ means $\exists f : A \to B$ with f one-to-one.

3. $\aleph_0$ is short notation for $\text{card}(\mathbb{N}^*)$.

4. c is short notation for $\text{card}(\mathbb{R})$.

5. The set A is *countably infinite* when: $\text{card}(A) = \aleph_0$.
   By item 1 this means: $\exists f : \mathbb{N}^* \to A$ with f bijection. Note, in that case $A = f(\mathbb{N}^*) = f(\{1, 2, \ldots\}) = \{f(1), f(2), \ldots\}$ and this means that all elements of A fit into one sequence $f(1), f(2), \ldots$.

6. Notation: $x < y$ is short for: $x \le y \wedge x \ne y$.

7. $\text{card}(A) < \text{card}(P(A))$.

8. Item 7 implies that not all infinite sets have the same cardinality!
   The cardinal number $\text{card}(\mathbb{N}^*) = \aleph_0$, is NOT the largest possible cardinality despite the fact that it is infinite! After all, $P(\mathbb{N}^*)$ has larger cardinality by item 7. And $P(P(\mathbb{N}^*))$ has larger cardinality still!

9. If $f : A \to B$ is onto then $\text{card}(B) \le \text{card}(A)$.

10. A is *countable* when either: A is countably infinite (defined in item 5) or A is finite.

11. A is countable when $\text{card}(A) \le \aleph_0$.

12. A subset of a countable set is again countable.

13. If $A \subseteq B$ then $\text{card}(A) \le \text{card}(B)$.

14. The ordering $\le$ on cardinal numbers is a *partial ordering*.
    In particular: whenever $d \le e$ and $e \le d$ we may conclude $d = e$.
    The proof is not easy! (Schroeder-Bernstein theorem on p 88–89).

15. The ordering $\le$ on cardinal numbers is a *total ordering*. So given any two cardinals d, e we have $d \le e$ or $d \ge e$. This means that one of these things must be true: $d < e$ or $d = e$ or $d > e$.

16. Set A is uncountable when $\text{card}(A) \not\le \aleph_0$. Using item 15 we can reformulate this by saying: A is uncountable when $\text{card}(A) > \aleph_0$.

17. Any infinite set contains a countably infinite subset. (note: That an uncountable set has a countably infinite subset follows from item 16).

18. $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

19. If you have countably many sets, and if each of these sets is countable, then their union is also countable.

20. $\mathbb{R}$ is uncountable. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*))$.

21. If $d = \text{card}(D)$ and $e = \text{card}(E)$ then $d + e$ is the cardinality of $D \bigcup E$ if we assume that $D \bigcap E = \emptyset$. Likewise, $d \cdot e$ is the cardinality of $D \times E$. d^e is the cardinality of D^E where $D^E = \{\text{all functions from } E \text{ to } D\}$.

22. If d, e are cardinal numbers, and if at least one of them is infinite, then $d + e = \max(d, e)$.

    If $d \neq 0$ and $e \neq 0$ and at least one of them is infinite, then $d \cdot e$ equals $\max(d, e)$ as well. So for non-zero cardinals with at least one infinite, the operations $+, \cdot, \max$ are the same!

23. There is a bijection between $P(A)$ and $\{0, 1\}^A$, and hence $\text{card}(P(A)) = \text{card}(\{0, 1\}^A) = \text{card}(\{0, 1\})^{\text{card}(A)} = 2^{\text{card}(A)}$.

24. $c = \text{card}(\mathbb{R}) = \text{card}(P(\mathbb{N}^*)) = \text{card}(\{0, 1\}^{\mathbb{N}^*}) = 2^{\text{card}(\mathbb{N}^*)} = 2^{\aleph_0}$.

25. $(d_1 d_2)^e = d_1^e d_2^e, \quad d^{e_1+e_2} = d^{e_1} d^{e_2}, \quad (d^e)^f = d^{ef}$

26. If you have d sets, and each of these sets has cardinality e, and if A is the union of all those sets, then $\text{card}(A) \leq de$ (if the d sets are disjoint, then you may replace the $\leq$ by $=$). Now if d or e is infinite, and both are non-zero, then we can also replace de by max(d,e), see item 22.

27. So far we have encountered these increasing cardinals:

    $$0,\ 1,\ 2,\ 3,\ \ldots\ \ \aleph_0,\ \ c = 2^{\aleph_0},\ \ 2^c,\ \ 2^{2^c},\ \ldots$$

    and we can wonder if there are any cardinals in between. Specifically, the *continuum hypothesis* asks if there is a cardinal d with $\aleph_0 < d < c$.

    From the axioms of set theory (= the only statements mathematicians accept without a proof) it is impossible to prove or disprove this.